Solutions to the Exercises in Chapter 10, Introduction to Commutative Algebra, M. F. Atiyah and I. G. MacDonald

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1. Since $p(\bigoplus \mathbb{Z}/p\mathbb{Z}) = 0$, clearly the p-adic completion of $\bigoplus \mathbb{Z}/p\mathbb{Z}$ is itself. Now the p-adic completion of B is clearly the inverse limit

$$\varprojlim_n B/p^n B = \varprojlim_n \bigoplus_{1 \leq i \leq n} \mathbb{Z}/p^i\mathbb{Z}.$$

Hence the completion of A in the p-adic completion of B is

$$\varprojlim_n (\underbrace{\mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}}_{n \text{ copies}}),$$

which is clearly the direct product of the $\mathbb{Z}/p\mathbb{Z}$. Now if we apply the p-adic completion functor $\mathcal{P}$ to the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\pi"] & B/A \arrow[r] & 0, \end{tikzcd}$$

we have that the kernel of $\mathcal{P}(\pi)$ is the direct product of the $\mathbb{Z}/p\mathbb{Z}$ rather than $\mathcal{P}(A)$. This shows that the functor is not right-exact.

2. Each A_n is of the form

$$\underbrace{0 \oplus \cdots \oplus 0}_{n \text{ copies of } 0} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots$$

Hence each A/A_n is the direct sum of n copies of $\mathbb{Z}/p\mathbb{Z}$, which should be regarded as an initial segment of A. The exact sequence of the three inverse systems looks like this

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, tail] & A \arrow[r] \arrow[d, "\mathrm{id}"] & A/A_{n+1} \arrow[r] \arrow[d, "\text{projection}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & A \arrow[r] & A/A_n \arrow[r] & 0
\end{tikzcd}$$

So we see that $\varprojlim A = A$, which is countable, and $\varprojlim A/A_n = \prod \mathbb{Z}/p\mathbb{Z}$, which is uncountable. So simply by cardinality considerations there cannot be a surjective map from the former to the latter.

Now it is easy to see that $\varprojlim A_n = 0$. So the the long exact sequence in 10.2 here turns into

$$\begin{aligned}
0 \longrightarrow 0 \longrightarrow A = \bigoplus \mathbb{Z}/p\mathbb{Z} \longrightarrow \prod \mathbb{Z}/p\mathbb{Z} \\
\xrightarrow{d} {\varprojlim}^1 A_n \longrightarrow {\varprojlim}^1 A = 0 \longrightarrow \operatorname{coker} d^C = 0 \longrightarrow 0.
\end{aligned}$$

So ${\varprojlim}^1 A_n = (\prod \mathbb{Z}/p\mathbb{Z})/(\bigoplus \mathbb{Z}/p\mathbb{Z})$.

3. Let's see: $x \in \bigcap \mathfrak{a}^n M$ if and only if there is an $\alpha \in \mathfrak{a}$ such that $(1-\alpha)x = 0$ (Krull's Theorem) if and only if there is a $\beta \notin \bigcup_{\mathfrak{a} \subseteq \mathfrak{m}} \mathfrak{m}$ such that $\beta x = 0$, i.e. $x \in \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \longrightarrow M_{\mathfrak{m}})$, where the "if" direction is by Ex. 14 of Chapter 3 and 2.5 since

$$\mathfrak{a}\left(\bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \longrightarrow M_{\mathfrak{m}})\right) = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \longrightarrow M_{\mathfrak{m}}).$$

For the second claim only note that $\hat{M} = 0$ if and only if $\mathfrak{a}M = M$, and hence, by the above result and Ex. 3 of Chapter 3, for all $\mathfrak{m} \supseteq \mathfrak{p} \supseteq \mathfrak{a}$ with $\mathfrak{m}$ maximal and $\mathfrak{p}$ prime we have

$$M_{\mathfrak{p}} = (M_{\mathfrak{m}})_{\mathfrak{p}_{\mathfrak{m}}} = 0_{\mathfrak{p}_{\mathfrak{m}}} = 0.$$

4. Following the hint consider $A \overset{x}{\rightarrowtail} A$ and use 10.12 to get $\hat{A} \overset{\hat{x}}{\rightarrowtail} \hat{A}$. Now if A is an integral domain then by the first claim clearly for every $x \in A$ $\hat{x}$ is not a zero-divisor in $\hat{A}$. So it seems that this does not necessarily imply that $\hat{A}$ has no zero-divisors. This intuition is of course correct. However, it is not hard to see that if $\mathfrak{a}$ is prime then $\hat{A}$ is an integral domain. Hence to construct a counterexample we need to find a prime ideal $\mathfrak{p}$ and another

ideal $\mathfrak{m} \supseteq \mathfrak{p}$ (for simplicity) of some Noetherian domain B, where we intend to set $A = B/\mathfrak{p}$, such that the $\mathfrak{m}$-adic completion of $\mathfrak{p}$ is no longer prime. This can be summarized as the following diagrams:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{p} \arrow[r] & B \arrow[r] & B/\mathfrak{p} \arrow[r] & 0, \end{tikzcd}$$

and by 10.12 the sequence of $\mathfrak{m}$-adic completions

$$\begin{tikzcd} 0 \arrow[r] & \hat{\mathfrak{p}} \arrow[r] & \hat{B} \arrow[r] & \widehat{B/\mathfrak{p}} \arrow[r] & 0 \end{tikzcd}$$

is exact, so the $(\mathfrak{m}/\mathfrak{p})$-adic completion of $B/\mathfrak{p}$ is isomorphic to $\hat{B}/\hat{\mathfrak{p}}$.

Following this idea, let $B = \mathbb{Q}[x, y]$, $\mathfrak{p} = (y^2 - x^3 - x^2)$, and $\mathfrak{m} = (x, y)$. So $A = \mathbb{Q}[x, y]/(y^2 - x^3 - x^2)$ is an integral domain. However $\mathbb{Q}[[x, y]]/(y^2 - x^3 - x^2)$ is not an integral domain since $y^2 - x^3 - x^2 = y^2 - x^2(1 + x)$ and $(1 + x)$ is a square now:

$$1 + x = (1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots)^2.$$

Of course this example is well-known; see [1, pp. 185-186].

One would think that the question that when the completion is an integral domain is of fundamental interest. However, this is not addressed even in [5] or [3]. (Maybe it is hidden deeply somewhere?)

5. Here is a more intuitive (that is, more diagrammatic) approach: It is not hard to see that $(M^{\mathfrak{a}})^{\mathfrak{b}}$ is the limit of the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[l] \arrow[d] & M/(\mathfrak{a}^3M + \mathfrak{b}^3M) \arrow[l] \arrow[d] \\
M/(\mathfrak{a}M + \mathfrak{b}^2M) \arrow[d] & M/(\mathfrak{a}^2M + \mathfrak{b}^2M) \arrow[l] \arrow[d] & \cdots \arrow[l] \arrow[d] \\
M/(\mathfrak{a}M + \mathfrak{b}M) & M/(\mathfrak{a}^2M + \mathfrak{b}M) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Clearly the limit of the diagonal of this diagram is the limit of the whole thing; that is, $(M^{\mathfrak{a}})^{\mathfrak{b}} = \varprojlim M/(\mathfrak{a}^nM + \mathfrak{b}^nM)$. Now using the last item in the hint we see that $\varprojlim M/(\mathfrak{a}^nM + \mathfrak{b}^nM) = \varprojlim M/(\mathfrak{a} + \mathfrak{b})^nM = M^{\mathfrak{a}+\mathfrak{b}}$.

6. Let $\mathfrak{J}$ be the Jacobson radical of A. So for every maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{J} \subseteq \mathfrak{m}$. If $\mathfrak{a} \subseteq \mathfrak{J}$ then for every maximal ideal $\mathfrak{m}$ and every $x \notin \mathfrak{m}$

the coset $x + \mathfrak{a}$ is disjoint from $\mathfrak{m}$. Conversely, suppose for contradiction that there is an $x \in \mathfrak{a} \setminus \mathfrak{J}$. So there is a maximal ideal $\mathfrak{m}$ such that $x \in \mathfrak{a} \setminus \mathfrak{m}$. For any $n > 1$, since $x^{n-1} \notin \mathfrak{m}$, there is a y such that $1 - x^{n-1}y \in \mathfrak{m}$. On the other hand, since $x^n y \in \mathfrak{a}^n$. we have $x - x^n y = x(1 - x^{n-1}y) \in \mathfrak{m}$. So $(x + \mathfrak{a}^n) \cap \mathfrak{m} \neq \emptyset$ for any n. So $\mathfrak{m}$ is not closed, contradiction.

7. By 10.14 $\hat{A}$ is a flat A-algebra. Applying Ex. 16(v) of Chapter 3 to the canonical embedding $1 \otimes \mathrm{id} : M \longrightarrow \hat{A} \otimes_A M \cong \hat{M}$, as in the hint, it is enough to show that $1 \otimes \mathrm{id}$ is injective for all finitely generated M if and only if A is Zariski. Now the "if" direction is immediate by 10.19. For the "only if" direction, by 10.15(iv) $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\hat{A}$. By Ex. 16(i, iii) of Chapter 3, for any maximal ideal $\mathfrak{m}$ of A, the extension $\mathfrak{m}^e$ in $\hat{A}$ is not $\hat{A}$ and hence is contained in some proper maximal ideal $\mathfrak{n}$ of $\hat{A}$, and $\mathfrak{n}^c = \mathfrak{m}$. Since $\hat{\mathfrak{a}} \subseteq \mathfrak{n}$, we have $\mathfrak{a} \subseteq \hat{\mathfrak{a}}^c \subseteq \mathfrak{n}^c = \mathfrak{m}$. So $\mathfrak{a}$ is contained in the Jacobson radical of A.

8. Clearly the ring A is just $\mathbb{C}[z_1, \ldots, z_n]_{(z_1,\ldots,z_n)}$. So by 7.4 A is Noetherian. For both A and B the maximal ideal is just the subset of the power series whose constant terms are 0. Let them be $\mathfrak{m}_A, \mathfrak{m}_B$. Since both A and B contain $\mathbb{C}[z_1, \ldots, z_n]$, it is not hard to see that, by 10.3, the completions of the two with respect to $\mathfrak{m}_A, \mathfrak{m}_B$ are C. Now if B is Noetherian then by Ex. 7 C is faithfully flat over both A and B, and hence by Ex. 17 of Chapter 3 B is flat over A.

9. The literature on Hensel's Lemma and its many variations is abundant, since they are of fundamental importance. So we merely give some references here. For the theory of valued fields [2] is highly recommended. Also, for its applications in algebraic number theory see [4].

10. (i) See [2].
(ii) A simple application of (i): Let $f(x) = x^2 - 2$, and observe that $f(3) = 7 \in (7)$ and $f'(3) = 6 \notin (7)$.
(iii) Think of $f(x, y)$ as a polynomial in $k[[x]][y]$. So the assumption is precisely saying that the projection of f in the residue field has a simple root a_0. Now use (i) again.

11. Observe that it is not clear what the converse of 10.26 is. The hint wants us to construct a Noetherian completion $\hat{A}$ of a non-Noetherian local ring A with respect to its maximal ideal.

Let A be the ring of germs of C^∞ functions on the real line of one variable at 0. It is convenient to think of each element of A as the stalk of a sheaf of smooth functions at 0. It is easy to see then that for any $[f]_0 \in A$, $[f]_0$ is a unit if and only of $f(0) \neq 0$. Hence the maximal ideal of A is actually a principal ideal generated by $[x]_0$. By the second remark after Krull's Theorem 10.17 A is not Noetherian. However, the completion of A with respect to $([x]_0)$ is isomorphic to the ring of power series $\mathbb{R}[[x]]$, and hence is Noetherian by 10.26.

12. By induction this reduces to showing that $A[[x]]$ is faithfully flat over A. It is flat by 10.14. By Ex. 16(ii) of Chapter 3 and Ex. 5(v) of Chapter 1 the flatness is also faithful.

References

[1] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1999.

[2] Antonio J. Engler and Alexander Prestel, *Valued fields*, Springer-Verlag, Berlin, 2005.

[3] Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[4] J. Neukirch, *Algebraic number theory*, Springer-Verlag, Berlin, 1999, translated from the German by N. Schappacher.

[5] Oscar Zariski and Pierre Samuel, *Commutative algebra*, Graduate Texts in Mathematics, vol. 28-29, Springer-Verlag, New York, 1960.