Solutions to the Exercises in Chapter 5, Introduction to Commutative Algebra, M. F. Atiyah and I. G. MacDonald

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1. Let $V(\mathfrak{a}) \subseteq \operatorname{Spec}(B)$ be a closed subset, where $\mathfrak{a}$ is an ideal of B. Let $\mathfrak{b} = f(A) \cap \mathfrak{a}$ and $V(\mathfrak{b})$ the closed subset in $\operatorname{Spec}(f(A))$. By Chapter 1, Ex. 21(iv), f induces a homeomorphism $f_0^* : \operatorname{Spec}(f(A)) \longrightarrow V(\ker(f))$, where the latter takes the restricted Zariski topology. So it is enough to show that $f^*(V(\mathfrak{a})) = f_0^*(V(\mathfrak{b}))$. Clearly $\mathfrak{p} \cap f(A) \in V(\mathfrak{b})$ for each $\mathfrak{p} \in V(\mathfrak{a})$. Conversely, since $B/\mathfrak{a}$ is integral over $f(A)/\mathfrak{b}$, by 5.10 for each $\mathfrak{p} \in V(\mathfrak{b})$ we can find a prime ideal $\mathfrak{q}$ of B such that $(\mathfrak{q}/\mathfrak{a}) \cap (f(A)/\mathfrak{b}) = \mathfrak{p}/\mathfrak{b}$. Clearly $\mathfrak{q} \in V(\mathfrak{a})$ and $\mathfrak{q} \cap f(A) = \mathfrak{p}$.

2. By 5.10 there is a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \ker(f)$. Let K and F be the fields of fractions of $A/\ker(f)$ and $B/\mathfrak{q}$, respectively. Since $A/\ker(f)$ can be embedded into $B/\mathfrak{q}$ and the latter is integral over the former, F is an algebraic field extension of K. The homomorphism f induces a field homomorphism $f' : K \longrightarrow \Omega$, which can be extended to a field homomorphism $f'' : F \longrightarrow \Omega$. These functions commute in the canonical way. So the function $B \longrightarrow B/\mathfrak{q} \longrightarrow F \longrightarrow \Omega$ is as required.

3. Every element in $B' \otimes C$ is of the form $\sum_i (b'_i \otimes c_i)$, where $b'_i \in B'$ and $c_i \in C$. So by 5.2 and 5.1(iii) it is enough to show that each $b'_i \otimes c_i$ is integral over $B \otimes C$. Let $b' \otimes c \in B' \otimes C$ and $(b')^n + f(b_1)(b')^{n-1} + \ldots + f(b_n) = 0$, where $b_1, \ldots, b_n \in B$. We have

$$\begin{aligned}
&(b' \otimes c)^n + (f(b_1) \otimes c)(b' \otimes c)^{n-1} + \ldots + f(b_n \otimes c^n) \\
&= (b')^n \otimes c^n + f(b_1)(b')^{n-1} \otimes c^n + \ldots + f(b_n) \otimes c^n \\
&= ((b')^n + f(b_1)(b')^{n-1} + \ldots + f(b_n)) \otimes c^n \\
&= 0.
\end{aligned}$$

Now 5.6(ii) is a special case since, with the notation there, $A \otimes S^{-1}A = S^{-1}A$ and $B \otimes S^{-1}A = S^{-1}B$.

4. A rather trivial remark first: in this book all fields without qualification are of characteristic 0. Now in the hint $x+1 \notin (x-1) = \mathfrak{n}$ but $(x-1) \cap k[x^2-1] = (x^2-1) = \mathfrak{m}$. So $A_\mathfrak{m} = k[x^2-1]$. Clearly $1/(x+1)$ cannot be integral over $k[x^2-1]$.

5. (i) See the last paragraph of the proof of 5.7.
(ii) By 5.8 the contraction of every maximal ideal of B is a maximal ideal of A. Conversely by 5.10 every maximal ideal $\mathfrak{m}$ of A can be extended to a maximal ideal of B whose contraction is $\mathfrak{m}$. The claim follows readily.

6. By induction this immediately reduces to the case $n = 2$. In that case, since addition and multiplication are defined componentwise, it is almost trivial to see that $B_1 \times B_2$ is an integral A-algebra.

7. Let $b \in B$ and $b^n + a_n b^{n-1} + \ldots + a_1 = 0$ with $a_1, \ldots, a_n \in A$. We may assume that the degree n is as small as possible. So $b(b^{n-1} + a_n b^{n-2} + \ldots + a_2) = -a_1$. Since $B \setminus A$ is multiplicatively closed, we have either $b \in A$, in which case we are done, or $b^{n-1} + a_n b^{n-2} + \ldots + a_2 \in A$, contradicting the choice of n.

8. (i) The hint says it all.
(ii) The idea is the same here: we want to find a bigger ring B^* that contains B such that f and g completely split into linear factors in B^*. For example, to add a linear factor for $f(x) = x^n + b_n x^{n-1} + \ldots + b_1$, consider the ring $B_1 = B[r]/(f(r))$, where r is a new symbol. Clearly B can be embedded into B_1 as a subring and $f(\overline{r}) = 0$, where $\overline{r}$ is the image of r in B_1. Now $x - \overline{r}$ is a linear factor of $f(x)$ in $B_1[x]$ since we may solve for $e_1, \ldots, e_{n-1}$ in the equation

$$(x-\overline{r})(x^{n-1} + e_{n-1}x^{n-2} + \ldots + e_1) = x^n + b_n x^{n-1} + \ldots + b_1.$$

Repeating this procedure we may then find a ring as desired and the rest of the argument is exactly the same as in (i).

9. For any $f \in C[x]$, since C is integral over A, clearly the ring $A[x][f]$ is a finitely generated $A[x]$-module and hence f is integral over $A[x]$. Now the claim is immediate by Ex. 7 and Ex. 8. (Why is the hint so complicated?)

10. (i) (a)⇒(b): For the conclusion of 5.11 clearly we only need to consider the case $m = 1$, $n = 2$. By Chapter 3, Ex. 21(iii) the associated sequence

$$\begin{tikzcd} \operatorname{Spec}(A/f^{-1}(\mathfrak{p}_1)) & \operatorname{Spec}(f(A)/\mathfrak{p}_1) \arrow[l, "(f/\mathfrak{p}_1)^*"'] & \operatorname{Spec}(B/\mathfrak{q}_1) \arrow[l, "\mathrm{id}^*"'] \end{tikzcd}$$

induced by the prime ideals $\mathfrak{p}_1, \mathfrak{q}_1$ may be identified with the sequence

$$V(f^{-1}(\mathfrak{p}_1)) \xleftarrow{(f/\mathfrak{p}_1)^*} V(\mathfrak{p}_1) \xleftarrow{\mathrm{id}^*} V(\mathfrak{q}_1),$$

where the closed subsets take the corresponding restricted Zariski topologies and $(f/\mathfrak{p}_1)^* \circ \mathrm{id}^* = f^* \restriction V(\mathfrak{q}_1)$. So $(f/\mathfrak{p}_1)^* \circ \mathrm{id}^*$ is also a closed mapping. Now by Chapter 1, Ex. 21(iv), f induces a homeomorphism

$$f_0^* : \mathrm{Spec}(f(A)) \longrightarrow V(\ker(f))$$

where $V(\ker(f))$ takes the restricted Zariski topology. So $V(f^{-1}(\mathfrak{p}_1))$ is homeomorphic to $V(\mathfrak{p}_1)$ via $(f/\mathfrak{p}_1)^*$. So id^* is a closed mapping as well. By Chapter 1, Ex. 21(v), $\mathrm{id}^*(V(\mathfrak{q}_1))$ is dense in $V(\mathfrak{p}_1)$. But it is also closed in $V(\mathfrak{p}_1)$ and hence must be $V(\mathfrak{p}_1)$ itself.
(b)⇔(c): As above $\mathrm{Spec}(A/\mathfrak{p})$ and $\mathrm{Spec}(f(A)/(\mathfrak{q} \cap f(A)))$ can be identified via the homeomorphism f_0^*. Then this is clear.
(ii) (a′)⇒(c′): The hint is pretty slick.
(b′)⇔(c′): Much as in (i)(b)⇔(c) above, but using Chapter 3, Ex. 21(ii).

11. Immediate by Ex. 10(i) and Chapter 3, Ex. 18.

12. The first claim is clear. Next, for any $\sigma \in G$ clearly the action on $S^{-1}A$ is given by $\sigma(a/s) = \sigma(a)/\sigma(s)$, where $a \in A$ and $s \in S$. It is clear that $(S^G)^{-1}A^G \subseteq (S^{-1}A)^G$. Conversely, suppose that $\sigma(a)/\sigma(s) = a/s$ for all $\sigma \in G$. Then there is a $t_\sigma \in S$ such that $(\sigma(a)s - \sigma(s)a)t_\sigma = 0$. Replacing it with $\prod_{\tau \in G} \tau(t_\sigma)$ if necessary, we may assume that $t_\sigma \in S^G$. Then

$$\left(\left(\sum_{\sigma \in G} \sigma(a)\right) s - \left(\sum_{\sigma \in G} \sigma(s)\right) a\right) \prod_{\sigma \in G} t_\sigma = 0.$$

Hence $a/s \in (S^G)^{-1}A^G$.

13. Follow the hint.

14. It is clear that $\sigma(B) = B$ since each σ is an automorphism. Since $B^G \subseteq K$ and is integral over A, we must have $B^G = A$.

15. Follow the hint.

16. For a more general proof, due to Nagata, that covers the case of k being finite, see [2, p. 357].

For the "geometrical interpretation", let $A \neq 0$ be the coordinate ring of X and $\xi_1, \dots, \xi_n$ the coordinate functions on X (i.e. the images of the x_i's under the projection $k[x_1, \dots, x_n] \longrightarrow A$). Let $\pi_1, \dots, \pi_r$ be certain linear combinations of $\xi_1, \dots, \xi_n$ as given by Noether's normalization lemma. Let $y_1, \dots, y_r$ be new variables. We think of $k[y_1, \dots, y_r]$ as the coordinate ring of a linear subspace $L \subseteq k^n$ of dimension r. Consider the injective integral k-algebra homomorphism $f : k[y_1, \dots, y_r] \longrightarrow A$ given by

$$y_i \longrightarrow \pi_i \ (1 \leq i \leq r).$$

By Chapter 1, Ex. 28 this corresponds to a regular mapping $\phi : X \longrightarrow L$, which clearly must be linear. Now by Ex. 1 and Ex. 10(i)

$$f^* : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(k[y_1, \dots, y_r])$$

is surjective. Since each point in L corresponds to a maximal ideal of the form $(y_1 - b_1, \dots, y_r - b_r)$ in $\mathrm{Spec}(k[y_1, \dots, y_r])$, where $b_1, \dots, b_r \in k$, we deduce that ϕ is surjective as well. In fact, since there are only finitely many automorphisms of A over $f(k[y_1, \dots, y_r])$, by 5.8, Ex. 13, and Ex. 17 below, ϕ has finite fibers (i.e. $\phi^{-1}(p)$ is finite for every $p \in L$). Now extend ϕ to k^n by linearity.

17. Since the linear space L in question is at least the trivial space 0, X cannot be empty. Of course this and the second claim can be proved together in a more direct way. Let $\mathfrak{m}$ be a maximal ideal containing $I(X) \neq (1)$. By Ex. 16 there is essentially a polynomial subring R of $k[t_1, \dots, t_n]/\mathfrak{m}$ such that the latter is integral over the former. By 5.7, R is a field, and hence must be k itself. But k is algebraically closed, so $k[t_1, \dots, t_n]/\mathfrak{m} = k$. Then it is easy to see that $\mathfrak{m}$ must be of the desired form, which corresponds to a point. Since this point is contained in X, we have $X \neq \emptyset$.

18, 19. By Hilbert's Nullstellensatz, with the notation in Ex. 17 above, $k[t_1, \dots, t_n]/\mathfrak{m}$ is a finite field extension of k, which must be equal to k since k is algebraically closed. The rest is the same as above.

20. Routine verification.

21. The hint is a complete proof.

22. The last bit of the hint perhaps needs a moment of reflection. The element $s \in A$ is obtained together with some $y_1, \dots, y_n \in B_v$ as in Ex. 20. Let Ω be the algebraic closure of $k = A/\mathfrak{m}$. The canonical mapping $A \longrightarrow \Omega$ can be extended along the sequence of embeddings

$$A \longrightarrow A_s \longrightarrow A[y_1, \dots, y_n]_s \longrightarrow (B_v)_s$$

to a homomorphism $f : (B_v)_s \longrightarrow \Omega$. But $s \notin \mathfrak{m}$ and $y_1, \dots, y_n$ are algebraically independent over A_s, we have $f(A) = f(A_s) = f(A[y_1, \dots, y_n]_s)$

and $f(B_v) = f((B_v)_s)$. So $f(B) \subseteq f(B_v)$ is integral over $f(A) = k$. Hence, by 5.7, $f(B)$ is a field. So $\ker(f) \cap B$ is a maximal ideal of B that does not contain v.

23. It is a pity that all these exercises are spoiled by the hints.

24. Let $f : A \longrightarrow B$ witness that B is an A-algebra. Let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} = f(A) \cap \mathfrak{q}$. Suppose that B is integral over A. Then $B/\mathfrak{q}$ is integral over $f(A)/\mathfrak{p}$. Passing to the quotient rings, we may assume that A, $f(A)$, and B are integral domains and the Jacobson radicals of A and $f(A)$ are 0. By Ex. 23(i) it is enough to show that the Jacobson radical $\mathfrak{J}$ of B is 0. Suppose for contradiction that this is not the case. Let $u \in \mathfrak{J}$ be a nonzero element. Since B is integral over $f(A)[u]$, by Ex. 5(i), u is in the Jacobson radical of $f(A)[u]$. But $f(A)[u]$ is finitely generated over $f(A)$ and, by Ex. 22, the Jacobson radical of $f(A)[u]$ is 0. This is a contradiction.

Passing to the quotient rings as above, the other case is a direct consequence of Ex. 22.

25. Note that the hint (i)⇒(ii) actually shows that, if $f : A \longrightarrow B$ witnesses that B is an A-algebra, then $f(A)$ is a subfield of B.

26. (1)⇒(2): Let U be an open set that is disjoint from $\overline{E \cap X_0}$. Then $U \cap E = \emptyset$, for otherwise we would have $U \cap E \cap X_0 \neq \emptyset$. So $\overline{E \cap X_0} = E$.
(2)⇒(3): Let $U_1 \neq U_2$ be open sets and E_1, E_2 their complements. So $E_1 \cap X_0 \neq E_2 \cap X_0$, for otherwise we would have $E_1 = \overline{E_1 \cap X_0} = \overline{E_2 \cap X_0} = E_2$. Hence the mapping is injective. It is surjective by definition.
(3)⇒(1): Let U, E be nonempty open and closed sets respectively such that $U \cap E \neq \emptyset$. If $U \cap E \cap X_0 = \emptyset$ then $U \cap (X \setminus E) \cap X_0 = U \cap X_0$, which is a contradiction since $U \cap (X \setminus E) \neq U$.
(i)⇒(ii): It is enough to show that every nonempty set of the form $X_a \cap V(E)$ contains a maximal ideal, where $a \in A$ and $E \subseteq A$. This is immediate by Ex. 23(i).
(ii)⇒(iii): Since the single point in question is a maximal ideal, this is clear.
(iiii)⇒(i): Let $\mathfrak{p}$ be a prime ideal of A that is not maximal and P the set of prime ideals that properly contain $\mathfrak{p}$. If there is an $a \in (\bigcap P) \setminus \mathfrak{p}$ then $X_a \cap V(\mathfrak{p})$ is a nonempty locally closed subset that consists of a single point $\mathfrak{p}$ and hence is closed, which is absurd. So $\bigcap P = \mathfrak{p}$ and by Ex. 23(iii) A is a Jacobson ring.

27. By Zorn's lemma and 5.21.

28. (1)⇒(2): If there are $a \in \mathfrak{a} \setminus \mathfrak{b}$ and $b \in \mathfrak{b} \setminus \mathfrak{a}$ then either $a/b \in A$ or $b/a \in A$. So either $a \in \mathfrak{b}$ or $b \in \mathfrak{a}$, contradiction.
(2)⇒(1): Clearly A is a local ring. Since K is the field of fractions of A, A

must be a maximal element of Σ, where Σ is as in Ex. 27. So A is a valuation ring of K.

Since $A \subseteq A_{\mathfrak{p}} \subseteq K$, by 5.18(ii), $A_{\mathfrak{p}}$ is a valuation ring of its field of fractions K. For $A/\mathfrak{p}$ we may use (2)$\Rightarrow$(1).

29. Clear by 5.18(i)(ii).

30, 31. These can be found in any introductory treatise on valued fields; see, for example, [1] or even [2].

32. These days Δ is called a convex subgroup. We shall use this modern terminology.

Since $\mathfrak{p}$ is a prime ideal, clearly by Ex. 31(1) $\pm v(A \setminus \mathfrak{p})$ is a group. Let $a \in A \setminus \mathfrak{p}$ and $b \in A$ such that $0 < \beta = v(b) < \alpha = v(a)$. Then $a/b \in A$. So $b \notin \mathfrak{p}$. This shows convexity.

Now let $\mathfrak{p}, \mathfrak{q}$ be two prime ideals of A. By Ex. 28(2) we may assume that $\mathfrak{p} \subsetneq \mathfrak{q}$. Let $a \in \mathfrak{q} \setminus \mathfrak{p}$. If there is a $b \in \mathfrak{p}$ such that $v(a) = v(b)$ then b/a is a unit in A. Since $\mathfrak{p}$ is a prime, either $a \in \mathfrak{p}$ or $b/a \in \mathfrak{p}$, which is a contradiction. So the mapping in question is injective. Conversely let Δ be any convex subgroup and $\mathfrak{a} = \{a \in A : v(a) > \Delta\}$. By Ex. 31, $\mathfrak{a}$ is an ideal of A. It is a prime ideal by Ex. 31(1).

For the last question, let Δ be the convex subgroup that corresponds to $\mathfrak{p}$. Then the value groups of $A/\mathfrak{p}$ and $A_{\mathfrak{p}}$ are Δ and Γ/Δ.

33. Routine verification.

34. Follow the hint.

35. Let $A' = f(A)$. Every A'-algebra is also an A-algebra. Let C be an A'-algebra. It is routine to check that $B \otimes_{A'} C \cong B \otimes_A C$ as A-algebras. This justifies passing from A to its image in B. The existence of a valuation ring A' containing A is shown by the same construction as in the text. The rest of the hint is clear.

In the hint for the second claim, $B/\mathfrak{N}$ can be embedded into $\prod(B/\mathfrak{p}_i)$ via

$$b + \mathfrak{N} \longmapsto (b + \mathfrak{p}_1, \ldots, b + \mathfrak{p}_n).$$

By Ex. 6, $A \longrightarrow \prod(B/\mathfrak{p}_i)$ is integral and hence $A \longrightarrow B/\mathfrak{N}$ is integral. So for any $b + \mathfrak{N} \in B/\mathfrak{N}$ we have

$$(b + \mathfrak{N})^m + e_1(b + \mathfrak{N})^{m-1} + \ldots + e_m = \mathfrak{N},$$

where $e_1, \ldots, e_m$ are in the image of A in $B/\mathfrak{N}$. So $b^m + e_1 b^{m-1} + \ldots + e_m = c \in \mathfrak{N}$. Since $c^l = 0$ for some l, we see that $A \longrightarrow B$ is integral.

References

[1] Antonio J. Engler and Alexander Prestel, *Valued fields*, Springer-Verlag, Berlin, 2005.

[2] S. Lang, *Algebra*, revised third ed., Springer-Verlag, New York, 2002.