Abstract. Let $\phi^n$ be an arbitrary compact topological manifold. We shall define a new topological object, $\Omega(t)\phi^n$, known as a lure, that alters the Minkowski Content of $\phi^n$ by acting on the boundary of $\phi^n$, $\partial\phi^n$ by stretching or shrinking the boundary. After the notion of a lure has been established for compact manifolds, we shall relax the requirement that the manifold be locally Euclidean, by defining $\Lambda^n$ to be an arbitrary topological manifold of domain in Euclidean Space. We shall then give a similar definition of $\Omega(t)\Lambda^n$ for these more general manifolds.

1. Introduction

Topology is the mathematical study of the properties that are invariant under homomorphisms. A circle is topologically equivalent to an ellipse (into which it can be deformed by stretching) and a sphere is equivalent to an ellipsoid. In the subsequent paper, we shall define a mathematical object, a lure, that continuously stretches a topological manifold. We shall define a homeomorphism in terms of an infinite set of lures acting on some arbitrary compact topological manifold.

A topological space $X$ is called locally Euclidean if $\exists n \in \mathbb{N}^+$ such that $\forall x \in X$, $x$ has a neighborhood which is homeomorphic to the Euclidean space $\mathbb{E}^n$. A topological space $X$ is said to be a Hausdorff Space if, $\forall q \neq q'$ there exists open sets of $X$ such that $q \in M$ and $q' \in N$ with $M \cap N = \emptyset$ Recall that an $n$-dimensional topological manifold is defined to be locally Euclidean Hausdorff Space, as they shall be the main topic of study in the subsequent paper.

We shall now introduce the notion of Minkowski Content, as it shall be the primary measure utilized in this paper. In order to do so, we must first introduce the concept of a Lebesgue Measure in $n$ dimensions.

Consider a box $B \subset \mathbb{R}^n \mid n \in \mathbb{N}$. $\forall i \in I := \{1, \ldots, n\}, a_i < b_i \in \mathbb{R}$, we define this box as

$$B := \prod_{i \in I} [a_i, b_i]$$

where we utilize the usual Cartesian Product. The volume of such a box is therefore simply

$$\text{Vol}(B) = \prod_{i \in I} (b_i - a_i)$$

notice $\text{Vol}(B) > 0$ as $a_i < b_i$. We utilize a direct product in this notion. So we shall consider some arbitrary subset of this box, namely any compact $n$-dimensional topological manifold, $\mathcal{A} \mid \mathcal{A} \subset B$. 

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**Definition** We define a Lebesgue Measure in the following manner:

$$\mu^n(A) := \inf \left\{ \sum_{B \in C} \text{Vol}(B) \mid \bigcup C = A \right\}$$

Consequently, we may now define the Minkowski Content for some subset $A$.

Let $(X, \mu, d)$ be a metric measure space where $d$ is a metric on $X$ and $\mu$ is the Lebesgue Measure. Define $A_\epsilon$, where $\{x \in X \mid d(A, x) \leq \epsilon\}$

**Definition** We may define the $m$-dimensional upper Minkowski Content of $A$ as

$$M^* m(A) := \lim_{\epsilon \to 0} \sup \frac{\mu((A)_\epsilon \setminus A)}{\gamma_{n-m} \epsilon^{n-m}}$$

we define the $m$-dimensional lower Minkowski Content of $A$ as

$$M_* m(A) := \lim_{\epsilon \to 0} \inf \frac{\mu((A)_\epsilon \setminus A)}{\gamma_{n-m} \epsilon^{n-m}}$$

where $\gamma_{n-m}$ is the volume of the $n-m$ dimensional ball.

Let $\phi^n$ be a $n$-dimensional compact topological manifold with boundary $\partial \phi^n$. Let $\partial \varphi(t)$ be the new boundary of $\phi^n$ defined later in the paper. Define explicitly the time dependent Minkowski Content as

**Definition** We may define the $m$-dimensional upper Minkowski Content of $A$ as

$$M^{*m}(A, t) := \lim_{\epsilon \to 0} \sup \frac{\mu((A)_\epsilon \setminus A, t)}{\gamma_{n-m} \epsilon^{n-m}}$$

we define the $m$-dimensional lower Minkowski Content of $A$ as

$$M_* m(A, t) := \lim_{\epsilon \to 0} \inf \frac{\mu((A)_\epsilon \setminus A, t)}{\gamma_{n-m} \epsilon^{n-m}}$$

with

$$\mu(A, 0) := \inf \left\{ \sum_{B \in C} \text{Vol}(B) \mid C \text{ covers } \phi^n \text{ up to } \partial \phi^n \right\}$$

and

$$\mu(A, t) := \inf \left\{ \sum_{B \in C} \text{Vol}(B) \mid C \text{ covers } \phi^n \text{ up to } (\partial \phi^n \setminus \partial \varphi(0)) \cup \partial \varphi(t) \right\}$$

notice that if $n = m$, then we are simply left with the Lebesgue Measure of $A$ at time $t$.

Notice that we want the Lebesgue measure to cover up to $(\partial \phi^n \setminus \partial \varphi(0)) \cup \partial \varphi(t)$ as we want the measure to cover up to the new boundary $\partial \varphi(t)$, but excluding the initial boundary $\partial \varphi(0)$.

### 2. One Dimensional Framework

#### 2.1. Measure Increasing Lures

Let $M$ be the Minkowski Content defined above, however, let $M$ be time dependent such that $M : \mathcal{A} \times \mathbb{R}^+ \to [0, \infty)$ where $\mathcal{A}$ is some arbitrary compact topological manifold, and $\mathbb{R}^+$ is the set of real time. Let $\phi := [a, b] | b > a$ be a compact subset of $\mathbb{R}$. Let $\mathcal{M}(t, \phi) := |b(t) - a|$ with $\mathcal{M}(0, \phi) = |b - a|

**Definition** We define an attractor $\alpha$ as set of properties towards which a system evolves towards as time progresses.
We call the attractor $\alpha$ a quasi-attractor as it does not behave in the same manner as a true attractor. A quasi-attractor $\alpha$ is a set of points in space $\beta_i$ towards which a system may move: $\cup_{i \in I} \beta_i \in \mathcal{R}_\infty$, where $\mathcal{R}_\infty$ is the infinity set of the lure, to be later defined in 2.1.

We define the basin of attraction for a quasi-attractor $\alpha$ as a compact subset $\zeta$ of $\mathbb{R}^n$ containing the set of points which shall be deformed, while excluding the points not of concern. For example, if we are working with some compact subset $[a, b] \subset \mathbb{R}$, a basin of attractor would be defined in the following way: Define $\mathcal{R}_i \cup_{i \in \mathbb{R}}$ as a true attractor. A quasi-attractor $\alpha$ to be the basin of attraction for $b$ such that $\mathcal{R}_\alpha$ is a set of points in space which shall be deformed, while excluding the points not of concern. For example, if we are working with some compact subset $[a, b] \subset \mathbb{R}$, a basin of attractor would be defined in the following way: Define $\mathcal{R}_i \cup_{i \in \mathbb{R}}$ as a true attractor. A quasi-attractor $\alpha$ to be the basin of attraction for $b$. Moreover, let $b(t)$ be a monotonically increasing function of $t$ such that $b(t) : \mathbb{R}^+ \to \mathbb{R}^+$

**Definition** The lure, $\Omega$, of $\phi$ is defined as

$$\Omega_\phi(t) := \bigcup_{i \in \mathbb{N}} \inf \{ \mathcal{R}_i(t) \mid b(t) \in \mathcal{R}_i(t), a \notin \mathcal{R}_i(t) \}$$

for $i \in \mathbb{N}$ where $\mathcal{R}_i(t)$ is a compact subset of $\mathbb{R}$ with $\forall t \in \Gamma, \mathcal{M}(\Omega_\phi(t)) = g \in \mathbb{R}^+$ and $\forall i > j, \mathcal{M}(\mathcal{R}_i(t)) > \mathcal{M}(\mathcal{R}_j(t))$ s.t. $\mathcal{R}_j(t) \subset \mathcal{R}_i(t)$. Additionally, $\forall T > t, \mathcal{M}(T - t, \Omega_\phi) = \Omega(T) - \Omega(t)$ so that $\forall \mathcal{R}_i(t) \mid |\mathcal{R}_i(T) - \mathcal{R}_i(t)| \geq \lambda, \bigcup_{i=1}^N \mathcal{R}_i(t) \notin \Omega(T)$ so that $\mathcal{R}_i(T) - \mathcal{R}_N(t) = \lambda$ where $\alpha$ is fixed in in $\mathcal{R}_\infty$, and is not independently dynamic, but simply dependent on the motion of $\Omega(t)_\phi$.

Recall that $\mu\left(\bigcup_{i=1}^\infty \mathcal{R}_i(t)\right) = \lim_{i \to \infty} \mathcal{R}_i(t) = g \in \mathbb{R}^+$.

**2.2. Measure Decreasing Lures.** Let $\mathcal{M}$ be the Minkowski Content defined above, however, let $\mathcal{M}$ be time dependent such that $\mathcal{M} : A \times \mathbb{R}^+ \to [0, \infty)$ where $A$ is some arbitrary compact topological manifold, and $\mathbb{R}^+$ is the set of real time. Let $\phi := [a, b] \mid b > a$ be a compact subset of $\mathbb{R}$. Let $\mathcal{M}(t, \phi) := |b(t) - a|$ with $\mathcal{M}(0, \phi) = |b - a|$. Let $b(t)$ be a monotonically decreasing function. The definition for a measure decreasing lure is synonymous with that of a measure increasing lure, simply with the condition that measure decreases over time and shall be denoted $\Omega^{-1}(t)_\phi$.

So we can think of a one dimensional lure as a dynamically moving infinite union of sets whose union is length $g$ where, after $\Omega$ has moved a distance $\lambda$ along the real line, the sets of the first $\lambda$ units drop out of the union $\Omega_\phi$, such that $\Omega_\phi$ lures $\phi$ by increasing its length. Notice that only one lure may exist in one dimension, as if we have a collection of lure $\{\Omega_i(t)_\phi\}_{i=1}^n$ will simply act as one lure $\Omega_\phi$.

**Theorem 2.1.** Given any two compact subsets of $\mathbb{R}$, $\mu(\phi_1) > \mu(\phi_2)$,

$$\exists \Omega^{-1}(t)_{\phi_1}, \Omega(t)_{\phi_2} \mid \Omega(t)_{\phi_2} \circ \Omega^{-1}(t)_{\phi_1} : \phi_1 \to \phi_2 \to \phi_1$$

**Proof.** By definition of a lure in one dimension, $\Omega(t)_{\phi_1} : \phi_1 \to \phi_2 \mid \mathcal{M}(0, \phi_1) < \mathcal{M}(t, \phi_1 = \phi_2)$. As a result, there exists a natural inverse that simply reduces the length of the subset $\phi_2$ to $\phi_1$. $\exists! T > t \mid \mathcal{M}(T, \phi_2 = \phi_1) = \mathcal{M}(t, \phi_1 = \phi_2)$

To see that this time is unique,

**Lemma 2.2.** Note that $\mathcal{M}$ depends continuously on $t$. Since $\mathcal{M}$ is continuous, the intermediate value theorem holds, $\therefore T$ is unique.

The inverse of this theorem also holds true by the same logic.
Definition Given two spaces $X$ and $Y$, we say they are homotopy equivalent if there exist functions $f, g : X \to Y$ such that $g \circ f \simeq I_X$ and $f \circ g \simeq I_Y$, where $\simeq$ indicates homotopic objects.

Corollary 2.3. All lures $\Omega(t)_{\phi}$ create a homotopy equivalence class with respect to $\phi$.

Proof. We see trivially that $\Omega(t)_{\phi_2} \circ \Omega^{-1}(t)_{\phi_1} \simeq g \circ f$ and $\Omega^{-1}(t)_{\phi_2} \circ \Omega(t)_{\phi_2} \simeq f \circ g$. 

3. Two Dimensional Framework

3.1. Measure Increasing Lures. Let $M^2$ be the two dimensional Minkowski Content defined in section 1, such that $M^2 : A \times R^+ \to [0, \infty)$ where $A$ is some arbitrary compact topological manifold, and $R^+$ is the set of real time. Let $\phi^2$ be a two dimensional compact topological manifold with boundary $\partial \phi^2$. Let $\partial \phi(t)$ be a subset of $\partial \phi^2$ that will be acted upon by $\alpha$. When acted upon, we have the necessary condition that $\forall T > t, M^2(T, \phi^2) > M^2(T, \phi^2)$. In two dimensions, we define the basin of attraction $R$ of $\alpha$ as a tubular neighborhood, equal in width to $diam(R) = diam(\partial \phi(t))$ as shown in Figure 1.

**Figure 1.** Depiction of a Two Dimensional Lure, $\Omega^2(\phi)$ at $t = 0$.

Definition The lure, $\Omega^2_{\phi^2}(t)$ is defined as

$$\Omega^2_{\phi^2}(t) := \bigcup_{i \in N} \inf \{R_i(t) \mid diam(R_i(t)) = diam(\partial \phi(t)), \partial \phi(t) \in R_i(t), \partial \phi^2 \setminus \partial \phi(t) \notin R_i(t) \}$$

for $i \in N$, where $R_i(t)$ is a compact subset of $\mathbb{R}^2$ with $M^2(\Omega^2(t)_{\phi^2}) = g \in \mathbb{R}^+$ and $\forall i = j, M^2(R_i) > M^2(R_j)$ and s.t. $R_j \subset R_i$. Fix a point $\omega_{\phi^2} \in \partial \phi(0)$ and define a point $\omega_{\phi^2} \in R_1(t)$. As $t$ progresses, there exists a path $\nu(t)$ that the lure takes, without loop, connection $\omega_{\phi^2}$ and $\omega_{\phi^2}$ with length $\lambda$, representing the smallest Euclidean distance in $\mathbb{R}^2$. Define a fixed point $\eta_{\phi^2}(t) \in R_i(t)$. $\forall \eta_{\phi^2}(t) \in R_i(t), \exists \eta_{\phi^2}(t) \in R_i(t) : |\eta_{\phi^2}(T) - \eta_{\phi^2}(t)| \geq \lambda$. Define $\eta_{\phi^2}(t) \notin \Omega^2(T)_{\phi^2}$ so that $|\eta_{\phi^2}(T) - \eta_{\phi^2}(t)| = \lambda$ and $\alpha$ is fixed in $R_{\infty}$, and is not independently dynamic, but dependent on the motion of $\Omega(t)_{\phi^2}$.
3.2. Measure Decreasing Lures. Let $\mathcal{M}^2$ be the Minkowski Content defined above, however, let $\mathcal{M}^2$ be time dependent such that $\mathcal{M}^2: \mathcal{A} \times \mathbb{R}^+ \rightarrow [0, \infty)$ where $\mathcal{A}$ is some arbitrary compact topological manifold, and $\mathbb{R}^+$ is the set of real time. Let $\phi^2$ be a compact two dimensional topological manifold. The definition for a measure decreasing lure is synonymous with that of a measure increasing lure, simply with the condition that measure decreases over time and shall be denoted $\Omega^{-2}(t)_\phi$.

We can therefore think of a two dimensional lure as a dynamically moving infinite union of two dimensional sets with diameter equal to the diameter of the boundary after $\Omega$ has moved a distance of $\lambda$, the sets that previously contained the first $\lambda$ units of $\nu(t)$ drop out of $\Omega^2(t)_\phi$, so that $\forall T \neq t, \mathcal{M}^2(\Omega^2(T)_{\phi^2}) = \mathcal{M}^2(\Omega^2(t)_{\phi^2})$.

Theorem 3.1. Given any two compact subsets of $\mathbb{R}^2$, $\mu^2(\phi^2_1) > \mu^2(\phi^2_2)$,

$$\exists \Omega^{-2}(t)_{\phi^2_1}; \Omega^2(t)_{\phi^2_2} \mid \Omega^2(t)_{\phi^2_1} \circ \Omega^{-2}(t)_{\phi^2_1} : \phi^2_1 \rightarrow \phi^2_2 \rightarrow \phi^2_1$$

Proof. By definition of a lure in two dimensions, $\Omega^2(t)_{\phi^2_1}, \phi^2_1 \rightarrow \phi^2_2 | \mathcal{M}^2(0, \phi^2_1) < \mathcal{M}^2(t, \phi^2_1 = \phi^2_2)$. As a result, there exists a natural inverse that simply reduces the length of the subset $\phi^2_1$ to $\phi^2_2$. $\exists ! T > t \mid \mathcal{M}^2(T, \phi^2_2 = \phi^2_1) = \mathcal{M}^2(t, \phi^2_1 = \phi^2_2)$

To see that this time is unique,

Lemma 3.2. Note that $\mathcal{M}^2$ depends continuously on $t$. Since $\mathcal{M}^2$ is continuous, the intermediate value theorem holds, '. $T$ is unique.

The inverse of this theorem also holds true by the same logic.

Corollary 3.3. All lures $\Omega^2(t)_{\phi^2}$ create a homotopy equivalence class with respect to $\phi$.

Proof. We see trivially that $\Omega^2(t)_{\phi^2} \circ \Omega^{-2}(t)_{\phi^2} \simeq g \circ f$ and $\Omega^{-2}(t)_{\phi^2} \circ \Omega^2(t)_{\phi^2} \simeq f \circ g$. $

3.3. Simple Two Dimensional Examples. We consider a simple two dimensional example involving a lure acting on the boundary of a circle $\partial \phi^2$, described explicitly as $x = \cos(t), y = \sin(t)$. Also define $\partial \phi^2(0) = (a \leq t \leq b)$ for $0 \leq a < b \leq 2\pi$. We define an attractor $\alpha : (x, y) \in \partial \phi^2(t) \mapsto (x + t, y + t)$. We define $\Omega^2_1(t)$ and $\Omega^2_2(t)$ as the lures containing $\alpha$ and acting on the arcs from $\frac{\pi}{2} \leq t \leq \pi$ and $\frac{3\pi}{2} \leq t \leq 2\pi$ respectively. The effect of the lures on $\phi^2$ is shown below in Figure 2, with the change in area at time $T$ given as $\Delta(\mathcal{M}^2)(T) = T\pi$.

We should also consider why indeed it is advantageous to consider a set of lures acting independently on a compact topological manifold, as opposed to a single continuous deformation. This becomes rather apparent if we consider a set $\{\Omega^2_i(t)\}_{i=1}^n$ acting on, for sake of simplicity, a circle in $\mathbb{R}^2$. We can define the attractors $\alpha_i \in \Omega^2_i \mid \alpha_i(t) : (x, y) \in \partial \phi^2(t) \mapsto (x + \gamma_i t, y + \gamma_i t)$ where $\forall i > j, \gamma_i > \gamma_j$ with $\gamma_i \in [0, 1]$ i.e. $\alpha$ increases the radius more and more as the radial degree increases. This independent deformation allows for the construction of such manifolds such as Figure 3. Note that theorem 2 should be rather evident in this case, from the figure.
4. Three Dimensional Framework

Let $\mathcal{M}^3$ be the three dimensional Minkowski Content defined in section 1, such that $\mathcal{M}^3$ be time dependent such that $\mathcal{M}^3 : \mathcal{A} \times \mathbb{R}^+ \rightarrow [0, \infty)$ where $\mathcal{A}$ is some arbitrary compact topological manifold, and $\mathbb{R}^+$ is the set of real time. Let $\phi^3$ be a three dimensional compact topological manifold with boundary $\partial \phi^3$. Let $\partial \phi(t)$ be a subset of $\partial \phi^3$ that will be acted upon by $\alpha$. When acted upon, we have the necessary condition that $\forall T > t, \mathcal{M}^3(T, \phi^3) > \mathcal{M}^3(t, \phi^3)$. In three dimensions, it becomes inherently more difficult to define $\mathcal{R}$ of $\alpha$. We shall utilize the notion of a refined topological cover for this purpose.
**Definition** We define a cover of a topological space \( X, C \) as the indexed family of sets 
\[
C = \{ U_\sigma : \sigma \in A \}
\]
the \( C \) is a cover of \( X \) if 
\[
X \subseteq \bigcup_{\sigma \in A} U_\sigma
\]
A refined cover \( C \) of \( X \) is the new cover \( D \) of \( X \) such that \( C = \{ V_\beta : \beta \in B \} \) is a refinement of \( U_{\sigma \in A} \) when 
\[
\forall \beta \exists V_\beta \subseteq U_\sigma
\]
Therefore we can define 
\[
\Omega := \{ R_i(t) = \{ U_\sigma : \sigma \in A \} \mid \partial \varphi(t) \subseteq \bigcup_{\sigma \in A} U_\sigma \}
\]
So each set \( R_i(t) \in \Omega^3(t)^3 \) is a refinement cover of the boundary on which we are acting.

**Definition** The lure, \( \Omega^3(t)^3 \) is defined as 
\[
\Omega^3(t)^3 := \bigcup_{i \in \mathbb{N}} \inf \{ R_i(t) \mid \partial \varphi(t) \in R_i(t), \partial \varphi(t) \notin R_i(t) \}
\]
for \( i \in \mathbb{N} \), where \( R_i(t) \) is a compact subset of \( \mathbb{R}^3 \), namely a refined cover of \( \partial \varphi(t) \) with \( \Omega^3(\Omega^3(t)^3) = g \in \mathbb{R}^+ \) and \( \forall i > j, \Omega^3(R_i) > \Omega^3(R_j) \) and s.t. \( R_j \subseteq R_i \).

Fix a point \( \omega \in \partial \varphi(0) \) and define a point \( \omega_{R_i}(t) \in R_i(t) \). As \( t \) progresses, there exists a path \( \nu \) that the lure takes, without loop, connection \( \omega \) and \( \omega_{R_i}(t) \) with length \( \lambda \), representing the smallest Euclidean distance in \( \mathbb{R}^3 \). Define a fixed point \( \eta_{R_i}(t) \in R_i(t) \). \( \forall \eta_{R_i}(t) \in R_i(t) : [\eta_{R_i}(T) - \eta_{R_i}(T)] \geq \lambda, \bigcup_{i=1}^N R_i(t) \notin \Omega^3(T)^3 \) so that \( \| \eta_{R_i}(T) - \eta_{R_i(T)} \| = \lambda \) and \( \alpha \) is fixed in \( R_\infty \), and is not independently dynamic, but dependent on the motion of \( \Omega(t)^3 \).

**4.1. Measure Decreasing Lures.** Let \( \mathcal{M}^3 \) be the three dimensional Minkowski Content defined in section 1, such that \( \mathcal{M}^3 \) be time dependent such that \( \mathcal{M}^3 : A \times \mathbb{R}^+ \to [0, \infty) \) where \( A \) is some arbitrary compact topological manifold, and \( \mathbb{R}^+ \) is the set of real time. Let \( \phi \) be a three dimensional compact topological manifold with boundary \( \partial \phi \). The definition for a measure decreasing lure is synonymous with that of a measure increasing lure, simply with the condition that measure decreases over time and shall be denoted \( \Omega^{-3}(t)^3 \).

We can therefore think of a three dimensional lure as a dynamically moving infinite union of refinement covers of the boundary which we are acting on with the quasi-attractor \( \alpha \).

**Theorem 4.1.** Given any two compact subsets of \( \mathbb{R}^3 \), \( \mu^3(\phi_1^3) > \mu^3(\phi_2^3) \),
\[
\exists \Omega^{-3}(t)^3, \Omega^3(t)^3 \circ \Omega^{-3}(t)^3 : \phi_1^3 \to \phi_2^3 \to \phi_1^3
\]
**Proof.** By definition of a lure in two dimensions, \( \Omega^3(t)^3 : \phi_1^3 \to \phi_2^3 \to \mathcal{M}(0, \phi_1^3) \)
\[
\mathcal{M}(t, \phi_1^2 = \phi_2^2) \}
\]
As a result, there exists a natural inverse that simply reduces the length of the subset \( \phi_2^3 \) to \( \phi_1^3 \). \( \exists ! T > t \) where \( \mathcal{M}(T, \phi_1 = \phi_1^2) = \mathcal{M}(t, \phi_1 = \phi_2^3) \)
To see that this time is unique,

**Lemma 4.2.** Note that \( \mathcal{M}^3 \) depends continuously on \( t \). Since \( \mathcal{M}^3 \) is continuous, the intermediate value theorem holds, \( \therefore T \) is unique.
The inverse of this theorem also holds true by the same logic.

**Corollary 4.3.** All lures $\Omega^3(t)\varphi_3$ create a homotopy equivalence class with respect to $\phi$.

**Proof.** We see trivially that $\Omega^3(t)\varphi_2 \circ \Omega^{-3}(t)\varphi_3 \simeq g \circ f$ and $\Omega^{-3}(t)\varphi_1 \circ \Omega^3(t)\varphi_2 \simeq f \circ g$. \qed

4.2. **Simple Three Dimensional Examples.**

5. **$N$ Dimensional Framework**

6. **Lures as $N$ Dimensional Homotopy Equivalences**

7. **Lures Acting on Non-Locally Euclidean Spaces**

**Definition** Given $N \in \mathbb{N}$, $n \geq 2$, a self-similar system $\Psi := \{\psi_j\}_{j=1}^N$ as a finite family of contracting similarities on a complete metric space $(X, d_X)$. Thus, $\forall x, y \in X$, and for each $j = 1, \ldots, N$, we have

$$d_X(\psi_j(x), \psi_j(y)) = r_j d_X(x, y)$$

where $0 < r_j < 1$ is known as the Lipschitz Constant (or scaling ratio) of $\psi_j$ for each $j = 1, \ldots, N$.

**Definition** We define the cantor set in the following manner:

Let $\mathcal{J} := \{0, 2, \ldots, 3^m - 1\}$ for $m \in \mathbb{N}$, then

$$C = [0, 1] \setminus \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \mathcal{J}} \left(\frac{3k + 1}{3^m}, \frac{3k + 2}{3^m}\right)$$

8. **An Alternate Definition Using Fixed Point Attractors**

We shall now define a discrete lure using fixed point attractors.

**Definition** Define $f$ such that $f : p \in \partial \varphi(0) \to x$ and

$$\bigcup_{i \in I} f_i(p) = \Phi(p) \in \partial \varphi(t)$$

i.e. the fixed point attractor is defined as

$$\Phi(\cdot) := \bigcup_{i \in I} f_i(\cdot)$$

Consequently, we may define a lure as

$$\bigcup_{r \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \Phi_{jr}(\cdot) \mid \Phi_k \neq \Phi_j \text{ for } j \neq k$$

where each $\Phi_{jr}$ maps to a unique time.

notice there is a bijective correspondence between $p$ and $\partial \varphi(t)$. Notice that this definition assumes the construction of a boundary $\partial \varphi(t)$ to be mapped to, before the mapping is executed.