

Solutions to the Exercises in Chapter 4,
Introduction to Commutative Algebra,
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1. Recall the definition of an irreducible component of a topological space and other related basic facts in Chapter 1, Ex. 20, especially (iv). Now if \mathfrak{a} is decomposable then there are only finitely many minimal prime ideals in the ring A/\mathfrak{a} by 4.6. The claim follows readily from this.

2. Suppose $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a minimal decomposition of \mathfrak{a} . Then $\mathfrak{a} = r(\mathfrak{a}) = \bigcap_{i=1}^n r(\mathfrak{q}_i)$, which must also be a minimal decomposition of \mathfrak{a} by 4.5. So every one of the prime ideals $r(\mathfrak{q}_i)$ is a minimal one belonging to \mathfrak{a} . So \mathfrak{a} has no embedded prime ideals.

3. Let \mathfrak{q} be a primary ideal of A . So A/\mathfrak{q} is also absolutely flat (Chapter 2, Ex. 28), so every nonunit of A/\mathfrak{q} is a zero-divisor. Since every zero-divisor in A/\mathfrak{q} is nilpotent, actually A/\mathfrak{q} is a local ring. Hence A/\mathfrak{q} is a field. So \mathfrak{q} is maximal.

4. \mathfrak{m} is maximal as $\mathbb{Z}[t]/(2, t) = \mathbb{Z}/2\mathbb{Z}$ is a field. \mathfrak{q} is primary as $\mathbb{Z}[t]/(4, t) = \mathbb{Z}/4\mathbb{Z}$ only has one zero-divisor, which of course must be nilpotent. It is almost trivial to see that $r(\mathfrak{q}) = \mathfrak{m}$ and $\mathfrak{m}^n \neq \mathfrak{q}$ for any n .

5. It is easy to see that $\mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Conversely, we have $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, yz)$, so $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2 = (x, yz) \cap (x^2, xy, xz, y^2, yz, z^2) = (x^2, xy, xz, yz) = \mathfrak{p}_1\mathfrak{p}_2$. So $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a primary decomposition. It is easy to check that this decomposition is indeed reduced. Clearly $\mathfrak{p}_1, \mathfrak{p}_2$ are isolated and \mathfrak{m} is embedded.

6. Note that, if the zero ideal of $C(X)$ were decomposable, then there would be only finitely many minimal prime ideals of $C(X)$. This certainly sounds strange since every maximal ideal \mathfrak{m}_x of $C(X)$ contains a minimal prime ideal. Hence to show that the zero ideal of $C(X)$ is not decomposable

it is enough to show that if $x \neq y \in X$ then any two minimal prime ideals $\mathfrak{p} \subseteq \mathfrak{m}_x$, $\mathfrak{q} \subseteq \mathfrak{m}_y$ are different: Since X is Hausdorff, there is an open set U such that $x \in U$ and $y \notin \bar{U}$. By Urysohn's lemma there are $f, g \in C(X)$ such that $f(\bar{U}) = 0$, $f(y) = 1$, $g(x) = 1$, and $g(X \setminus U) = 0$. So $fg = 0$. So $f \in \mathfrak{p}$ but $f \notin \mathfrak{q}$, so $\mathfrak{p}, \mathfrak{q}$ are different.

7. (i) Clear.

(ii) Since $A[x]/\mathfrak{p}[x] = (A/\mathfrak{p})[x]$ and A/\mathfrak{p} is a domain, $(A/\mathfrak{p})[x]$ is also a domain. Hence $\mathfrak{p}[x]$ is prime.

(iii) Consider $A[x]/\mathfrak{q}[x] = (A/\mathfrak{q})[x]$. Suppose that $f \in (A/\mathfrak{q})[x]$ is a zero-divisor. By Chapter 1, Ex. 2(iii) there is a nonzero $a \in A/\mathfrak{q}$ such that $af = 0$. So every coefficient of f is a zero-divisor in A/\mathfrak{q} . Since \mathfrak{q} is primary, every coefficient of f is nilpotent. By Chapter 1, Ex. 2(ii) f is nilpotent in $(A/\mathfrak{q})[x]$. So $\mathfrak{q}[x]$ is primary.

(iv) Since $\mathfrak{q}_i^e = \mathfrak{q}_i[x]$, we have $\mathfrak{a}^e = \mathfrak{a}[x] = (\bigcap_{i=1}^n \mathfrak{q}_i)[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x] = \bigcap_{i=1}^n \mathfrak{q}_i^e$. The rest of the claim follows from (ii) and (iii) above.

(v) Clear.

8. Fix an i and consider the ideal $\mathfrak{p}_i = (x_1, \dots, x_i)$. Since $k[x_1, \dots, x_n]/\mathfrak{p}_i = k[x_{i+1}, \dots, x_n]$ is a domain, \mathfrak{p}_i is prime. On the other hand let us consider the natural embedding $k[x_1, \dots, x_i] \longrightarrow k[x_1, \dots, x_n]$. All the powers of the maximal ideal of $k[x_1, \dots, x_i]$ are primary by 4.2. Since the powers of \mathfrak{p}_i are the extensions of the powers of the maximal ideal of $k[x_1, \dots, x_i]$, by (possibly repeatedly applying) Ex. 7 we see that all the powers of \mathfrak{p}_i are primary.

9. Let $x \in A$ be a zero-divisor and $xy = 0$. Then the set of prime ideals that contain $(0 : y)$ is not empty, and hence has a minimal one, which clearly contains x . Conversely if $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$, then there is an $a \in A$ such that $(0 : a) \subseteq \mathfrak{p}$. Suppose for contradiction that x is not a zero-divisor. Then the set $\{yx^n : y \in A \setminus \mathfrak{p}, n \geq 0\}$ is multiplicatively closed and is disjoint from $(0 : a)$. So \mathfrak{p} cannot be minimal in the set of prime ideals containing $(0 : a)$.

Next, let $S^{-1}\mathfrak{p} \in D(S^{-1}A) \subseteq \text{Spec}(S^{-1}A)$. So $S^{-1}\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a/s)$ for some $a/s \in S^{-1}A$. By 3.15, $(0 : a/s) = S^{-1}(0 : a)$. So by the one-to-one correspondence between the prime ideals of A and $S^{-1}A$ we see that \mathfrak{p} must be minimal in the set of prime ideals containing $(0 : a)$. Conversely, let $\mathfrak{p} \in D(A) \cap \text{Spec}(S^{-1}A)$ be minimal in the set of prime ideals containing $(0 : a)$ for some $a \in A$. Then $S^{-1}(0 : a) = (0 : a/1) \subseteq S^{-1}\mathfrak{p}$. Again by the one-to-one correspondence between the prime ideals of A and $S^{-1}A$ we deduce that $S^{-1}\mathfrak{p}$ is minimal in the set of prime ideals containing $(0 : a/1)$.

Let $0 = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the associated prime ideals. By the first remark after 4.5 for each i there is an $a_i \in A$ such that $(0 : a_i)$ is \mathfrak{p}_i -primary. So $r(0 : a_i) = \mathfrak{p}_i$ must be the minimal ideal in the set of prime ideals containing $(0 : a_i)$. So $\mathfrak{p}_i \in D(A)$. Conversely let $\mathfrak{p} \in D(A)$ be minimal in the set of prime ideals containing $(0 : a)$. We have a primary decomposition $r(0 : a) = \bigcap_{a \notin \mathfrak{q}_j} \mathfrak{p}_j$ by 4.4. This is not necessarily a minimal primary decomposition, but clearly the \mathfrak{p}_j 's contain the associated prime ideals of $r(0 : a)$. Since $r(0 : a) \subseteq \mathfrak{p}$, by 4.6 \mathfrak{p} contains \mathfrak{p}_j for some j . But $(0 : a) \subseteq \mathfrak{p}_j$, so we must have $\mathfrak{p}_j = \mathfrak{p}$.

10. (i) If $a \in S_{\mathfrak{p}}(0)$, then there is an $s \in S_{\mathfrak{p}} = A \setminus \mathfrak{p}$ such that $as = 0$. Since \mathfrak{p} is prime, $a \in \mathfrak{p}$.

(ii) The argument in (i) shows that any prime ideal of A avoiding $S_{\mathfrak{p}}$ contains $S_{\mathfrak{p}}(0)$ and hence $r(S_{\mathfrak{p}}(0))$, so if $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ then it must be a minimal prime ideal. Conversely, if there is an $a \in \mathfrak{p} \setminus r(S_{\mathfrak{p}}(0))$, then the set $\{sa^n : s \in S_{\mathfrak{p}}, n \geq 0\}$ is multiplicatively closed. Hence \mathfrak{p} properly contains another prime ideal, contradiction.

(iii) Since $A \setminus \mathfrak{p} \subseteq A \setminus \mathfrak{p}'$, this is clear by the argument in (i).

(iv) Suppose for contradiction that there is a nonzero $a \in \bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0)$. So there is a prime ideal $\mathfrak{q} \in D(A)$ that is minimal in the set of prime ideals containing $(0 : a)$. Since $a \in S_{\mathfrak{q}}(0)$, there is an $s \in A \setminus \mathfrak{q}$ such that $as = 0$. So $s \in (0 : a) \subseteq \mathfrak{q}$, contradiction.

11. By Ex. 10(ii) above we have $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$. By 3.11, in $S_{\mathfrak{p}}^{-1}A$, $r(0) = r(S_{\mathfrak{p}}^{-1}S_{\mathfrak{p}}(0)) = S_{\mathfrak{p}}^{-1}r(S_{\mathfrak{p}}(0)) = S_{\mathfrak{p}}^{-1}\mathfrak{p}$. Since $S_{\mathfrak{p}}^{-1}\mathfrak{p}$ is the maximal ideal of $S_{\mathfrak{p}}^{-1}A$, by 4.2 the zero ideal of $S_{\mathfrak{p}}^{-1}A$ is $S_{\mathfrak{p}}^{-1}\mathfrak{p}$ -primary, and of course must be the smallest one. Now by 4.8(ii) we deduce that $S_{\mathfrak{p}}(0)$ is the smallest \mathfrak{p} -primary ideal.

Now if $a \in A$ is not nilpotent then there is a minimal prime ideal \mathfrak{q} that avoids a . Since $S_{\mathfrak{q}}(0) \subseteq \mathfrak{q}$, clearly $a \notin S_{\mathfrak{q}}(0)$. So $a \notin \mathfrak{a}$, i.e. \mathfrak{a} must be contained in the nilradical.

Finally, if the zero ideal is decomposable, then by Ex. 9 $D(A)$ is the set of the associated prime ideals of 0. Let $E \subseteq D(A)$ be the set of the minimal prime ideals in $D(A)$. So actually E is the set of the minimal prime ideals of A . So $\mathfrak{a} = \bigcap_{\mathfrak{p} \in E} S_{\mathfrak{p}}(0)$. Suppose now that $\mathfrak{a} = 0$. By the above result $S_{\mathfrak{p}}(0)$ is \mathfrak{p} -primary for each $\mathfrak{p} \in E$. So we must have $E = D(A)$, i.e. every prime ideal of 0 is isolated. The other direction follows from Ex. 10(iv).

12. (i) $c \in S(\mathfrak{a}) \cap S(\mathfrak{b})$ if and only if $S^{-1}(c) \subseteq S^{-1}\mathfrak{a} \cap S^{-1}\mathfrak{b}$ if and only if $S^{-1}(c) \subseteq S^{-1}(\mathfrak{a} \cap \mathfrak{b})$ if and only if $c \in S(\mathfrak{a} \cap \mathfrak{b})$.

(ii) $c \in S(r(\mathfrak{a}))$ if and only if $S^{-1}(c) \subseteq S^{-1}S(r(\mathfrak{a}))$ if and only if $S^{-1}(c) \subseteq S^{-1}(r(\mathfrak{a}))$ if and only if $S^{-1}(c) \subseteq r(S^{-1}\mathfrak{a})$ if and only if there is an n such that $(c/1)^n \in S^{-1}\mathfrak{a}$ if and only if there is an n such that $c^n \in S(\mathfrak{a})$ if and only if $c \in r(S(\mathfrak{a}))$.

(iii) $S(\mathfrak{a}) = (1)$ if and only if $S^{-1}\mathfrak{a} = S^{-1}A$ if and only if \mathfrak{a} meets S by 3.11(ii).

(iv) Let U_2 be the image of S_2 in $S_1^{-1}A$ and U_1 the image of S_1 in $S_2^{-1}A$. We have: $c \in S_1(S_2(\mathfrak{a}))$ if and only if $S_1^{-1}(c) \subseteq S_1^{-1}(S_2(\mathfrak{a}))$ if and only if $U_2^{-1}(S_1^{-1}(c)) \subseteq U_2^{-1}(S_1^{-1}(S_2(\mathfrak{a})))$ if and only if, by Chapter 3, Ex. 3, $(S_1S_2)^{-1}(c) \subseteq (S_1S_2)^{-1}(S_2(\mathfrak{a}))$. Since $(S_1S_2)^{-1}(S_2(\mathfrak{a})) = U_1^{-1}(S_2^{-1}(S_2(\mathfrak{a}))) = U_1^{-1}(S_2^{-1}\mathfrak{a}) = (S_1S_2)^{-1}\mathfrak{a}$, we conclude that $c \in S_1(S_2(\mathfrak{a}))$ if and only if $(S_1S_2)^{-1}(c) \subseteq (S_1S_2)^{-1}\mathfrak{a}$ if and only if $c \in (S_1S_2)(\mathfrak{a})$.

For the last claim, only observe that by 4.9 $S(\mathfrak{a})$ is the intersection of some primary ideals. For a fixed minimal decomposition of \mathfrak{a} there are only finitely many such intersections.

13. (i) Since $S_{\mathfrak{p}}^{-1}\mathfrak{p}$ is the maximal ideal in $S_{\mathfrak{p}}^{-1}A$ and $r(S_{\mathfrak{p}}^{-1}\mathfrak{p}^n) = S_{\mathfrak{p}}^{-1}r(\mathfrak{p}^n) = S_{\mathfrak{p}}^{-1}\mathfrak{p}$, by 4.2 $S_{\mathfrak{p}}^{-1}\mathfrak{p}^n$ is $S_{\mathfrak{p}}^{-1}\mathfrak{p}$ -primary. By the last claim of 4.8 $\mathfrak{p}^{(n)}$ is \mathfrak{p} -primary.

(ii) Consider the ring A/\mathfrak{p}^n . Clearly $\mathfrak{p}/\mathfrak{p}^n$ is the nilradical of A/\mathfrak{p}^n and hence is the smallest prime ideal of A/\mathfrak{p}^n . By the first claim of Ex. 11 we deduce that $\mathfrak{p}^{(n)}$ is the smallest \mathfrak{p} -primary ideal that contains \mathfrak{p}^n , which must be the \mathfrak{p} -primary component of \mathfrak{p}^n .

(iii) By 1.18 we have $\mathfrak{p}^{m+n} \subseteq \mathfrak{p}^{(m)}\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{(m+n)}$. As in (ii) above $\mathfrak{p}^{(m+n)}$ is the smallest \mathfrak{p} -primary ideal that contains \mathfrak{p}^{m+n} . So $\mathfrak{p}^{(m+n)}$ is the \mathfrak{p} -primary component of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$.

(iv) There is a typo here: Clearly the right-hand side of the biconditional should read “ \mathfrak{p}^n is \mathfrak{p} -primary”. Then the claim follows from (i) and (ii) above.

14. Without loss of generality $\mathfrak{a} = 0$. Suppose that $\mathfrak{p} = (0 : x)$ is maximal in the set, then \mathfrak{p} must be prime, for otherwise there are $a, b \notin \mathfrak{p}$ such that $abx = 0$, but then $\mathfrak{p} \subsetneq (0, ax)$, contradiction. Now trivially \mathfrak{p} is the smallest prime that contains $(0 : x)$, so by Ex. 9 \mathfrak{p} is a prime ideal belonging to \mathfrak{a} .

15. Without loss of generality $\mathfrak{a} = 0$. Clearly $S_f(0) = \bigcup_n (0 : f^n)$. But why does the chain $(0 : f) \subseteq (0 : f^2) \subseteq \dots$ stabilize? Here is the argument: By the choice of f , for all large n we have $f^n \in \mathfrak{q}$ for each primary component \mathfrak{q} of 0 with $r(\mathfrak{q}) \notin \Sigma$; on the other hand $f^n \notin \mathfrak{p}$ for each $\mathfrak{p} \in \Sigma$. So for each primary component \mathfrak{q} of 0, if $r(\mathfrak{q}) \in \Sigma$ then $(\mathfrak{q} : f^n) = \mathfrak{q}$ and if $r(\mathfrak{q}) \notin \Sigma$ then $(\mathfrak{q} : f^n) = A$. So $(0 : f^n) = (\bigcap_{r(\mathfrak{q}) \in \Sigma} \mathfrak{q} \cap \bigcap_{r(\mathfrak{q}) \notin \Sigma} \mathfrak{q} : f^n) = \bigcap_{r(\mathfrak{q}) \in \Sigma} (\mathfrak{q} : f^n) = \mathfrak{q}_{\Sigma} = S_f(0)$.

16. Since every ideal of $S^{-1}A$ is an extended ideal, the claim follows immediately from 4.9.

17. Just follow the hint, noting that, as usual, starting with $\mathfrak{a} = 0$ would simplify the situation. Also, at any transfinite limit stage α we have $\mathfrak{a} = \bigcap_{i < \alpha} \mathfrak{q}_i \cap \bigcup_{i < \alpha} \mathfrak{a}_i$.

18. (i) \Rightarrow (ii): Passing from A to A/\mathfrak{a} we may always assume $\mathfrak{a} = 0$. Let \mathfrak{p} be an isolated prime ideal of 0. Then By Ex. 11 $S_{\mathfrak{p}}(0)$ is the smallest \mathfrak{p} -primary ideal, and hence must be the \mathfrak{p} -primary component of 0. Choose an f as in Ex. 15 (possible since \mathfrak{p} is a minimal prime ideal), consequently $S_{\mathfrak{p}}(0) = S_f(0) = (0 : f^n)$ for all large n . (L2) follows immediately from the last claim of Ex. 12.

(ii) \Rightarrow (i): Those who avert considering transfinite chains (i.e. indexed by arbitrary transfinite numbers) would probably get stuck on this part of the problem even with a truly revealing hint. The point is that we must take the condition (L2) as formulated for all descending chains, including transfinite ones.

Now by Ex. 17 \mathfrak{a} is an intersection of primary ideals, say, $\mathfrak{a} = \bigcap_{i < \alpha} \mathfrak{q}_i$ for some transfinite number α . Notice that the identity in Ex. 12(i) does not generalize to arbitrary intersections. However, for $S_{\alpha} = A \setminus \bigcup_{i < \alpha} \mathfrak{p}_i$ we do have $S_{\alpha}(\mathfrak{a}) = S_{\alpha}(\bigcap_{i < \alpha} \mathfrak{q}_i) \subseteq \bigcap_{i < \alpha} S_{\alpha}(\mathfrak{q}_i) = \bigcap_{i < \alpha} \mathfrak{q}_i$. By (L2) and the hint $S_{\alpha}(\mathfrak{a}) = S_n(\mathfrak{a}) = \bigcap_{i \leq n} \mathfrak{q}_i$ for some integer n and all $\alpha \geq n$. So $\bigcap_{i \leq n} \mathfrak{q}_i \subseteq \bigcap_{i < \alpha} \mathfrak{q}_i \subseteq \bigcap_{i \leq n} \mathfrak{q}_i$, so $\mathfrak{a} = \bigcap_{i \leq n} \mathfrak{q}_i$.

19. The first claim follows from the one-to-one correspondence between the primary ideals in A and $A_{\mathfrak{p}}$ (see 4.8).

Next we show the ending claim of the hint. Since $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are distinct, we only need to show that no \mathfrak{q}_i is redundant. Suppose for contradiction that $\bigcap_{j \neq i} \mathfrak{q}_j \subseteq \mathfrak{q}_i$ for some i . Clearly $i \neq n$ by the choice of \mathfrak{q}_n . But if $i < n$ then, by taking the radicals, we see that $\mathfrak{p}_n \subseteq \mathfrak{p}_i$. Since \mathfrak{p}_n is maximal in the lot, we have a contradiction $\mathfrak{p}_n = \mathfrak{p}_i$.

20. It is straightforward to check the identity. As for the analogues of 1.13, not all formulas there can be reformulated for r_M in a natural way. So different answers are inevitable, which, perhaps, reflect different understandings of what these formulas are really about.

21. For any $a \notin (Q : M)$, if $a + (Q : M) \in A/(Q : M)$ is a zero-divisor then there is a $b + (Q : M) \in A/(Q : M)$ such that $b \notin (Q : M)$ and $ab \in (Q : M)$. Since $b(M/Q) \neq 0$, $\phi_a : M/Q \rightarrow M/Q$ is not injective. So by the assumption there is an n such that $a^n(M/Q) = 0$, i.e. $a + (Q : M)$ is nilpotent in $A/(Q : M)$. So $(Q : M)$ is primary.

Analogue of 4.3: If $Q_i (1 \leq i \leq n)$ are \mathfrak{p} -primary, then $Q = \bigcap_{i=1}^n Q_i$ is \mathfrak{p} -primary. (If $a \in A$ is not nilpotent in M/Q then $a \notin \mathfrak{p}$, since $r_M(Q) = r_M(\bigcap_{i=1}^n Q_i) = r(\bigcap_{i=1}^n Q_i : M) = r(\bigcap_{i=1}^n (Q_i : M)) = \bigcap_{i=1}^n r(Q_i : M) = \mathfrak{p}$. So for every $x \in M$ and each i , $ax \in Q_i$ if and only if $x \in Q_i$. So $ax \in Q$ if and only if $x \in \bigcap_{i=1}^n Q_i = Q$. So a is not a zero-divisor in M/Q .)

The analogue of 4.4 is straightforward.

22. The proof of 4.5 works almost word by word for the analogue of 4.5 in this context of modules. For the second claim only note that, since $N \subseteq Q_i$ for each i , Q_i/N is a primary submodule of M/N and $(Q_i/N : M/N) = (Q_i : M)$.

23. These shall be omitted.