Y. Rabin[1]
Department of Mechanical Engineering,
Carnegie Mellon University,
5000 Forbes Avenue,
Pittsburgh, PA 15213-3890
and
Department of Human Oncology,
Allegheny University of the
Health Sciences,
320 East North Avenue,
Pittsburgh, PA 15212-4772

P. S. Steif
Department of Mechanical Engineering,
Carnegie Mellon University,
5000 Forbes Avenue,
Pittsburgh, PA 15213-3890
Mem. ASME.

Thermal Stresses in a Freezing Sphere and its Application to Cryobiology

Thermal stresses in an inwardly solidifying sphere are studied analytically. A closed-form solution is given which accounts for thermal expansion associated with temperature gradients and volume changes associated with phase transition. Consistent with the target application of cryopreservation of biological solutions and tissues, the material is modeled as elastic-perfectly plastic. Parametric studies using appropriate material properties and typical cryopreservation protocols suggest that strains associated with phase transition lead to far higher stresses than those associated with thermal expansion, with important implications for cryopreservation procedures.

Introduction

It is well known that freezing biological tissues can introduce severe damage. Sometimes this damage is intentional and desired, as is the case in cryosurgery. In other situations, such as cryopreservation, this damage is an undesired byproduct. This paper arises from efforts to quantify the development of mechanical stresses in freezing biological solutions and tissues and to relate such stresses to tissue damage. While mechanical stress is one mechanism of tissue destruction in cryobiology applications (Ishiguro and Rubinsky, 1994; Hunt et al., 1994; Gao et al., 1995), there are other mechanisms related to crystal growth and to the kinetics of freezing at the cellular level (Meryman, 1974; Mazur, 1984; Taylor, 1987).

Mechanical stress development in this class of problems has been analyzed by a number of researchers. Analyses of thermal stress associated with cryopreservation and with water freezing have been reported by Rubinsky et al. (1980) and Rubinsky (1982), who modeled the frozen tissue as a linear elastic material. An inverse-engineering analysis has been performed by Lin et al. (1990) for the estimation of mechanical properties of frozen water by comparison of predicted strains and experimental measurements. Expansion of the analysis to aqueous solutions containing cryoprotectants has been reported by Gao et al. (1995). Analysis of thermal stresses during cryosurgery has been reported by Rabin and Steif (1996), assuming an elastic-perfectly plastic model for the freezing tissue.

In this paper we reconsider the inward freezing of a body and propose several alternative modeling assumptions. While these assumptions are at odds with some of the analyses in the literature, particularly those carried out by workers motivated by cryobiological applications, we will argue the physical sensibility of these assumptions. In particular, we propose: (i) that the pressure in the core liquid region is important and cannot be neglected; (ii) that the nonhydrostatic part of the stress in the just-frozen region must be zero; and (iii) that the volume strain associated with freezing cannot be neglected and, in fact, appears to produce stresses that overwhelm stresses due to other sources. We note that the first two assumptions have been well appreciated by workers in the area of metal solidification and casting (Boley and Weiner, 1963; Tien and Richmond, 1982; Heinlein et al., 1986; Zabaras and Ruan, 1990; Zabaras et al., 1991). The third assumption may not be significant in metal solidification; however, in contrast to metals, aqueous solutions, and presumably biological tissues, expand severely upon solidification.

The purpose of this paper, therefore, is primarily to present a credible set of modeling assumptions, which are new to the calculation of freezing solutions and tissues, and to show that these assumptions radically change the accepted picture of stress development during cryopreservation. We note that the solution is based on the highly idealized spherically symmetric problem, which, with respect to the nonsymmetric freezing problem, likely leads to upper bounds on stresses and pressures. Nevertheless, the spherical symmetry simplifies the solution and allows the analysis and discussion to focus on the new approach to calculating mechanical stresses in freezing materials.

Analysis

The material domain under consideration is the sphere occupying the region $0 < r < R$, as illustrated in Fig. 1. The entire domain is initially liquid at the phase transition temperature, and a freezing front then starts to propagate inwards. The freezing front location, h_f, as well as the temperature distribution, are assumed to be determined purely by the heat transfer problem, which is discussed in the next section.

Solid Mechanics Problem. Within the material domain, the usual field equations of continuum mechanics, specialized to the spherically symmetric geometry, are assumed to prevail. These equations include mechanical equilibrium

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \tag{1}$$

and the strain displacement relations

$$\epsilon_r = \frac{du}{dr}; \quad \epsilon_\theta = \frac{u}{r} \tag{2}$$

where σ is the stress, ϵ is the strain, and u is the radial displacement.

[1] To whom correspondence should be addressed. Currently at the Department of Mechanical Engineering, Technion-Israel Institute of Technology, Technion City, Haifa 32000, Israel. e-mail: yoed@tx.technion.ac.il.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

Manuscript received by the ASME Applied Mechanics Division, Feb. 17, 1997; final revision, Aug. 11, 1997. Associate Technical Editor: S. A. Berger.

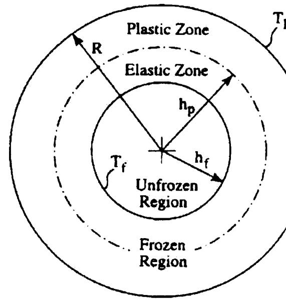

Fig. 1 Geometry of the phase transition problem of an inwardly freezing sphere, where R is the sphere outer surface, h_f is the freezing front, h_p is the interface between the elastic and plastic zones (also referred to as plastic front), T_f is the freezing temperature, and T_R is the forced temperature at the sphere outer surface

The constitutive relations are most conveniently expressed in terms of deviatoric and hydrostatic components of stress and strain. To that end, we define

$$\sigma_{ij} = S_{ij} - p\delta_{ij} \tag{3}$$

and

$$\epsilon_{ij} = \epsilon'_{ij} + \tfrac{1}{3}e\delta_{ij} \tag{4}$$

where S is the deviatoric stress, p is the hydrostatic pressure, ϵ' is the deviatoric strain, and e is the volume strain.

The liquid is assumed to be inviscid and therefore sustains only a hydrostatic pressure p, which is related to the volume strain according to

$$p = -\kappa(e - e_{th}) \quad 0 < r < h_f \tag{5}$$

where κ is the bulk modulus ($= E/[3(1 - 2\nu)]$) and the thermal strain is

$$e_{th} = \int^{T} \beta dT \tag{6}$$

where β is the thermal expansion coefficient.

The mechanical behavior of frozen biological tissues is very complex and not well understood. In the case of cryopreservation, it is common to employ cryoprotectants; these cause the unfrozen tissue to partly vitrify (become glassy) rather than freeze (transform to a crystalline solid). Vitrified material is likely to experience time-dependent deformation due to relaxation processes. Here, we ignore vitrification presuming solidification without cryoprotectants. There is evidence that the solid material, at least under compression and without cryoprotectants, deforms elastically until a critical level of stress at which point a pseudo-plasticity occurs, associated with the formation of microcracks (Rabin, et al., 1996). Therefore, we will consider two possible constitutive relations for the solid: either an elastic model or an elastic-perfectly plastic model.

A critical step in formulating the constitutive relation for the solid is to recognize that arbitrary deviatoric (volume preserving) strains could have occurred in the liquid state prior to solidification. The strains from which one calculates the deviatoric stress must be the total accumulated strains since the beginning of the contemplated process minus the strains accumulated up to the instant just after solidification. In the case of an elastic solid, the deviatoric stress can be written as

$$S_{ij} = 2G(\epsilon'_{ij} - \epsilon'^{*}_{ij}) \quad h_f < r < R \tag{7}$$

where ϵ'^{*}_{ij} is the deviatoric strain just after solidification, and G is the shear modulus ($= E/[2(1 + \nu)]$). Consistent with the assumption of isotropy, we have presumed the thermal strains to have zero deviatoric part.

The hydrostatic pressure in the solid is given by

$$p = -\kappa[(e - e_{th}) - \Delta e] \quad h_f < r < R \tag{8}$$

where Δe is the volumetric strain upon freezing. The bulk modulus in the solid, κ, is assumed to be equal to the value in the liquid; more discussion of this assumption will be given below. As noted above, the volume strain upon freezing does not seem to have been incorporated previously, at least the context of cryobiology.

The boundary conditions consist of zero radial stress on the outer surface,

$$\sigma_r(R) = 0, \tag{9}$$

and continuity of radial stress and displacement at the freezing front, respectively,

$$\sigma_r(h_f)|_1 = \sigma_r(h_f)|_2, \tag{10}$$

and

$$\epsilon_\theta(h_f)|_1 = \epsilon_\theta(h_f)|_2 \tag{11}$$

where the indexes 1 and 2 indicate the unfrozen and frozen regions, respectively. The displacement continuity in Eq. (11) has been rewritten in terms of the hoop strain using the strain-displacement relations.

The forgoing equations are readily solved as follows. The liquid is initially at the phase transition temperature and, thus, the thermal strains in the liquid region are zero. The solution in liquid can be written in terms of the uniform liquid pressure p_l:

$$u = -\frac{p_l}{3\kappa} r \quad 0 < r < h_f. \tag{12}$$

It will be useful for the ensuing analysis to recast the condition of radial stress continuity as a condition on the radial strains; by combining Eqs. (5), (8), (10), and (11), one finds

$$\epsilon_r(h_f)|_2 = \epsilon_r(h_f)|_1 + \Delta e. \tag{13}$$

To derive an equation governing the displacement in the solid domain, it is necessary to evaluate the quantity ϵ'^{*}_{ij}, which for the case of spherical symmetry corresponds to $\epsilon_r - \epsilon_\theta$. From Eqs. (11)–(13), the difference $(\epsilon_r - \epsilon_\theta)^*$ is found to be simply

$$(\epsilon_r - \epsilon_\theta)^* = \epsilon_r(h_f)|_2 - \epsilon_\theta(h_f)|_2 = \Delta e. \tag{14}$$

We recapitulate the significance of this term: the liquid undergoes a deviatoric strain equal to Δe prior to becoming solid, and, as reflected in Eq. (7), this strain should not be counted in determining the deviatoric stresses in the solidified material.

It is now possible to express S_r and p in terms of the radial displacement, u, and the various other known quantities and substitute into the equation of equilibrium; the result is

$$\frac{4}{3}G\left[u'' - \left(\frac{u}{r}\right)'\right] + \kappa\left[u'' + 2\left(\frac{u}{r}\right)' - e'_{th}\right] + \frac{4G}{r}\left[u' - \frac{u}{r} - \Delta e\right] = 0. \tag{15}$$

The solution of this equation is

$$u = \frac{3\kappa}{4G + 3\kappa}\frac{1}{r^2}\int_{h_f}^{r} e_{th}r^2 dr + \frac{4G\Delta e}{4G + 3\kappa} r\left[\ln\left(\frac{r}{h_f}\right) - \frac{1}{3}\right] + B_1 r + \frac{B_2}{r^2} \quad h_f < r < R \tag{16}$$

where the integration constant B_1 is calculated from boundary condition (10), by expressing the hydrostatic pressure in the solid in terms of the displacement, while taking in account zero deviatoric stress in the solid region at the freezing front,

$$B_1 = \frac{1}{3}\left(\Delta e - \frac{p_1}{\kappa}\right). \tag{17}$$

The integration constant B_2 is determined from condition (11), by comparing Eqs. (12) and (16) at the freezing front,

$$B_2 = -\frac{h_f^3 \kappa \Delta e}{4G + 3\kappa}. \tag{18}$$

Note that the previous solutions related to cryobiology do not meet the assumption of zero deviatoric stress at the freezing front.

The solution now depends only on the liquid pressure p_1, via constant B_1. This quantity is determined by setting the radial stress equal to zero at the sphere outer surface, Eq. (9), which leads to

$$p_1 = -\frac{4G\kappa}{4G + 3\kappa}\left\{\frac{3}{R^3}\int_{h_f}^{R} e_{th} r^2 dr + \Delta e\left[1 - 3\ln\left(\frac{R}{h_f}\right) - \left(\frac{h_f}{R}\right)^3\right]\right\}. \tag{19}$$

Final forms for the deviatoric stress and hydrostatic pressure in the solid region are, respectively,

$$S_r = -\frac{1}{2}S_\theta = \frac{4G\kappa}{4G + 3\kappa}\left[-\frac{3}{r^3}\int_{h_f}^{r} e_{th} r^2 dr + e_{th} + \Delta e\left(\frac{h_f^3}{r^3} - 1\right)\right] \tag{20}$$

and

$$p = \frac{4G\kappa}{4G + 3\kappa}\left[e_{th} - 3\Delta e \ln\left(\frac{r}{h_f}\right)\right] + p_1. \tag{21}$$

It is worth noting that the solution depends linearly both on the phase transition volume strain, Δe, and on the thermal strain, e_{th}. In the results and discussion presented below, two typical cases related to cryopreservation will be considered in order to study in a quantative manner the effects of the phase transition volume strain and of the thermal strain.

The solution assuming an elastic-perfectly plastic response in the solid turns out to be quite simple, assuming the Mises yield condition (Hill, 1950) specialized to spherical symmetry,

$$|\sigma_r - \sigma_\theta| = \tfrac{3}{2}|S_r| = \sigma_y, \tag{22}$$

where σ_y is the yield strength. As will be seen in the results and discussion section, the deviatoric stress S_r is maximal at the outer boundary, which is therefore where plasticity initiates. As the process continues, the plastic front, h_p, which separates the zones of elastic and plastic deformations, moves inwards from the outer surface, Fig. 1. Under the assumption that the flow stress remains constant independent of the pressure, one finds the stress distribution in the plastic zone to be fully determined by equilibrium, Eq. (1), by the yield condition, Eq. (22), and by the stress-free condition on the outer boundary, Eq. (9). Specifically, the pressure in the plastic region is given by

$$p = 2\sigma_y\left[\ln\left(\frac{R}{r}\right) - \frac{1}{3}\right] \quad h_p < r < R. \tag{23}$$

The location of the plastic front is also readily determined: the previous solution for the deviatoric stress in the elastic zone, Eq. (20), which was independent of the pressure in the liquid, p_1, remains valid. Thus, the location of the plastic front h_p is that position r at which the absolute value of S_r from Eq. (20) just reaches the yield value of $2\sigma_y/3$. The elastic-plastic interface merely progresses inwards as the freezing front moves inwards.

The pressure in the elastic zone is fully determined once the pressure in the liquid is known. The liquid pressure is chosen so as to match the pressure at the plastic front coming from the elastic side, Eq. (21), with the pressure from the plastic side, Eq. (23); the resulting equation is

$$p_1 = 2\sigma_y\left[\ln\left(\frac{R}{h_p}\right) - \frac{1}{3}\right] + \frac{4G\kappa}{4G + 3\kappa}\left[3\Delta e \ln\left(\frac{h_p}{h_f}\right) - e_{th}\right]. \tag{24}$$

Heat Transfer Problem. A perturbation solution of the inward freezing sphere, initially at the phase-change temperature and subject to a step-like change of the outer surface temperature, has been given by Riley et al. (1974). The perturbation variable is the Stefan number, F, which is the ratio of sensible heat to latent heat for the phase-change process. With the following dimensionless variables and parameters

$$\eta = \frac{R - h_f}{R};\quad \zeta = \frac{r}{R};\quad F = \frac{C(T_f - T_R)}{L};\quad \tau = \frac{\alpha t F}{R^2};\quad \phi = \frac{T - T_R}{T_f - T_R}, \tag{25}$$

the dimensionless temperature distribution, ϕ, is given by

$$\phi = \frac{(1-\eta)(1-\zeta)}{\eta\zeta} + \frac{F(1-\zeta)}{6\zeta\eta(1-\eta)}\left[1 - \left(\frac{1-\zeta}{\eta}\right)^2\right] - \frac{F^2(1-\zeta)}{\zeta\eta(1-\eta)^3}\left\{\frac{1}{36}\left[1 - \left(\frac{1-\zeta}{\eta}\right)^2\right] + \frac{(3-4\eta)}{120}\left[1 - \left(\frac{1-\zeta}{\eta}\right)^4\right]\right\} + O(F^3) \tag{26}$$

where C is the specific heat, L is the latent heat, and α is the thermal diffusivity. The dependancy of the freezing front location on time is given implicitly by

$$\tau = \frac{3\eta^2 - 2\eta^3}{6} + \frac{\eta^2 F}{6} - \frac{\eta^2 F^2}{45(1-\eta)} + O(F^3). \tag{27}$$

The perturbation solution given here is accurate up to the third order, which provides a very good approximation for the purpose of the current study. This approximation is fairly good until the ratio of freezing front location to sphere radius, h_f/R, reaches the value of 0.2 (when more than 99 percent of the sphere volume has already changed phase). A modified solution has been derived by Riley et al. (1974) for the end of the phase-change process, a short period during which the freezing front approaches the origin.

Material Properties. It is widely assumed that soft biological tissues have physical properties similar to those of aqueous solutions. Recent experiments (Rabin et al., 1996) have shown that frozen tissues loaded in compression in cryogenic temperatures respond roughly like an elastic-perfectly plastic solid, with the plasticity apparently stemming from distributed microcracking (Rabin et al., 1997a). The thermal expansion was found (Rabin et al., 1997b) to be similar to that of water with the coefficient of thermal expansion being almost a linear function of temperature. These observations motivate the adoption of an elastic-perfectly plastic model for the frozen material, with the thermal expansion coefficient varying linearly with temperature. Since data is not available on the mechanical properties of biological tissues within the phase transition temperature range, we will assume that the behavior is similar to that of a pure material (freezing at a single temperature).

There are at least nine known different pure water ice phases (Fletcher, 1970). Ice *I* is the most relevant ice phase for the current study; this is the only phase in which water expands upon freezing. Solidification of ice *I* takes place between pressures of 5 kPa and 207 MPa, while the phase transition temperature decreases monotonously from 0.01°C to −22°C, respectively. At higher pressures water contracts upon freezing and the freezing temperature increases continually with pressure (up to at least 4.4 GPa and 440°C). Due to lack of relevant data for biological materials, the volume strain upon phase transition is assumed constant, and equal to that of water at standard conditions of temperature and pressure, that is, $\Delta e = 0.0907$. Other physical properties are assumed to be constant and uniformly distributed in each phase; relevant physical properties values for water, soft frozen biological tissues, and the chosen values for the current study, are listed in Table 1.

The assumption of equal bulk moduli in the liquid and solid phases is largely one of convenience. There is much uncertainty regarding the bulk modulus of ice, although if inferred from measured elastic moduli and a reasonable value of Poisson's ratio, it may be up to four times that of water at standard conditions. Choosing differing bulk moduli substantially complicates the constitutive description as well as the solution, since the strain difference at phase change $(\epsilon_r - \epsilon_\theta)^*$ will no longer be constant. Since rather strong qualitative conclusions are arrived at in this paper, we believed it unwise to introduce a second-order effect based on rather uncertain physical properties.

Finally, we note that biological tissues are composite materials which may have different physical properties in different orientations. The simplified analysis presented here presumes a homogeneous material only, which can be identified with, say, the average property values. The modeling of a biological tissue as a composite material is beyond the scope of this paper.

Table 1 Typical properties of polycrystalline ice water and soft frozen biological tissues

	Polycristalline Ice Water @ 101.3 kPa	Frozen Biological Tissues	Current Study
Poisson's Ratio, ν	0.31 - 0.36 (Fletcher, 1970)	-	0.33
Volume Strain of Phase Transition, Δe	0.0907 @ 0°C (Sohnel and Novotny, 1995)	-	0.0907
Thermal Expansion Coefficient: $\beta_3+\beta_4 T$ [1/°C]	$\beta_3=56.3\cdot10^{-6}$ $\beta_4=2.53\cdot10^{-7}$ -180°C<T<0°C (Powell, 1958)	$\beta_3=65\cdot10^{-6}$ $\beta_4=2.89\cdot10^{-7}$ -180°C<T<-20°C (Rabin et al., 1997b)	$\beta_3=60\cdot10^{-6}$ $\beta_4=2.5\cdot10^{-7}$
Elastic Modulus, E [GPa]	8.9 - 9.9 @ -5°C (Fletcher, 1970)	14 - 132 @ -196°C (Rabin et al., 1996)	10

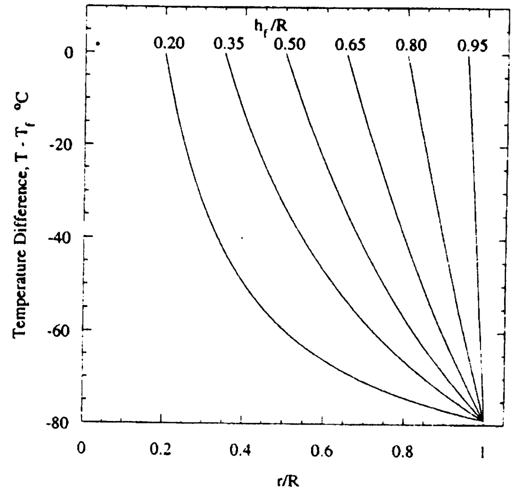

Fig. 2 Temperature distribution in case *A*, where a step-like temperature function is imposed at the sphere outer surface ($\Delta T = -79°C$)

Results and Discussion

Since the solution for the elastic response in the solid depends independently on the two "driving forces," that is, the thermal expansion in the solid state and the volume expansion due to phase transition, we consider these separately. Case A corresponds to neglecting the volume expansion due to phase transition and accounting for a temperature history which is the consequence of immersion in dry ice, i.e., a step-like temperature function is imposed at the sphere outer surface having a magnitude of −79°C. Case B, which approximates a typical cryopreservation protocol, corresponds to maintaining the material at nearly the phase transition temperature and allowing the phase transition to occur, presuming the volume strain is that which occurs at standard atmospheric pressure, i.e., $\Delta e = 0.0907$.

The temperature distribution for case A is shown in Fig. 2, where the corresponding Stephen number for pure water ice equals 0.27. As mentioned above, this solution is valid until the freezing front reaches 20 percent of the sphere radius (when more than 99 percent of the sphere volume is frozen). The deviatoric stresses and hydrostatic pressures for this case are shown in Figs. 3 and 4, respectively. Note first that the hydrostatic pressure in the liquid is by no means zero, and that it increases as the freezing process proceeds. Not surprisingly, this is at odds with the assumption that the pressure in the liquid is zero, as has been suggested previously for the analysis of a similar problem (Rubinsky et al., 1980). A second clear distinction between the current solution and previously reported solutions, most of which are for cryobiology applications (Rubinsky et al., 1980; Rubinsky, 1982; Lin et al., 1990; Gao et al., 1995; Rabin and Steif, 1996), is that the deviatoric stress here is zero at the freezing front (and increases monotonously as one approaches the sphere outer surface). This stems from the assumption that the deviatoric stress must be zero in recently solidified material, since no straining has yet occurred in the solid state. Finally, it is important to point to the magnitude of the predicted deviatoric stress: it is maximum at the initiation of the process, when the step-like temperature function is imposed at the sphere outer surface. This stress decreases with time, as the freezing front propagates inwards. The strains to initiate plasticity are in the order of 0.005 in frozen biological tissues (tested in compression; Rabin et al., 1996) and are not expected to exceed this value in polycrystalline water ice at low strain rates (Fletcher, 1970). It appears, in conclusion, that relative little plasticity would occur, if at all, during a sudden

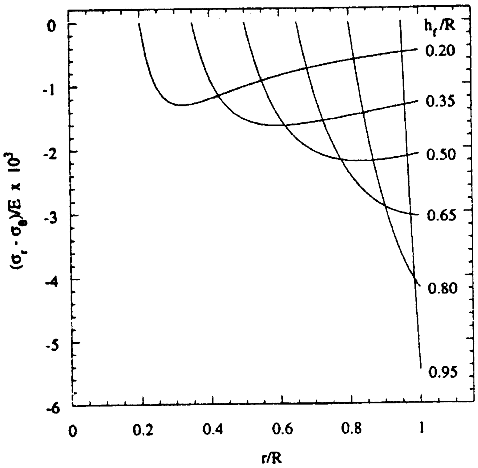

Fig. 3 Deviatoric stress distribution for case *A*, where no phase transition volume changes are included

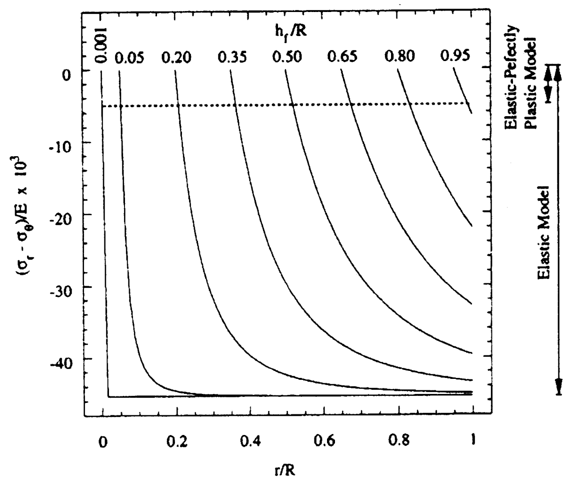

Fig. 5 Deviatoric stress distribution for case *B* (Δe = 0.0907; ΔT = 0). The dashed line represents the extreme deviatoric stress in the case of an elastic-perfectly plastic model and a yield strain of 0.005.

cool down to dry ice temperatures, when volume straining due to phase transition is prevented. In practice, the stresses are expected to be somewhat lower in magnitude at the initiation of the process than those presented in Fig. 3. The idealized boundary condition of a step change in temperature produces infinite gradients initially, while in practice a finite time is necessary to bring the surface down to the lower temperature.

Consider now case B featuring only volume strain due to phase transition and no temperature gradients. With a linear elastic model for the solid, the stresses reach enormous values (Fig. 5); this might be expected given the rather large imposed strain associated with phase transition. Clearly, the linear elastic model for case B is unrealistic, considering a typical yield strain in the order on 0.005. The deviatoric stress, as in case A, varies from zero at the freezing front to a maximum at the sphere outer surface. This allows one to use the elastic-plastic solution derived above. The deviatoric strains for the case of an elastic-perfectly plastic solid is also presented in Fig. 5, assuming a yield strain of 0.005. The deviatoric stresses in the plastic zone are represented by the dashed line in Fig. 5. Note that almost the entire frozen region experiences plastic deformations as the freezing front approaches the origin.

The hydrostatic pressures for the more realistic case of elastic-plastic deformations are shown in Fig. 6. The analysis present in this study is valid for solidification of ice *I* only, which is the only case of volume expansion upon freezing. Taking into account the maximal hydrostatic pressure of ice *I*, i.e. 207 MPa, and assuming a typical elastic modulus of 10 GPa, one can see from Fig. 6 that the solution is not valid as the freezing front approaches the origin due to the high pressure magnitude. It was found that the hydrostatic pressure in this case reaches 207 MPa when the ratio of freezing front location to sphere radius reaches the value of 0.1 approximately, i.e., when 99.9 percent of the sphere volume is frozen, although the size of this region increases with the yield strength. The sequence of events after this late stage is unclear.

Finally, in case B freezing was assumed to occur when the temperature was held at the freezing point. However, it was

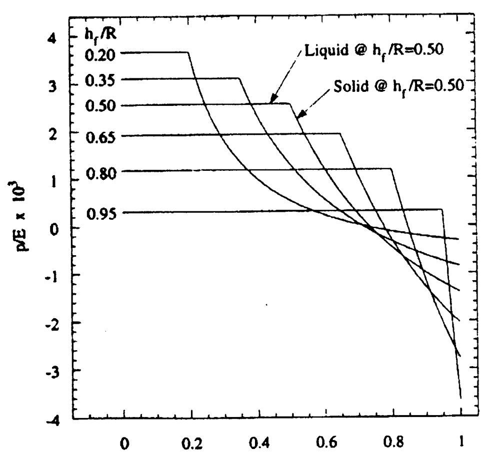

Fig. 4 Hydrostatic pressure distribution for case *A*

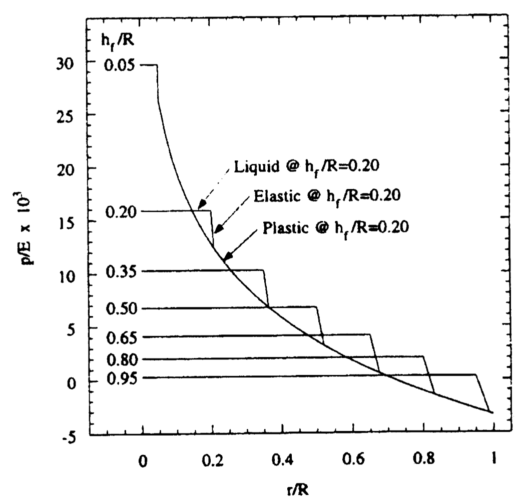

Fig. 6 Hydrostatic pressure distribution for case *B*, assuming an elastic-perfectly plastic model and yield strain of 0.005 (Δe = 0.0907; ΔT = 0)

demonstrated that the hydrostatic pressures in the liquid may reach the critical pressure which corresponds to a transition temperature of −22°C. For this reason the imposed surface temperature will need to be reduced accordingly (but slowly) to complete freezing the sphere.

Summary and Conclusions

A closed-form solution for the thermal stresses in an inward solidifying sphere has been presented. The formulation accounts for thermal expansion associated with temperature gradients and volume changes associated with phase transition. The frozen medium is modeled as an elastic-perfectly plastic.

For physical properties similar to those of water, parametric studies have shown that the effect of volume expansion upon phase transition is much more significant than the effect of thermal expansion in generating stresses, at least in a typical cryopreservation protocol. It has been shown that a plastic zone is likely to occupy most of the frozen region after the completion of the freezing process. It follows that the attendant potential for tissue destruction is unavoidable regardless of how slowly the freezing is carried out, provided there is a substantial expansion associated with phase transition. Finally, the phase transition temperature may significantly decrease during the cryopreservation process, due to the elevated hydrostatic pressure in the unfrozen region.

References

Boley, B. A., and Weiner, J. H., 1963, "Elasto-Plastic Thermal Stress in a Solidifying Body," *J. Mech Phys Solids*, Vol. 11, pp. 145–154.

Fletcher, N. H., 1970, *The Chemical Physics of Ice*, Cambridge University Press, Cambridge, UK.

Gao, D. Y., Lin, S., Watson, P. F., and Critser, J. K., 1995, "Fracture Phenomena in an Isotonic Salt Solution During Freezing and Their Elimination Using Glycerol," *Cryobiology*, Vol. 32, pp. 270–284.

Heinlein, M., Mukherjee, S., and Richmond, O., 1986, "A Boundary Element Method Analysis of Temperature Fields and Stresses During Solidification," *Acta Mechanica*, Vol. 59, pp. 59–81.

Hill, R., 1950, *Mathematical Theory of Plasticity*, Oxford University Press, London.

Hunt, C. J., Song, Y. C., Bateson, A. J., and Pegg, E. D., 1994, "Fractures in Cryopreserved Arteries," *Cryobiology*, Vol. 31, pp. 506–515.

Ishiguro, H., and Rubinsky, B., 1994, "Mechanical Interaction Between Ice Crystals and Red Blood Cells During Directional Solidification," *Cryobiology*, Vol. 31, pp. 483–500.

Lin, S., Gao, D. Y., and Yu, X. C., 1990, "Thermal Stress Induced by Water Solidification in a Cylindrical Tube," ASME *Journal of Heat Transfer*, Vol. 112, pp. 1079–1082.

Mazur, P., 1984, "Freezing of Living Cells: Mechanisms and Implications," *Am J Physiol.*, Vol. 247, pp. C125–C142.

Meryman, H. T., 1974, "Freezing Injury and its Prevention in Living Cells," *Annual Review of Biophysics and Bioengineering*, Vol. 3, pp. 341–363.

Powell, R. W., 1958, "Thermal Conductivity and Expansion Coefficients of Water and Ice," *Adv. Phys.*, Vol. 7, pp. 276–297.

Rabin, Y., and Steif, S. P., 1996, "Analysis of Thermal Stresses Around Cryosurgical Probe," *Cryobiology*, Vol. 33, pp. 276–290.

Rabin, Y., Steif, S. P., Taylor, M. J., Julian, T. B., and Wolmark, N., 1996, "An Experimental Study of the Mechanical Response of Frozen Biological Tissues at Cryogenic Temperatures," *Cryobiology*, Vol. 33, pp. 472–482.

Rabin, Y., Olson, P., Taylor, M. J., Steif, P. S., Julian, T. B., and Wolmark, N., 1997a, "Gross Damage Accumulation in Frozen Rabbit Liver Due to Mechanical Stress at Cryogenic Temperatures," *Cryobiology*, Vol. 34, pp. 394–405.

Rabin, Y., Taylor, M. J., and Wolmark, N., 1997b, "Thermal Expansion Measurements of Frozen Biological Tissues at Cryogenic Temperatures," ASME *Journal of Biomechanical Engineering*, in press.

Riley, D. S., Smith, F. T., and Poots, G., 1974, "The Inward Solidification of Spheres and Circular Cylinders," *International J. of Heat and Mass Transfer*, Vol. 17, pp. 1507–1516.

Rubinsky, B., Cravalho, E. G., and Mikic, B., 1980, "Thermal Stress in Frozen Organs," *Cryobiology*, Vol. 17, pp. 66–73.

Rubinsky, B., 1982, "Thermal Stress During Solidification," ASME *Journal of Heat Transfer*, Vol. 104, pp. 196–199.

Sohnel, O., and Novotny, P., 1995, *Densities of aqueous solution of inorganic substances*, Elsevier, New York.

Taylor, M. J., 1987, "Physico-chemical Principles in Low Temperature Biology," *The Effect of Low Temperature on Biological Systems*, B. W. W. Grout, and G. J. Morris, eds., pp. 3–71.

Tien, R. H., and Richmond, O., 1982, "Theory of Maximum Tensile Stresses in the Solidifying Shell of a Constrained Rectangular Casting," ASME JOURNAL OF APPLIED MECHANICS, Vol. 49, pp. 481–486.

Zabaras, N., and Ruan, Y., 1990, "Front Tracking Thermomechanical Model for Hypoelastic-Viscoplastic Behavior in a Solidifying Body," *Computer Methods in Applied Mechanics and Engineering*, Vol. 81, pp. 333–364.

Zabaras, N., Ruan, Y., and Richmond, O., 1991, "On the Calculation of Deformations and Stresses During Axially Symmetric Solidification," ASME JOURNAL OF APPLIED MECHANICS, Vol. 58, pp. 865–871.