Some Basic Facts about $I[\lambda]$ Ideal

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These notes on $I[\lambda]$ are based on Shelah's talks in Jerusalem from 12/94 to 3/95, as recorded by James Cummings.

Definition 0.1. Let λ be a regular uncountable cardinal. $S \in I^1[\lambda]$ iff $S \subseteq \lambda$ and there is a sequence $\langle p_\alpha : \alpha < \lambda \rangle$ and a club $E \subseteq \lambda$ such that

(1) $p_\alpha \subseteq \mathcal{P}(\alpha)$;
(2) $|p_\alpha| < \lambda$;
(3) If $\delta \in S \cap E$, then there is a $C \subseteq \delta$ such that
  (a) C is unbounded in δ;
  (b) $\text{otp}(C) < \delta$;
  (c) for every $\alpha \in C, C \cap \alpha \in p_\alpha$.

An example: let $S^\lambda_\omega = \{\delta < \lambda : \text{cf}(\delta) = \omega\}$, then $S^\lambda_\omega \in I^1[\lambda]$, where λ is uncountable, $E = \lambda$, $p_\alpha = [\alpha]^{<\omega}$. For any $\delta \in S^\lambda_\omega$, take C to be any unbounded subset of δ of order type ω.

Below we will show $I^1[\lambda]$ is a normal ideal. Clearly it is downward closed. Also if $S \subseteq \lambda$ is non-stationary, then $S \in I^1[\lambda]$. Under some cardinal arithmetic assumptions $I^1[\lambda]$ contains stationary subsets of λ. For example, let κ be a cardinal, if $\forall \alpha < \lambda\ (|\alpha|^{<\kappa} < \lambda)$, then $\{\delta < \lambda : \text{cf}(\delta) \leq \kappa\} \in I^1[\lambda]$. (Why? Simply set $p_\alpha = [\alpha]^{<\kappa}$.) On the other hand, if λ is a Mahlo cardinal, then we know some stationary $S \subseteq \lambda$ is not in $I^1[\lambda]$ (because of 3(b)). However, if λ is merely an inaccessible cardinal but not a Mahlo cardinal, then $\lambda \in I^1[\lambda]$ (hence $I^1[\lambda]$ is not a proper ideal). We are not very much concerned with the properness of $I^1[\lambda]$ here. We only want to find some (nice) sufficient conditions under which $I^1[\lambda]$ is not just the non-stationary ideal.

Claim 0.2. *(1) If $\alpha < \lambda$ and $S_i \in I^1[\lambda]$ for $i < \alpha$, then $\bigcup_{i<\alpha} S_i \in I^1[\lambda]$.*
(2) $I^1[\lambda]$ is normal.

Proof. (1) Clear.
(2) Let $\bar{S} = \langle S_\alpha : \alpha < \lambda, S_\alpha \in I^1[\lambda] \rangle$. Define

$$S = \bigtriangledown \bar{S} = \{\delta : \exists \alpha < \delta \ (\delta \in S_\alpha)\}.$$

Let $\bar{p}^\alpha$, E_α witness $S_\alpha \in I^1[\lambda]$. Define

$$E = \triangle_{\alpha<\lambda} E_\alpha = \{\delta : \forall \alpha < \delta \ (\delta \in E_\alpha)\},$$
$$p_\epsilon = \{x \setminus \gamma : \gamma < \epsilon, \ \exists \alpha < \epsilon \ (x \in p^\alpha_\epsilon)\}.$$

Clearly E is a club, $p_\epsilon \subseteq \mathcal{P}(\epsilon)$, and $|p_\epsilon| < \lambda$.

Let $\delta \in S \cap E$, then $\delta \in S_\alpha \cap E_\alpha$ for some $\alpha < \epsilon$. So there exists an unbounded $C \subseteq \delta$ such that $\mathrm{otp}(C) < \delta$ and $\forall \gamma \in C \ (C \cap \gamma \in p^\alpha_\gamma)$. Let $\bar{C} = C \setminus (\alpha + 1)$. Then $\bar{C}$ is still unbounded below δ and $\mathrm{otp}(\bar{C}) < \delta$. It is easy to see that by the definition of $\bar{p} = \langle p_\epsilon : \epsilon < \lambda \rangle$ for all $\gamma \in \bar{C}$ we have $\bar{C} \cap \gamma \in p_\gamma$. So E, $\bar{p}$ witness $S \in I^1[\lambda]$. □

Lemma 0.3. *In the definition of $I^1[\lambda]$ we may demand "$\mathrm{otp}(C) = \mathrm{cf}(\delta) < \delta$ and $\min(C) > \mathrm{otp}(C)$" (replacing (3)(b) there).*

Proof. Let $\bar{p}$, E witness $S \in I^1[\lambda]$. We shall refine $\bar{p}$ and produce a C as required in the lemma. For each limit ordinal $\alpha < \lambda$ choose a club $\rho_\alpha \subseteq \alpha$ with $\mathrm{otp}(\rho_\alpha) = \mathrm{cf}(\alpha)$. Define

$$p^1_\epsilon = \{x \setminus \gamma : \gamma < \epsilon, \ x \in p_\epsilon\},$$
$$p^2_\epsilon = p^1_\epsilon \cup \left\{x^{[\rho_\alpha]} : \alpha < \min(C), \ x \in p^1_\epsilon\right\},$$

where $x^{[\rho_\alpha]} = \{\beta \in x : \mathrm{otp}(C \cap \beta) \in \rho_\alpha\}$. Clearly $p_\epsilon \subseteq p^1_\epsilon \subseteq p^2_\epsilon \subseteq \mathcal{P}(\epsilon)$, $|p^2_\epsilon| < \lambda$.

Now let $\delta \in S \cap E$, then there exists an unbounded $C \subseteq \delta$ such that $\mathrm{otp}(C) < \delta$ and $\forall \gamma \in C \ (C \cap \gamma \in p_\gamma)$. If $\mathrm{otp}(C) = \mathrm{cf}(\delta) < \delta$, then clearly $C \setminus (\mathrm{cf}(\delta) + 1)$ is as required. Otherwise $\mathrm{cf}(\delta) < \mathrm{otp}(C) < \delta$. Let $C_1 = C \setminus (\mathrm{otp}(C) + 1)$, $\alpha = \mathrm{otp}(C_1)$, $C_2 = C_1^{[\rho_\alpha]}$. So $\mathrm{otp}(C_2) = \mathrm{cf}(\alpha) = \mathrm{cf}(\delta) < \delta$. Also $\mathrm{otp}(C_2) < \mathrm{otp}(C) < \min(C_1) \leq \min(C_2)$. Finally, if $\gamma \in C_2$, then $C_2 \cap \gamma = (C_1 \cap \gamma)^{[\rho_\alpha]}$. Since $C_1 \cap \gamma = [C \setminus (\mathrm{otp}(C) + 1)] \cap \gamma \in p^1_\gamma$, $C_2 \cap \gamma \in p^2_\gamma$, and we are done. □

Lemma 0.4. *Define $I^2[\lambda]$ by replacing "for every $\alpha \in C$, $C \cap \alpha \in p_\alpha$" by "for every $\alpha \in C$, $C \cap \alpha \in \bigcup_{\beta<\delta} p_\beta$" in the definition of $I^1[\lambda]$. Then $I^2[\lambda] = I^1[\lambda]$.*

Proof. Clearly $I^1[\lambda] \subseteq I^2[\lambda]$.

Conversely, let $\bar{p}$, E witness $S \in I^2[\lambda]$. Without loss of generality we may assume $\alpha < \beta \Rightarrow p_\alpha \subseteq p_\beta$ and p_α is closed under initial segments for all $\alpha < \lambda$. (Why? Simply replace each old p_α by $\bigcup_{\alpha \leq \beta} \{x \cap \gamma : x \in p_\beta \text{ and } \gamma \leq \beta\}$.)

Given a set of ordinals x, define a function

$$f_x(\gamma) = \begin{cases} \min \{\delta < \lambda : x \cap \gamma \in p_\delta\} & \text{if it exists,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\bar{x} = \operatorname{ran}(f_x)$. So $x_1 \subseteq x_2 \Rightarrow \bar{x}_1 \subseteq \bar{x}_2$. Let $\delta \in S \cap E$, and choose C unbounded in δ such that $\operatorname{otp}(C) < \delta$ and $\forall \gamma \in C$ $(C \cap \gamma \in \bigcup_{\beta < \delta} p_\beta)$. Define

$$p^1_\delta = \{\bar{x} \cap \delta : x \in p_\delta\}.$$

Now, clearly $\forall \gamma \in C$ $(f_C(\gamma) < \delta)$ and $\operatorname{otp}(\bar{C}) \leq \operatorname{otp}(C) < \delta$. Notice that $C^* = \bar{C} \cap \delta$ is unbounded in δ (simply because C is unbounded in δ). Hence it is enough to show $\forall \gamma \in C^*$ $(C^* \cap \gamma \in p^1_\gamma)$. Let $\gamma \in C^*$, and fix a $\beta \in C$ such that $f_C(\beta) = \gamma$ (such a β exists because $f_C(C)$ is the first and the last ordinal in $\bar{C}$ that is not smaller than δ). Let $a = C \cap \beta \in p_\gamma$. So $\bar{a} = \overline{C \cap \beta} = (C^* \cap \gamma) \cup \{\gamma\}$, so $C^* \cap \gamma = \bar{a} \cap \gamma \in p^1_\gamma$. □

We can improve the definition of $I^1[\lambda]$ in the sense that $|p_\alpha|$ is lowered to 1 and $\langle p_\alpha : \alpha < \lambda \rangle$ is "coherent".

Definition 0.5. Let λ be a regular uncountable cardinal. $S \in I[\lambda]$ iff $S \subseteq \lambda$ and there is a sequence $\langle p_\alpha : \alpha < \lambda \rangle$ and a club $E \subseteq \lambda$ such that

(1) $p_\alpha \subseteq \alpha$ (not necessarily closed) and $\operatorname{otp}(p_\alpha) < \alpha$;
(2) (Coherence) $\beta \in p_\alpha \Rightarrow p_\beta = p_\alpha \cap \beta$;
(3) $\delta \in S \cap E \Rightarrow \delta = \sup(p_\delta)$ & $\operatorname{cf}(\delta) = \operatorname{otp}(p_\delta)$.

Theorem 0.6. $I[\lambda] = I^1[\lambda]$.

Proof. Clearly $I[\lambda] \subseteq I^1[\lambda]$.

Conversely, let $\bar{p}$, E witness $S \in I^1[\lambda]$. By Lemma 0.3 we may assume $x \in p_\alpha \Rightarrow \operatorname{otp}(x) < \alpha$ for all $\alpha < \lambda$. We may also assume $\beta \in x \in p_\alpha \Rightarrow x \cap \beta \in p_\beta$, since we can find a new witness $\bar{r}$ by setting $r_\alpha = \{x \in p_\alpha : \beta \in x \Rightarrow x \cap \beta \in p_\beta\}$. Define

$$E_1 = \left\{ \delta \in E : \forall \gamma < \delta \ (\gamma + \left| \bigcup_{\epsilon \leq \gamma} p_\epsilon \right| < \delta) \right\}.$$

E_1 is a club because $|p_\epsilon| < \lambda$ and λ is regular. Let $\langle \delta_\xi : \xi < \lambda \rangle$ be the canonical enumeration of E_1. Since $\forall \xi\ (\delta_\xi + \left|\bigcup_{\epsilon \le \delta_\xi} p_\epsilon\right| < \delta_{\xi+1})$, we can fix a surjective function $f_\xi : \mathrm{Succ} \cap [\delta_\xi, \delta_{\xi+1}) \longrightarrow \bigcup_{\epsilon \le \delta_\xi} p_\epsilon$. Also fix a g_ξ such that $f_\xi \circ g_\xi = \mathrm{id}$.

Now define p^1_α by cases:

Case 1: α is a successor. So $\exists \xi\ (\delta_\xi < \alpha < \delta_{\xi+1})$. Let $a = f_\xi(\alpha)$, so $a \in \bigcup_{\epsilon \le \delta_\xi} p_\epsilon$. For each η such that $\eta + 1 < \xi$ and $a \cap [\delta_\eta, \delta_{\eta+1}) \neq \emptyset$, let $\beta(\eta) = \min(a \setminus \delta_\eta)$. So $a \cap \beta(\eta) \in p_{\beta(\eta)} \subseteq \bigcup_{\epsilon \le \delta_{\eta+1}} p_\epsilon$, so $g_{\eta+1}(a \cap \beta(\eta)) \in \mathrm{Succ} \cap [\delta_{\eta+1}, \delta_{\eta+2})$ (notice $\delta_{\eta+2} \le \delta_\xi$). Let

$$p^1_\alpha = \{g_{\eta+1}(a \cap \beta(\eta)) : \eta + 1 < \xi \ \&\ a \cap [\delta_\eta, \delta_{\eta+1}) \neq \emptyset\}.$$

Clearly $\mathrm{otp}(p^1_\alpha) \le \mathrm{otp}(a) < \delta_\xi < \alpha$.

Case 2: $\alpha \in S \cap E_2$, where $E_2 = \lim(E_1)$. As $\alpha \in S \cap E$ we have an unbounded $a \subseteq \alpha$ such that $\mathrm{otp}(a) = \mathrm{cf}(\alpha)$ and $\forall \gamma \in a\ (a \cap \gamma \in p_\gamma)$. Use a to define p^1_α exactly as in Case 1. Notice that, as α is an accumulation point in E_1 and a is unbounded in α, p^1_α is also unbounded in α and $\mathrm{otp}(p^1_\alpha) = \mathrm{otp}(a) = \mathrm{cf}(\alpha) < \alpha$.

Case 3: Otherwise set $p^1_\alpha = \emptyset$.

Clearly we only need to check "coherence". So let $\gamma \in p^1_\alpha$. So γ is a successor, so p^1_γ is defined in Case 1. Let a be the subset of α from which p^1_α is constructed. So $\gamma = g_{\eta+1}(a \cap \beta(\eta))$ for some η. So $\gamma \in \mathrm{Succ} \cap [\delta_{\eta+1}, \delta_{\eta+2})$. Since $f_{\eta+1}(\gamma) = a \cap \beta(\eta)$, it is clear $p^1_\alpha \cap \gamma = p^1_\gamma$. So "coherence" holds and $\bar{p}^1 = \langle p^1_\alpha : \alpha < \lambda \rangle$, E_2 witness $S \in I[\lambda]$. □

The next two theorems show that in many cases, without any assumption on cardinal arithmetic, $I[\lambda]$ $(= I^1[\lambda] = I^2[\lambda])$ contains stationary subsets of λ.

Theorem 0.7. *If λ is a regular uncountable cardinal, then $I^2[\lambda^+]$ contains a stationary subset of λ^+, namely $\{\delta < \lambda^+ : \omega \le \mathrm{cf}(\delta) < \lambda\} \in I^2[\lambda^+]$. Moreover, the subset $C \subseteq \delta$ as required in the definition can be made closed and unbounded in δ (so in particular C is a club in δ if δ is uncountable).*

Proof. For each $\alpha \in [\lambda, \lambda^+)$ fix an increasing sequence $\langle a^\alpha_\xi : \xi < \lambda \rangle$ such that

(1) $|a^\alpha_\xi| < \lambda$;
(2) a^α_ξ is closed;
(3) $\alpha = \bigcup_{\xi < \lambda} a^\alpha_\xi$.

For $\alpha < \beta$ define a function $g_{\alpha,\beta} : \lambda \longrightarrow \lambda$ as follows: for each $\epsilon \in \lambda$ let $g_{\alpha,\beta}(\epsilon)$ be the least ordinal such that $a^\alpha_\epsilon \subseteq a^\beta_{g_{\alpha,\beta}(\epsilon)}$ and $a^\beta_\epsilon \cap \alpha \subseteq a^\alpha_{g_{\alpha,\beta}(\epsilon)}$. Let $E_{\alpha,\beta} = \{\delta : g_{\alpha,\beta}[\delta] \subseteq \delta\}$. So $E_{\alpha,\beta}$ is a club in λ. So $\delta \in E_{\alpha,\beta} \Rightarrow a^\alpha_\delta = a^\beta_\delta \cap \alpha$, and if δ is big enough then $\alpha \in a^\beta_\delta$. For each $\beta \in [\lambda, \lambda^+)$ define

$$p_\beta = \left\{ a^\alpha_\xi \cap \gamma : \gamma \leq \alpha \leq \beta \ \& \ \xi < \lambda \right\}.$$

For $\beta < \lambda$ let p_β be any subset of β. Clearly $|p_\beta| < \lambda^+$ for all $\beta < \lambda^+$.

Now we claim that for all $\delta \in [\lambda, \lambda^+)$ with $\omega \leq \mathrm{cf}(\delta) < \lambda$ there is an unbounded $C \subseteq \delta$ such that $\mathrm{otp}(C) < \lambda \leq \delta$ and $\forall \gamma \in C\ (C \cap \gamma \in \bigcup_{\beta<\delta} p_\beta)$. To see this, fix an unbounded $a \subseteq \delta$ with $\mathrm{otp}(a) = \mathrm{cf}(\delta)$. As $|a| < \lambda$, $E = \bigcap_{\substack{\alpha \in a \\ \beta \in a}} E_{\alpha,\beta}$ is a club in λ, and we can find a $\xi \in E$ such that $\alpha < \beta \ \& \ \alpha,\ \beta \in a \Rightarrow a^\alpha_\xi = a^\beta_\xi \cap \alpha \ \& \ \alpha \in a^\beta_\xi$. Let $C = \bigcup_{\alpha \in a} a^\alpha_\xi$. Clearly $C \subseteq \delta$ and $|C| < \lambda$. Since $a \subseteq C$, C is unbounded in δ. Now Let $\gamma \in C$. Let $\beta \in a$ be the least such that $\gamma < \beta$. Since $\beta < \alpha \Rightarrow a^\beta_\xi = a^\alpha_\xi \cap \beta$ for all $\alpha \in a$, we have $\gamma \in a^\beta_\xi$. So $C \cap \gamma = \left(\bigcup_{\alpha\in a} a^\alpha_\xi\right) \cap \gamma = a^\beta_\xi \cap \gamma \in p_\beta$.

Also, given any increasing sequence of ordinals $\bar{\alpha} \subseteq C$ with $\sup(\bar{\alpha}) < \delta$, let $\beta \in a$ be the least such that $\sup(\bar{\alpha}) < \beta$, then $\bar{\alpha} \subseteq a^\beta_\xi$, since a^β_ξ is closed, we have $\sup(\bar{\alpha}) \in a^\beta_\xi \subseteq C$. So C is closed and unbounded in δ. □

More generally:

Theorem 0.8. *Let κ, λ be regular cardinals. If $\kappa^+ < \lambda$, then there is a stationary $S \subseteq S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ such that $S \in I^2[\lambda]$. And, again, the subset $C \subseteq \delta$ as required in the definition can be made closed and unbounded in δ.*

Before we prove the theorem, we need to recall a combinatorial principle:

Definition 0.9. If $S \subseteq \lambda$ is stationary, then we say S carries a club guessing sequence if there is a sequence $\langle C_\delta : \delta \in S,\ C_\delta \subseteq \delta$ closed and unbounded$\rangle$ such that for every club $D \subseteq \lambda$ there exists some $\delta \in S$ with $C_\delta \subseteq D$.

Fact 0.10. *(Club Guessing) For every regular κ, if λ is a cardinal with $\mathrm{cf}(\lambda) \geq \kappa^{++}$, then any stationary $S \subseteq S^\lambda_\kappa$ carries a club guessing sequence $\langle C_\delta : \delta \in S\rangle$ such that $\mathrm{otp}(C_\delta) = \mathrm{cf}(\delta)$, and the set $\{\delta \in S : C_\delta \subseteq C\}$ is stationary for all club $C \subseteq \lambda$.*

For a proof of this fact see e.g. Abraham and Magidor [1].

Proof of 0.8. If $\kappa^{++} = \lambda$, then by Theorem 0.7 $S^\lambda_{\leq\kappa} \in I^2[\lambda]$. Clearly the method used in the proof there also works for S^λ_κ.

So w.l.o.g. $\kappa^{++} < \lambda$. Let $S = S^{\kappa^{++}}_\kappa$. By Fact 0.10 fix a guessing sequence $\langle C_\delta : \delta \in S\rangle$. For all $\alpha < \lambda$ & $\mathrm{cf}(\alpha) = \kappa^{++}$, let $a_\alpha \subseteq \alpha$ be closed and unbounded with $\mathrm{otp}(\alpha) = \kappa^{++}$. Define

$$p_\alpha = \left\{a^{[C_\delta]}_\beta \cap \gamma : \gamma \leq \beta \leq \alpha,\ \delta \in S\right\},$$

$$S^* = \Bigg\{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa,\ \text{and there is a closed and unbounded } C \subseteq \alpha$$
$$\text{with } \mathrm{otp}(C) = \kappa \text{ such that } \forall\gamma \in C \left(C \cap \gamma \in \bigcup_{\beta<\alpha} p_\beta\right)\Bigg\}.$$

Manifestly $S^* \in I^2[\lambda]$. So it will suffice to show S^* is stationary. Let $D \subseteq \lambda$ be a club. Define an increasing and continuous sequence $\langle\alpha_\xi : \xi < \kappa^{++}\rangle \subseteq D$ as follows:

(1) $\alpha_0 = \min(D)$;
(2) $\alpha_\xi = \sup\{\alpha_\gamma : \gamma < \xi\}$ for ξ limit;
(3) Having defined α_ξ, let $b_\xi = \{\alpha_\gamma : \gamma \leq \xi\}$ and $A_\xi = \left\{b^{[C_\delta]}_\xi : \delta \in S\right\}$. So $|A_\xi| \leq \kappa^{++} < \lambda$, so fix a $\beta_\xi < \lambda$ such that $A_\xi \cap \bigcup_{\gamma<\lambda} p_\gamma = A_\xi \cap \bigcup_{\gamma\leq\beta_\xi} p_\gamma$. Let $\alpha_{\xi+1} = \min(D \setminus \beta_\xi)$.

Let $\rho = \sup\{\alpha_\gamma : \gamma < \kappa^{++}\}$. Since $\mathrm{cf}(\rho) = \kappa^{++}$, let $\langle\alpha^\xi_\rho : \xi < \kappa^{++}\rangle$ be the canonical enumeration of a_ρ. Define

$$E = \left\{\alpha_\xi : \xi < \kappa^{++} \ \&\ \alpha_\xi = \alpha^\xi_\rho\right\}.$$

Clearly $E \subseteq \rho$ is a club. Since $E \subseteq a_\rho$, there is a $\delta \in S$ such that $C = \left\{\alpha^\xi_\rho : \xi \in C_\delta\right\} \subseteq E$. Let $\rho^* = \sup(C)$. We claim that C witnesses $\rho^* \in S^*$. To see this, let $\alpha_\gamma \in C$. Since $\alpha_\xi = \alpha^\xi_\rho$ for all $\alpha_\xi \in C$, it follows $C \cap \alpha_\gamma = b^{[C_\delta]}_\gamma \cap \alpha_\gamma = a^{[C_\delta]}_\rho \cap \alpha_\gamma \in p_\rho$. But $\alpha_{\gamma+1} < \rho^* < \rho$, so by the choice of $\alpha_{\gamma+1}$ there is a β with $\alpha_\gamma < \beta < \alpha_{\gamma+1} < \rho^*$ such that $b^{[C_\delta]}_\gamma \in p_\beta$. Since p_β is closed under initial segments, $C \cap \alpha_\gamma \in p_\beta$. So $\rho^* \in S^*$. Since $\rho^* \in D$, it follows that S^* is stationary. □

Theorem 0.11. *Let κ, θ, and λ be regular cardinals with $\kappa^+ < \theta < \lambda$. Then there is a sequence $\langle p_\alpha : \alpha < \lambda,\ p_\alpha \subseteq \alpha,\ \mathrm{otp}(p_\alpha) \leq \kappa\rangle$ and a stationary $S \subseteq S^\lambda_\kappa$ with $S \in I[\lambda]$ such that*

(1) $\delta \in S \Rightarrow \sup(p_\delta) = \delta$ & $\forall \gamma \in p_\delta \ (p_\delta \cap \gamma = p_\gamma)$*;*

(2) $\{\delta < \lambda : \mathrm{cf}(\delta) = \theta,\ S \cap \delta$ *is stationary*$\}$ *is stationary.*

0.11(1) is a direct consequence of Thereom 0.8. Why? Fix a stationary $S^* \subseteq S^\lambda_\kappa$ as given by Thereom 0.8. By Theorem 0.6 and Lemma 0.4 $S^* \in I[\lambda]$. Let $\bar{p}$, E witness this. Define $S = S^* \cap E$ and $q_\alpha = p^{[\kappa]}_\alpha$ for all $\alpha < \lambda$. Clearly S, $\bar{q}$ are as required. In fact, under the assumptions of Thereom 0.8, it can be proved directly that there is a stationary $S \subseteq S^\lambda_\kappa$ such that $S \in I[\lambda]$. The proof involves the use of internally approachable chains of submodels. Below we will give a very brief description of this technique, and will use it to prove 0.11(1), (2).

Let $\mathcal{A} = \langle H_\chi, \in, <_\chi \rangle$ be a model of everything that have been said so far in this paper, where χ is a cardinal that is much larger than the cardinals used in this paper (except χ, of course, which is also used in this paper). For a regular uncountable cardinal κ and an ordinal γ with $\mathrm{cf}(\gamma) = \kappa$, an *IA chain of substructures of* $\mathcal{A}$ *of length* κ *at* γ is a continuous and increasing sequence $\langle M_i : i < \kappa \rangle$ of elementary substructures of $\mathcal{A}$ such that

(1) M_0 contains whatever objects we want it to contain (in practice exactly what M_0 contains should be clear in context);
(2) $|M_i| \leq \kappa$ for all $i < \kappa$;
(3) $M_i \cap \gamma \in \gamma$ for all $i < \kappa$, and $\bigcup_{i<\kappa} M_i \cap \gamma = \gamma$;
(4) if $\gamma = \kappa$, then $|M_i| < \kappa$ for all $i < \kappa$;
(5) $\langle M_i : i \leq j \rangle \in M_{j+1}$ for all $j < \kappa$ (so $j \subseteq M_j$ and $M_i \in M_j$ if $i < j$).

(Abusing notation standardly "M" refers to both the structure and the universe of the structure, and "$|M|$" refers to the cardinality of the universe of M.) These five requirements are sufficient for our purpose. The interested reader can find more about IA chains in [3] Chapter 4.

Proof of 0.11. Let $\bar{C} = \langle C_\delta : \delta \in S^\theta_\kappa \rangle$ be a club guessing sequence. Let $\langle M_i : i < \lambda \rangle$ be an IA chain of length λ at λ such that $\bar{C}$, $\theta \subseteq M_0$ and $\lambda \in M_0$. Let $M = \bigcup_{i<\lambda} M_i$, $\epsilon_i = M_i \cap \lambda$. Let $\langle a_i : i < \lambda \rangle$ be an enumeration of all the bounded subsets of λ in M such that $a_i \subsetneq a_j \Rightarrow i < j$. Define

$$b_i = \{j < \lambda : \exists \alpha \in a_i \ (a_i \cap \alpha = a_j)\}$$

if $\mathrm{otp}(a_i) < \kappa$. So $b_i \subseteq i$ for all $i < \lambda$ (if defined). Clearly the set

$$E = \{\gamma < \lambda : \epsilon_\gamma = \gamma \ \& \ (a_i \in M_\gamma \Rightarrow i < \gamma)\}$$

is a club in λ. Let

$$S = \{\gamma \in E \cap S^\lambda_\kappa : \text{ there is a closed and unbounded } D \subseteq \gamma \text{ with } \operatorname{otp}(D) = \kappa \text{ such that every } d \sqsubset D \text{ is in } M_\gamma\},$$

where "$d \sqsubset D$" stands for "d is a proper initial segment of D".

Now, if $\gamma \in S$, fix a $D \subseteq \gamma$ as required above, then $\forall d \sqsubset D$ $(d = a_j)$ for some $j < \gamma$. Obviously the set $C = \{j < \gamma : \exists d \sqsubset D\ (d = a_j)\}$ is unbounded in γ. (Why? Otherwise all initial segments of D are contained in some M_{γ^*} with $\gamma^* < \gamma$, since $\kappa \subseteq M_{\gamma^*}$, this is impossible.) Clearly $\forall j \in C$ $(C \cap j = b_j)$. Set $b_\gamma = C$. So $\langle b_i : i < \lambda \rangle$, S exemplifies (1). Clearly $S \in I[\lambda]$. So it is sufficient to prove (2).

For (2), suppose for a contradiction that the set

$$A = \{\delta < \lambda : \operatorname{cf}(\delta) = \theta,\ S \cap \delta \text{ is stationary in } \delta\}$$

is non-stationary in λ. Fix a club $D \subseteq E$ such that $D \cap A = \emptyset$. We may build an IA chain $\langle N_j : j < \theta \rangle$ of length θ such that $\langle M_i : i < \lambda \rangle$, $\lambda \in N_0$, $\bar{C}$, $\theta \subseteq N_0$, $|N_j| = \theta$ for all $j < \theta$, and $\delta = \sup(\bigcup_{j<\theta} N_j \cap \lambda) \in D$. So $S \cap \delta$ is not stationary in δ.

Let $\sigma_j = \sup(N_j \cap \lambda)$. So $\langle \sigma_i : i < j \rangle \in N_{j+1}$ for all $j < \theta$, and the σ_j's are increasing, continuous, and cofinal in δ. Choose $\langle \gamma_j : j < \theta \rangle \in M_{\delta+1}$ increasing, continuous, and cofinal in δ. Let $e = \{j < \theta : \sigma_j = \gamma_j\}$. Since e is a club in θ and $\bar{C}$ is a club guessing sequence, the set $G = \big\{j \in S^\theta_\kappa : C_j \cup \{j\} \in e\big\}$ is stationary. Pick any $j \in G$, let $c = \{\sigma_i : i \in C_j\} = \{\gamma_i : i \in C_j\}$. So every proper initial segment of c can be computed from C_j and a proper initial segment of $\{\sigma_i : i < j\}$, so it is in N_j; similarly it is also in $M_{\delta+1}$. Now let $d \sqsubset c$, so N_j knows that $d \in M$, so N_j knows that $d \in M_l$ for some $l \in N_j$, since $l < \sigma_j$, it follows that $d \in M_{\sigma_j}$. So c witnesses that $\sigma_j \in S$. So $S \cap \delta$ is stationary, contradicting the assumption above. □

References

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