Solutions to the Exercises in Chapter 7, Introduction to Commutative Algebra, M. F. Atiyah and I. G. MacDonald

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1. Following the hint, for contradiction, we have that $\mathfrak{a} + (x)$ is a finitely generated proper ideal. Clearly in this case we can choose a set of generators that consists of $a_1, \ldots, a_n \in \mathfrak{a}$ and x. Let $\mathfrak{a}_0 = (a_1, \ldots, a_n) \subseteq \mathfrak{a}$. Then any $a \in \mathfrak{a}$ may be written as $a = a_0 + bx$ with $a_0 \in \mathfrak{a}_0$ and $b \in A$. Since $bx \in \mathfrak{a}$, we have $b \in (\mathfrak{a} : x)$. Since $\mathfrak{a} \subsetneq (\mathfrak{a} : x)$ (the latter contains y), it follows that $\mathfrak{a}$ is finitely generated, contradiction.

2. The "only if" direction is Chapter 1, Ex. 5(ii). For the other direction, since A is Noetherian, for some large m and any n we have $a_n = \sum_{i=0}^{m} b_i a_i$. If $a_0, \ldots, a_m$ are nilpotent (this implies that each a_n is nilpotent) then for sufficiently large l we must have $f^l = 0$.

3. (i)$\Rightarrow$(ii): If $a \in S \cap \mathfrak{a}$ then clearly $(S^{-1}\mathfrak{a})^c = A = (\mathfrak{a} : a)$. If $S \cap \mathfrak{a} = \emptyset$ then, since $\mathfrak{a}$ is primary, $S \cap r(\mathfrak{a}) = \emptyset$ by 4.6. Hence $(\mathfrak{a} : s) = \mathfrak{a}$ for each $s \in S$. By 3.11(ii) we have $(S^{-1}\mathfrak{a})^c = \bigcup_{s \in S} (\mathfrak{a} : s) = \mathfrak{a}$.
(ii)$\Rightarrow$(iii): Let $S = \{1, x, \ldots, x^n, \ldots\}$. The claim is clear if $S \cap r(\mathfrak{a}) \neq \emptyset$. If $S \cap r(\mathfrak{a}) = \emptyset$ then by (ii) $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : x^n)$ for some n. But $(S^{-1}\mathfrak{a})^c = \bigcup_n (\mathfrak{a} : x^n)$ and $(\mathfrak{a} : x^i) \subseteq (\mathfrak{a} : x^j)$ if $i \leq j$. So $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : x^m)$ for all $m \geq n$.
(iii)$\Rightarrow$(i): As in 7.12.

4. (i) Yes. (Let S be the set of polynomials that are not divisible by any $x - a$ with $|a| = 1$. Then S is a multiplicatively closed subset of $\mathbb{C}[X]$. So by 7.3 and 7.5 the ring in question is Noetherian.)
(ii) Yes. (Any power series f of the form $a + zf_1(z)$ with $a \in \mathbb{C}, a \neq 0$ is (formally) invertible. Hence if $f(z)$ has a positive radius of convergence, then so is f^{-1}. Since any power series g may be written as $z^n g'$ with g' of the above form, we see that every ideal of the ring in question is of the form (z^n)

for some $n > 0$.)
(iii) No. (Let $f_n = \prod_{i=n}^{\infty}(1 - \frac{z}{2^i})$. Each f_n may be written as a power series whose radius of convergence is infinite. For any power series $g_0, \ldots, g_n$ we have $\sum_{i=0}^{n} g_i f_i \neq f_{n+2}$, because the left-hand side has a root 2^{n+1} and the right-hand side does not. So the chain of ideals $(f_0, \ldots, f_n)$ does not stabilize.)
(iv) Yes. (This ring is generated over $\mathbb{C}$ by $z^{k+1}, \ldots, z^{2k+1}$ and hence is Noetherian by 7.7.)
(v) No. (It is not hard to see that the chain of ideals $(z, zw, \ldots, zw^n)$ does not stabilize.)

5. By Chapter 5, Ex. 12 and 7.8.

6. Follow the hint.

7. Clear by 7.6.

8. Yes. (Consider the exact sequence $0 \longrightarrow (x) \longrightarrow A[x] \longrightarrow A \longrightarrow 0$. By 6.3 A is a Noetherian $A[x]$-module and hence a Noetherian A-module.)

9. Follow the hint.

10. The proof of 7.5 may be easily adapted here.

11. No. (Consider the ring R in the solution of Chapter 6, Ex. 12. This ring is not Noetherian since the chain of ideals $\underbrace{k \times \ldots \times k}_{n \text{ copies}} \times 0 \times \ldots$ does not stabilize. However, each local ring is isomorphic to k and hence is Noetherian.)

12. Clear by Chapter 3, Ex. 16(i).

13. Every fiber of f^* is of the form $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of A. Since $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is a $k(\mathfrak{p})$-algebra, where $k(\mathfrak{p})$ is the residue field of $A_\mathfrak{p}$, it follows that if B is a finitely generated A-algebra then $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is a finitely generated $k(\mathfrak{p})$-algebra and hence is Noetherian by 7.7. Now the claim follows from Chapter 6, Ex. 8.

14. Follow the hint.

15. (i)$\Rightarrow$(ii): If M is free then it is a direct sum of finitely many copies of A and hence is obviously flat.
(ii)$\Rightarrow$(iii): Clear.
(iii)$\Rightarrow$(iv): Applying the $\mathrm{Tor}^A(-, M)$ functor to the short exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$$

we get a segment of the long exact Tor-sequence

$$\mathrm{Tor}_1^A(k, M) \longrightarrow \mathfrak{m} \otimes M \longrightarrow A \otimes M \longrightarrow k \otimes M \longrightarrow 0,$$

which readily implies that $\mathrm{Tor}_1^A(k, M) = 0$.
(iv)⇒(i): Follow the hint and note its similarity to the hint of Chapter 3, Ex. 15.

16. (i)⇒(ii): By 7.3, 3.10, and Ex. 15.
(ii)⇒(iii): Trivial.
(iii)⇒(i): Again by 7.3, 3.10, and Ex. 15.

17. It is straightforward to define irreducible submodules of M and get an analogue of 7.11. It remains to show that if M is Noetherian then every irreducible submodule N of M is primary. Suppose that $a \in A$, $x \in M \setminus N$, and $ax \in N$. Let $Q_i = \{x \in M : a^i x \in N\}$. Since M is Noetherian, for some n we have $Q_n = Q_{n+1} = \dots$. Let $N_1 = a^n M + N$. If $Q_n \cap N_1 \neq N$ then there is a $y \in M$ such that $a^n y \notin N$ and $a^{2n} y \in N$, so $y \in Q_{2n} = Q_n$, so $a^n y \in N$, contradiction. So $Q_n \cap N_1 = N$. Since $x \in Q_n$ and N is irreducible, we conclude that $N_1 = N$, i.e. $a^n M \subseteq N$.

18. (i)⇒(ii): Let $0 = \bigcap_{i=1}^n Q_i$ be a minimal primary decomposition in M and $r_M(Q_1) = \mathfrak{p}$. Consider any nonzero $x \in \bigcap_{i=2}^n Q_i$. Since Q_1 is primary, $(Q_1 : M) \subseteq (Q_1 : x) \subseteq \mathrm{Ann}(x) \subseteq \mathfrak{p}$. If there is an $a \in \mathfrak{p} \setminus \mathrm{Ann}(x)$ then there is a natural number $n > 1$ such that $a^n \in \mathrm{Ann}(x)$ but $a^m \notin \mathrm{Ann}(x)$ for any $m < n$. So $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(ax) \subseteq \mathfrak{p}$ as $a^{n-1} \in \mathrm{Ann}(ax) \setminus \mathrm{Ann}(x)$. Iterating this we shall reach $\mathfrak{p}$ in finitely many steps because A is a Noetherian ring.
(ii)⇒(iii): If $\mathrm{Ann}(x) = \mathfrak{p}$ then $Ax = A/\mathfrak{p}$.
(iii)⇒(i): If $N \subseteq M$ is isomorphic to $A/\mathfrak{p}$ then clearly there is an $x \in N$ such that $N = Ax$ and $\mathrm{Ann}(x) = \mathfrak{p}$. Let $0 = \bigcap_{i=1}^n Q_i$ be a minimal primary decomposition in M (it exists by Ex. 17). Then $\mathfrak{p} = r(\mathrm{Ann}(x)) = r(\bigcap_{i=1}^n Q_i : x) = r(\bigcap_{i=1}^n (Q_i : x)) = \bigcap_{i=1}^n r(Q_i : x) = \bigcap_{x \notin Q_i} r(Q_i : x)$. Since $r_M(Q_i) = r(Q_i : x)$ whenever $x \notin Q_i$ (the analogue of 4.4(ii) for modules), we have $\mathfrak{p} = \bigcap_{x \notin Q_i} r_M(Q_i)$. So $\mathfrak{p}$ belongs to 0 in M by 1.11.

For the last claim, let $\mathfrak{p}_0$ be a prime ideal belonging to 0 (again it exists by Ex. 17) and $M_0 = A/\mathfrak{p}_0$ a submodule of M. Since M/M_0 is again a Noetherian A-module, we may repeat the procedure until M is reached.

19. Without loss of generality $r \leq s$. Let $\mathfrak{d}_i = \mathfrak{c}_i \cap \bigcap_{j=2}^r \mathfrak{b}_j$ for $1 \leq i \leq s$. We claim that $\mathfrak{b}_1 = \bigcap_{i=1}^s (\mathfrak{b}_1 + \mathfrak{d}_i)$. If this is not the case then for every i there are $b_i \in \mathfrak{b}_1, d_i \in \mathfrak{d}_i \setminus \mathfrak{a}$ such that $b_1 + d_1 = \ldots = b_s + d_s \notin \mathfrak{b}_1$. So $d_1 - d_2, \ldots, d_1 - d_s \in (\mathfrak{b}_1 \cap \bigcap_{j=2}^r \mathfrak{b}_j) = \mathfrak{a}$. So $d_1 \in \mathfrak{c}_i$ for every i. But this is a contradiction because $d_1 \notin \mathfrak{a}$ and $\bigcap_{i=1}^s \mathfrak{c}_i = \mathfrak{a}$. Now, since $\mathfrak{b}_1$ is irreducible, we have that $\mathfrak{b}_1 = \mathfrak{b}_1 + \mathfrak{d}_i$ for some i, i.e. $\mathfrak{d}_i \subseteq \mathfrak{b}_1$. So $\mathfrak{d}_i = \mathfrak{a}$. After re-indexing we may assume that $\mathfrak{d}_1 = \mathfrak{a}$. Since the decompositions are minimal, there is a $b \in (\bigcap_{j=2}^r \mathfrak{b}_j) \setminus \mathfrak{a}$. Since $b \notin \mathfrak{b}_1, \mathfrak{c}_1$, we have $(\mathfrak{a} : b) = (\mathfrak{b}_1 : b) = (\mathfrak{c}_1 : b)$. Since

the ring A is Noetherian, by 7.12 $\mathfrak{b}_1, \mathfrak{c}_1$ are primary ideals, hence by 4.4(ii) $r(\mathfrak{b}_1) = r(\mathfrak{b}_1 : b) = r(\mathfrak{c}_1 : b) = r(\mathfrak{c}_1)$.

Now by discarding redundant ones we may obtain a minimal decomposition of $\mathfrak{a}$ from the lot $\mathfrak{c}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_r$. Then we may repeat the procedure to get rid of the next b_i with i minimal. After at most r steps we obtain a minimal decomposition of $\mathfrak{a}$ from $\mathfrak{c}_1, \ldots, \mathfrak{c}_s$, which clearly must contain every $\mathfrak{c}_i$. So $r = s$ and $r(\mathfrak{b}_i) = r(\mathfrak{c}_i)$ for all i.

For the analogue for modules we only need to assume that the A-module M is Noetherian (A could be any ring), as in Ex. 17 above. The proof is a straightforward adaptation of the argument above.

20. (i) Using elementary relations among set operations we may "push" the complement signs inwards as far as possible in the formal expression of any $E \in \mathcal{F}$. Then the claim easily follows.
(ii) Let $E = \bigcup_{i=1}^n (U_i \cap C_i)$, where U_i is a non-empty open set and C_i a closed set for all i. If E is dense in X then $X = \overline{E} = \bigcup_{i=1}^n C_i$. So either there is an i such that $C_i = X$ and $U_i \subseteq E$ or X is a nontrivial union of close sets, which is impossible since X is irreducible. The other direction is clear.

21. The "only if" direction is by Ex. 20(ii). For the other direction, suppose for contradiction that $E \notin \mathcal{F}$. Then the collection of closed sets $X' \subseteq X$ such that $E \cap X' \notin \mathcal{F}$ is not empty and therefore has a minimal element X_0. If $X_0 = \bigcup_i Y_i$ is a nontrivial decomposition of X into closed subsets Y_i, then by the choice of X_0 each $E \cap Y_i \in \mathcal{F}$ and hence $E \cap X_0 = \bigcup_i (E \cap Y_i) \in \mathcal{F}$, contradiction. So X_0 is irreducible. If $\overline{E \cap X_0} \subsetneq X_0$ then again by the choice of X_0 we have $E \cap X_0 = E \cap \overline{E \cap X_0} \in \mathcal{F}$, contradiction. If $E \cap X_0$ contains a nonempty open subset of X_0, say U_0, then $(X_0 \setminus U_0)$ is a closed subset of X strictly contained in X_0 and hence $(X_0 \setminus U_0) \cap E \in \mathcal{F}$. But then $E \cap X_0 = U_0 \cup ((X_0 \setminus U_0) \cap E) \in \mathcal{F}$, contradiction again.

22. The "only if" direction is trivial. For the other direction, suppose for contradiction that E is not open. Then the collection of closed sets $X' \subseteq X$ such that $E \cap X'$ is not open in X' is not empty and therefore has a minimal element X_0. As in the above proof of Ex. 21 X_0 must be irreducible. Then clearly $E \cap X_0 \neq \emptyset$. So $E \cap X_0$ contains a nonempty open subset U_0 of X_0. But then $E \cap (X_0 \setminus U_0)$ is open in $X_0 \setminus U_0$ and hence $E \cap X_0$ is open in X_0, contradiction.

23. By Ex. 20(i) it is enough to take $E = U \cap C$, where U is open and C is closed. Let $C = V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of B. Replacing B with $B/\mathfrak{a}$ we may now assume that E is open. By Chapter 6, Ex. 8 and Ex. 6, E is a finite union of open sets of the form X_g and hence we may reduce to the

case where $E = X_g$ for some $g \in B$. By 7.3 we may replace B with B_g and hence may assume that $E = Y$. From this point forth we may proceed in two ways. The first basically follows the hint. The second is a more direct proof, which perhaps sheds more light on why canonical spectrum mappings between Noetherian rings preserve constructibility.

(1) Since each irreducible closed subset of X is of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A, by Ex. 21 and Chapter 3, Ex. 21(iii), it is enough to replace A, B with $A/\mathfrak{p}, B/\mathfrak{p}^e$ respectively and show that either $f^*(Y)$ is not dense in X or it contains a non-empty open subset of X, where X is irreducible. So suppose that $f^*(Y)$ is dense in X. By Chapter 6, Ex. 7 there are only finitely many irreducible components of Y, say $Y_1, \ldots, Y_n$. Note that some $f^*(Y_i)$ is dense in X, for otherwise $X = \overline{f^*(Y)} = \bigcup_{i=1}^n \overline{f^*(Y_i)}$ is a nontrivial decomposition of X into closed subsets. Now it is enough to show that $f^*(Y_i)$ contains a non-empty open subset of X. If $Y_i = \operatorname{Spec}(B/\mathfrak{p})$ for some minimal prime ideal of B then, since $f^*(Y_i)$ is dense in X, by Chapter 1, Ex. 21(iv)(v) we may replace B with $B/\mathfrak{p}$ and assume that A, B are integral domains and f is injective. Finally, by Chapter 6, Ex. 7 there is an $a \in A$ such that, for any prime ideal $\mathfrak{p}$ of A that does not contain a, the canonical mapping of A into the algebraic closure of the field of fractions of $A/\mathfrak{p}$ may be extended to B. The kernel of this extension is a prime ideal of B whose contraction in A is clearly $\mathfrak{p}$. So $f^*(Y)$ contains the open subset $X_a \subseteq X$.

(2) By Chapter 6, Ex. 7 and Chapter 1, Ex. 20(iv), instead of A and B, we may now consider $A_1 = A/\mathfrak{p}^c$ and $B_1 = B/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime ideal of B that corresponds to an irreducible component of Y, because $X_1 = \operatorname{Spec}(A_1)$ is a closed subset of X and hence any constructible subset of X_1 is a constructible subset of X by Ex. 20(i). Note that A_1, B_1 are integral domains and the induced morphism f_1 between them is injective. So by Chapter 5, Ex. 21 there is an $a_1 \in A_1$ such that, for any prime ideal $\mathfrak{p}$ of A_1 that does not contain a_1, the canonical mapping of A_1 into the algebraic closure of the field of fractions of $A_1/\mathfrak{p}$ may be extended to B_1. This means that $f_1^*(\operatorname{Spec}(B_1))$ contains the open subset $X_{a_1} \subseteq X_1$. Since every prime ideal in $f_1^*(\operatorname{Spec}(B_1)) \setminus X_{a_1}$ contains a_1, we may now proceed to consider $A_1/(a_1), B_1/(a_1)^e$. Again let $\mathfrak{p}$ be a minimal prime ideal of $B_1/(a_1)^e$ that corresponds to an irreducible component of $\operatorname{Spec}(B_1/(a_1)^e)$. Let $A_2 = (A_1/(a_1))/\mathfrak{p}^c$, $(B_1/(a_1)^e)/\mathfrak{p}$, and f_2 the induced morphism between them. As above we may find an open subset of $\operatorname{Spec}(A_2)$ that is contained in $f_2^*(\operatorname{Spec}(B_2))$. We iterate this process and note that it must terminate in finitely many steps since A is a Noetherian ring. Also note that if we choose a different irreducible component to work

with at certain step then the resulting chain of open subsets may be different. Essentially we have constructed a tree structure such that at each node of the tree we have produced an open subset that actually is a constructible subset of X. Since this tree is finitely branching and has no infinite path, it actually is finite (this is known as König's lemma and is easily proved by induction). So the union of the open sets produced at each node is a constructible subset of X and is equal to $f^*(Y)$.

24. The forward direction is proved in Chapter 5, Ex. 10. For the other direction, as in Ex. 23, we reduce to proving that $E = f^*(Y)$ is open in X. By Ex. 23 E is a constructible subset of X. Let $X_0 = V(\mathfrak{p})$ be any irreducible closed subset of X. If $E \cap X_0 \neq \emptyset$ then by the going-down property $\mathfrak{p} \in E \cap X_0$ and hence $\overline{E \cap X_0} = X_0$. By Ex. 21 E contains a non-empty open subset of X_0. So by Ex. 22 E is open.

25. Clear by the exercises indicated in the hint.

26. (i) Since any additive function λ is constant on any isomorphism class of finitely generated A-modules, we may regard λ as a function from $F(A)$ into G. The free abelian group generated by $F(A)$ is also written as $\mathbb{Z}^{(F(A))}$ (see p. 21). Now define a group homomorphism $\lambda' : \mathbb{Z}^{(F(A))} \longrightarrow G$ by first setting $n(M) \longmapsto n\lambda(M)$ for each $(M) \in F(A)$, $n \in \mathbb{Z}$ and then extending to the whole $\mathbb{Z}^{(F(A))}$ by linearity. Since λ is additive, clearly $D \subseteq \ker(\lambda')$. So there is a canonical homomorphism

$$\lambda_0 : K(A) = \mathbb{Z}^{(F(A))}/D \longrightarrow \mathbb{Z}^{(F(A))}/\ker(\lambda') \rightarrowtail G.$$

It is routine to check that λ_0 is as required.
(ii) The chain in Ex. 18 yields a finite expression $\gamma(M) = \sum_{i=0}^{r} \gamma(A/\mathfrak{p}_i)$.
(iii) If A is a PID then every prime ideal is maximal and is isomorphic to A as A-modules. So $\gamma(\mathfrak{p}) = \gamma(A)$ and hence $\gamma(A/\mathfrak{p}) = 0$ for all prime ideal $\mathfrak{p}$. So $K(A)$ is generated by a single element $\gamma(A)$ by (ii). So $K(A) \cong \mathbb{Z}$.
(iv) Since f is finite, every isomorphism class in $F(B)$ is a subclass of an isomorphism class in $F(A)$. This determines a function $\phi : F(B) \longrightarrow F(A)$, which in turn determines a group homomorphism $\widehat{\phi} : \mathbb{Z}^{(F(B))} \longrightarrow \mathbb{Z}^{(F(A))}$. Let $K(B) = \mathbb{Z}^{(F(B))}/D_B$ and $K(A) = \mathbb{Z}^{(F(A))}/D_A$. Since $\widehat{\phi}(D_B) \subseteq D_A$, we have a canonical homomorphism

$$f_! : K(B) \longrightarrow \mathbb{Z}^{(F(B))}/\widehat{\phi}^{-1}(D_A) \longrightarrow K(A),$$

which is easily checked to be as required. From the construction it should be clear that we have a contravariant functor between the corresponding categories. So in particular we have $(g \circ f)_! = f_! \circ g_!$.

27. (i) Since tensor with flat modules preserves exact sequences, this becomes routine checking. For example, $\gamma_1(M)(\gamma_1(N_1)+\gamma_1(N_2)) = \gamma_1(M)\gamma_1(N_1 \oplus N_2) = \gamma_1(M \otimes (N_1 \oplus N_2)) = \gamma_1((M \otimes N_1) \oplus (M \otimes N_2)) = \gamma_1(M \otimes N_1) + \gamma_1(M \otimes N_2) = \gamma_1(M)\gamma_1(N_1) + \gamma_1(M)\gamma_1(N_2)$.
(ii) Similar to (i).
(iii) By Ex. 15.
(iv) The tensor map $B \otimes_A - : F_1(A) \longrightarrow F_1(B)$ preserves exact sequences and determines a group homomorphism $\mathbb{Z}^{(F_1(A))} \longrightarrow \mathbb{Z}^{(F_1(B))}$. The rest of the argument is as in Ex. 26(iv).
(v) By linearity it is enough to consider the case $x = \gamma_1^A(M)$ and $y = \gamma^B(N)$. We then have $f_!(f^!(\gamma_1^A(M))\gamma^B(N)) = f_!(\gamma_1^B(B \otimes_A M)\gamma^B(N)) = f_!(\gamma^B(B \otimes_A M \otimes_B N)) = f_!(\gamma^B(M \otimes_A N)) = \gamma^A(M \otimes_A N) = \gamma_1^A(M)\gamma^A(N) = \gamma_1^A(M) f_!(\gamma^B(N))$.