Solutions to the Exercises in Chapter 2, Introduction to Commutative Algebra, M. F. Atiyah and I. G. MacDonald

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To warm up we do some of the exercises in the text (or rather just to get these annoying diagram templates in place at once).

Proposition 2.9. (i) Suppose

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0 \tag{0.1}$$

is exact. Then we define $\bar{v} : \mathrm{Hom}(M'', N) \longrightarrow \mathrm{Hom}(M, N)$ by $g \longmapsto g \circ v$. Since v is surjective, it is clear that $\bar{v}$ is injective. The whole situation can be summarized in the following diagram

$$\begin{tikzcd}
M \arrow[d, "v"'] \arrow[dr, "h"] & M' \arrow[l, "u"'] \arrow[d, "f"] \\
M'' \arrow[r, "g"'] \arrow[d] & N \\
0 \arrow[ur] &
\end{tikzcd} \tag{0.2}$$

Now since $v \circ u = 0$, we have $g \circ v \circ u = 0$ for all $g \in \mathrm{Hom}(M'', N)$. On the other hand if $h \circ u = 0$ for $h \in \mathrm{Hom}(M, N)$ then $\mathrm{im}(u) \subseteq \ker(h)$, so $\ker(v) \subseteq \ker(h)$, so there is an induced map $g = (h/\ker(v)) : M'' \cong (M/\ker(v)) \longrightarrow N$ such that $g \circ v = h$. So $\mathrm{im}(\bar{v}) = \ker(\bar{u})$. So

$$0 \longrightarrow \mathrm{Hom}(M'', N) \xrightarrow{\bar{v}} \mathrm{Hom}(M, N) \xrightarrow{\bar{u}} \mathrm{Hom}(M', N) \tag{0.3}$$

is exact.

Conversely assume 0.3 is exact for every A-module N. First of all take $N = \operatorname{coker}(v)$. Let g be the projection map. So if v is not surjective then $g \neq 0$ but $g \circ v = 0$, contradiction. Next take $N = M''$. Let $g = \mathrm{id}$. So $g \circ v \circ u = 0$, so $\operatorname{im}(u) \subseteq \ker(v)$. Lastly take $N = M/\operatorname{im}(u)$. Let h be the projection map. So there is a $g \in \operatorname{Hom}(M'', N)$ such that $g \circ v = h$, so $\ker(v) \subseteq \ker(g \circ v) = \ker(h) = \operatorname{im}(u)$. So $\operatorname{im}(u) = \ker(v)$.
(ii) We erect a similar diagram

$$\begin{tikzcd}
N \arrow[r, "v"] & N'' \\
N' \arrow[u, "u"] & M \arrow[l, "g"] \arrow[u, "f"'] \arrow[ul, "h"] \arrow[dl] \\
0 \arrow[u] &
\end{tikzcd}$$

This is dual to 0.2 and the argument is completely analogous.

Proposition 2.10. We illustrate how the boundary homomorphism d is constructed. First we have a naturally induced diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(f') \arrow[r, "\bar{u}"] \arrow[d, hook] & \ker(f) \arrow[r, "\bar{v}"] \arrow[d, hook] & \ker(f'') \arrow[d, hook] & \\
& M' \arrow[r, "u"] \arrow[d, "f'"'] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[d, "f''"] & \\
& N' \arrow[r, "u'"'] \arrow[d] & N \arrow[r, "v'"'] \arrow[d] & N'' \arrow[d] & \\
& \operatorname{coker}(f') \arrow[r, "\overline{u'}"'] & \operatorname{coker}(f) \arrow[r, "\overline{v'}"'] & \operatorname{coker}(f'') \arrow[r] & 0
\end{tikzcd}$$

Then in order to move from $\ker(f'')$ to $\operatorname{coker}(f')$ we construct the following diagram

$$\begin{tikzcd}
 & \ker(f) \arrow[r, hook] & v^{-1}(\ker(f'')) \arrow[r, "v"] \arrow[dl, "f"'] \arrow[d, hook] & \ker(f'') \arrow[d, hook] \\
 & \operatorname{im}(u') \arrow[dd] & M \arrow[r, "v"'] \arrow[dd] & M'' \arrow[ddl, "\cong"] \\
 & & & \\
\operatorname{coker}(f') \arrow[r, "\cong"] & \operatorname{im}(u')/u' \circ f'(M') \arrow[d, hook] & M/\operatorname{im}(u) \arrow[d] & \\
\operatorname{coker}(f) & N/u' \circ f'(M') \arrow[l] & M/f^{-1}(u' \circ f'(M')) \arrow[l, "f^*"'] &
\end{tikzcd} \tag{0.4}$$

where f^* is defined by $m + f^{-1}(u' \circ f'(M')) \longmapsto f(m) + u' \circ f'(M')$. One may chase the diagram to check that the images of $\ker(f'')$ and $\operatorname{coker}(f')$ in $N/u' \circ f'(M')$ induce the desired boundary homomorphism d. That $\operatorname{im}(\bar{v}) = \ker(d)$ is essentially because $\ker(f) + \operatorname{im}(u) = f^{-1}(u' \circ f'(M'))$. That $\operatorname{im}(d) = \ker(\overline{u'})$ is essentially because $f^{-1}(\operatorname{im}(u')) = v^{-1}(\ker(f''))$. Contrary to what the book claims, one needs to do some mental gymnastics to get the picture here. For example, since $u(\ker(f')) \subseteq \ker(f)$ and u is injective, one may think of f as a coarse extension of f' to M. The map f need not be strictly coarser: if $u(\ker(f')) = \ker(f)$ then d becomes injective. On the other hand if $\operatorname{im}(u') = f(v^{-1}(\ker(f'')))$ then d becomes surjective.

Note that if we cut off the first and the fourth 0's in the assumed diagram then 2.10 still holds, with the head and the tail cut off as well. That is, the sequence

$$\ker(f') \xrightarrow{\bar{u}} \ker(f) \xrightarrow{\bar{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\bar{u}'} \operatorname{coker}(f) \xrightarrow{\bar{v}'} \operatorname{coker}(f'')$$

is exact. This version is used more often.

1. Since $am + bn = 1$ for some $a, b \in \mathbb{Z}$, for any $i \in \mathbb{Z}/m\mathbb{Z}$, $j \in \mathbb{Z}/n\mathbb{Z}$ we have $i \otimes j = bni \otimes j = i \otimes bnj = 0$.

2. Follow the hint and use the canonical isomorphism in 2.14.

3. Let $\mathfrak{m}$ be the maximal ideal. Let $K = A/\mathfrak{m}$ be the residue field. $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are finitely generated K-modules, i.e. finite-dimensional vector spaces over K. We have

$$\mathrm{Hom}(M/\mathfrak{m}M \otimes_K N/\mathfrak{m}N, N/\mathfrak{m}N) \cong \mathrm{Hom}(M/\mathfrak{m}M, \mathrm{Hom}(N/\mathfrak{m}N, N/\mathfrak{m}N)).$$

If $M/\mathfrak{m}M, N/\mathfrak{m}N \neq 0$ then the righthand side is not 0. But by 2.14, 2.15, and 2 above $M/\mathfrak{m}M \otimes_K N/\mathfrak{m}N \cong (K \otimes_A M) \otimes_K (K \otimes_A N) \cong K \otimes_A (M \otimes_A N) = 0$, hence the lefthand side is 0, contradiction. So one of $M/\mathfrak{m}M, N/\mathfrak{m}N$ is 0. By Nakayama's lemma one of M, N is 0.

4. We claim that $N \otimes \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} N \otimes M_i$ for any A-module N. Using this and 2.19 the result is readily derived. First define a bilinear map $f : N \times \bigoplus_{i \in I} M_i \longrightarrow \bigoplus_{i \in I} N \otimes M_i$ by $(b, \ldots, a_i, \ldots) \longmapsto (\ldots, b \otimes a_i, \ldots)$. For any A-module P and any bilinear map $h : N \times \bigoplus_{i \in I} M_i \longrightarrow P$ there is a restricted bilinear map $h_i : N \times M_i \longrightarrow P$ for each $i \in I$. Hence each h_i factors through $N \otimes M_i$, say, $h_i = f_i \circ g_i$, where $f_i : N \times M_i \longrightarrow N \otimes M_i$ is the canonical map. Define an A-linear map $g : \bigoplus_{i \in I} N \otimes M_i \longrightarrow P$ by $(\ldots, b \otimes a_i, \ldots) \longmapsto \sum_{i \in I} g_i(b \otimes a_i)$. It is easily checked that $f \circ g = h$. By the defining property of a tensor product we establish the desired isomorphism.

5. Notice $A[x] = \bigoplus_{i \in \mathbb{N}} (x^i)_A$ as an A-module and $M \otimes (x^i)_A \cong M \otimes A \cong M$ for any A-module M and any i. Now use 4.

6. That $M[x]$ is an $A[x]$-module is obvious. Define an A-bilinear map $f : A[x] \times M \longrightarrow M[x]$ in the obvious way. For any A-module P and any bilinear map $h : A[x] \times M \longrightarrow P$, set $mx^i \longmapsto h(x^i, m)$. This induces an A-linear map $g : M[x] \longrightarrow P$ such that $f \circ g = h$. So $M[x] \cong A[x] \otimes_A M$.

7. Do induction on the degree of $f(x) \in \mathfrak{p}[x]$. Suppose $f(x) = (g(x) + ax^n)(h(x) + bx^m)$. Since $ab \in \mathfrak{p}$ we have $a, b \in \mathfrak{p}$. So $g(x)bx^m, ax^n h(x) \in \mathfrak{p}[x]$, so $g(x)h(x) \in \mathfrak{p}[x]$. But the degree of $g(x)h(x)$ is less than $n + m$, so by the inductive hypothesis $g(x), h(x) \in \mathfrak{p}[x]$. Now suppose $\mathfrak{m}$ is a proper maximal ideal. Then $\mathfrak{m}[x] \subsetneq \mathfrak{m} + xA[x]$, which is an ideal.

8. (i) If $f : P \longrightarrow Q$ is injective, then $f \otimes 1 : P \otimes M \longrightarrow Q \otimes M$ is injective, so $(f \otimes 1) \otimes 1 : (P \otimes M) \otimes N \longrightarrow (Q \otimes M) \otimes N$ is injective, i.e. $f \otimes 1 : P \otimes (M \otimes N) \longrightarrow Q \otimes (M \otimes N)$ is injective.

(ii) The argument is almost identical to the last one, only now we use 2.14(iv), 2.15, and the fact that $M \otimes_A B$ carries a B-module structure.

9. Clear.

10. Let $\bar{u} : M/\mathfrak{a}M \longrightarrow N/\mathfrak{a}N$. Consider the diagram

$$\begin{tikzcd}
M \arrow[r, "u"] \arrow[d, "f"'] & \mathrm{im}(u) \arrow[r] \arrow[d, "g"] & 0 \\
M/\mathfrak{a}M \arrow[r, "\bar{u}"'] & N/\mathfrak{a}N \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and f, g are the corresponding quotient maps. Since $\bar{u}$ is surjective, the sequence

$$\begin{tikzcd}
\mathrm{coker}(f) \arrow[r] & \mathrm{coker}(g) \arrow[r] & 0
\end{tikzcd}$$

is exact. But $\mathrm{coker}(f) = 0$, so $\mathrm{im}(u) + \mathfrak{a}N = N$. By 2.7 $\mathrm{im}(u) = N$, i.e. u is surjective.

11. The hint reveals it all again. 2 is needed here, of course. Using the same technique we immediately see that if $\phi : A^m \longrightarrow A^n$ is surjective, then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \longrightarrow (A/\mathfrak{m}) \otimes A^n$ is surjective, hence $m \geq n$. However, if $\phi : A^m \longrightarrow A^n$ is injective, then there is no guarantee that $1 \otimes \phi$ is injective. The whole point of defining a flat module is to single out those modules that do so (Cf. 2.19). So if for some maximal ideal $\mathfrak{m}$ $A/\mathfrak{m}$ is flat then $1 \otimes \phi$ is injective and hence $m \leq n$.

However this last bit does hold in general for all commutative rings. The argument is similar to the proof of 2.4. Suppose for contradiction that $\phi : A^m \longrightarrow A^n$ is injective and $m > n > 0$. In fact without loss of generality we may assume that $m = n + 1$ and A^n is a submodule of A^m. Let $\{e_i : 1 \leq i \leq m\}$ be the canonical free basis for A^m. Then for each e_i we have $\phi(e_i) = \sum_{j=1}^{n} a_{ij} e_j$, where $a_{ij} \in A$, i.e.,

$$\sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij}) e_j = 0 \text{ and } \phi(e_m) - \sum_{j=1}^{n} a_{mj} e_j = 0,$$

where δ_{ij} is the Kronecker delta. From this we may obtain an $m \times m$ matrix

$$P = \begin{pmatrix} Q & 0 \\ 0 & \phi \end{pmatrix},$$

where Q is a suitable $n \times n$ upper triangular matrix, such that $P(e_1, \ldots, e_m)^T = 0$. So in particular $\phi(e_m) = 0$, which is a contradiction as ϕ is assumed to be injective.

12. Let's look at this from a slightly different angle. By 2.3 $M \cong A^n/B$ for some n and some submodule B of A^n, hence we may have $\phi : A^n/B \longrightarrow A^m$. So $\ker(\phi) = C/B$ for some submodule $C \subseteq A^n$, so $A^n/C \cong A^m$. By 11 we know $n \geq m$, so $A^n/C \cong A^n/A^{n-m}$, so $C \cong A^{n-m}$ (maybe this is not that obvious). So C/B is finitely generated.

13. Since $1 \otimes N \subseteq A \otimes N \cong N$ via the canonical map, clearly g is injective. Define $p : B \otimes_A N \longrightarrow N$ by $(b, y) \longmapsto by$. So $p \circ g = \mathrm{id}$. In this situation we say that the short exact sequence $0 \longrightarrow \ker(p) \longrightarrow N_B \longrightarrow N \longrightarrow 0$ splits. It is a standard result then that $N_B \cong \ker(p) \oplus N \cong \mathrm{im}(g) \oplus \ker(p)$. Explicitly define $\pi : \ker(p) \oplus N \longrightarrow N_B$ by setting $(d, n) \longmapsto d + g(n)$.

14. There is no problem here. Advertisement (since I am a logician): Direct systems are investigated on a more general footing in model theory, where one sees that the reason that direct systems exist is logical rather than algebraic. See any classic textbook on model theory for a taste of this, e.g. [1].

15. Clear.

16. Think of each $\alpha_i : M_i \longrightarrow N$ as an A-module. This makes sense because each α_i is linear and everything commutes. Now taking the direct limit of this system we obtain a uniquely determined direct limit of the form $\alpha : M \longrightarrow N$ where α is linear.

17. Use 16.

18. Use 16 again: the homomorphisms ϕ_i's can be viewed as objects, and they form a direct system, and arrows among them are the corresponding pairs of homomorphisms. (Life is easy as long as as things commute!)

19. We use the trick in 18 again. Consider the sequence $M \longrightarrow N \longrightarrow P$ as one object and by 15 the elements in it all take the form $\mu_i(x_i) \longmapsto \nu_i(y_i) \longmapsto \rho_i(z_i)$ for some $i \in I$, hence the exactness is preserved.

20. Clearly $\varinjlim(M_i \times N) \cong M \times N$. Using the canonical maps we construct

a diagram:

$$\begin{tikzcd}
 & \varinjlim(M_i \times N) \arrow[dl, "f"'] \arrow[d, "g"] & \\
\varinjlim(M_i \otimes N) \arrow[r, shift left, "\psi"] & M \otimes N \arrow[l, shift left, "\phi"] & M_i \times N \arrow[l, "h_i"] \arrow[ul, "\nu_i"'] \arrow[dl, "g_i"] \\
 & M_i \otimes N \arrow[ul, "\mu_i^*"] \arrow[u, "\mu_i \otimes 1"] &
\end{tikzcd}$$

for every $i \in I$. Now $\phi \circ \psi \circ \mu_i^* \circ g_i = \phi \circ \mu_i \otimes 1 \circ g_i = \phi \circ h_i = \phi \circ g \circ \nu_i = f \circ \nu_i = \mu_i^* \circ g_i$. This is enough to establish $\phi \circ \psi = \mathrm{id}$ as all the generators of $\varinjlim(M_i \otimes N)$ are fixed. The other direction is similar.

21. As we have remarked in 14, from the logical point of view, this type of construction preserves whatever first-order structure one starts with. This is essentially due to the compactness of first-order logic. Now if $A = 0$ and no A_i is trivial, then the 1's in the A_i's will be mapped all the way into A via α_i's and is not 0 in A, contradiction.

22. Observe that each $\mathfrak{R}_i$ is a subgroup of the additive group A_i and hence $\varprojlim \mathfrak{R}_i$ is a subgroup of the additive group $\varprojlim A_i$, which must be in the nilradical of the ring $\varprojlim A_i$. For the other direction use 15.

23. See 21 if you are not sure this is well-defined.

24. We review the definitions and the basic properties of both the Ext and the Tor functors (assuming a minimal amount of knowledge of category theory), which are (loosely) dual notions and are to the heart of homological algebra. These are copied from [2]. This review will be long. But the point is of course that it is not that long as in [2] where you need to work through half of the book to get to the defnitions. Presumably [3] is a better modernized source. But I do not have it handy. Atiyah and MacDonald's book itself was published in 1969 anyway!

To simplify the diagrams we use $\rightarrowtail$ for injective maps (monomorphisms) and $\twoheadrightarrow$ for surjective maps (epimorphisms). Hence we truncate the 0's and say "$M \rightarrowtail N \twoheadrightarrow P$ is exact."

Fix a commutative ring A and let $\mathfrak{A}$ be the category of A-modules. All the definitions will happen in this category, even though more generally they can be formulated for noncommutative rings.

1 Projective and Injective Modules

We say a module P is *projective* if for any given diagram of the following shape

$$\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dl, dotted, "h"'] \\
N \arrow[r, two heads, "g"'] & M
\end{tikzcd} \tag{1.1}$$

there is a homomorphism h which makes the diagram commute. Similarly the dual notion of an *injective* module I is such that for any diagram of the shape

$$\begin{tikzcd}
 & I \\
N \arrow[ur, dotted, "h"] & M \arrow[l, tail, "g"] \arrow[u, "f"']
\end{tikzcd}$$

there is a homomorphism h which makes the diagram commute.

Fact 1.1. *The following are equivalent:*

(1) P is projective;

(2) for every short exact sequence $M \rightarrowtail N \twoheadrightarrow Q$ the induced sequence $\mathrm{Hom}(P, M) \rightarrowtail \mathrm{Hom}(P, N) \twoheadrightarrow \mathrm{Hom}(P, Q)$ is exact;

(3) every short exact sequence $0 \longrightarrow M \longrightarrow N \xrightarrow{f} P \longrightarrow 0$ splits, i.e. there is a $g : P \longrightarrow N$ such that $f \circ g = \mathrm{id}$;

(4) P is a direct summand in every module of which it is a quotient;

(5) P is a direct summand in a free module.

Fact 1.2. *(1) Every free module is projective.*

(2) Every module is a quotient of a free module.

(3) A direct sum $\bigoplus_{i\in I} P_i$ *is projective if and only if each* P_i *is.*

(4) If A *is a principal ideal domain then every submodule of a projective module is projective.*

No need to panic here if you do not know them. I have listed these properties of a projective module only to convince you that this is a natural concept (all that one needs to remember is the defining diagram 1.1 of a projective module and, occasionally, every projective module is a direct summand in a free module) and more importantly we will have plenty of projective modules to work with. This last point is essential for homological algebra for we will need the following kind of creature all the time, an ample supply of which is guaranteed by the above facts:

Definition 1.3. A short exact sequence $M \rightarrowtail P \twoheadrightarrow N$ with P projective is called a *projective presentation* of N.

Using the notion of an injective module one can dualize much of what has been said above about projective modules, though it should be noted that "dualization" here is a rather vague and hence only heuristic procedure. For the record we shall list these dualizations below, even though they will not be used in the sequel.

First recall that an A-module M is *divisible* if for every $a \in A$ and every $m \in M$ there is an $n \in M$ such that $an = m$. Note that in general n is not required to be unique.

Fact 1.4. *If* A *is a principal ideal domain then* M *is injective if and only if it is divisible.*

The proof of this fact uses an argument similar to what is known in model theory as the back-and-forth method, only here the traffic is one-way. From the logical point of view the first-order theory of divisible ordered abelian groups (divisible as $\mathbb{Z}$-modules) is well-understood. In particular it admits quantifier elimination.

Definition 1.5. M is a *cofree* module if it is the direct product of modules $A^* = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

It hardly needs to be pointed out that this definition makes sense because for any A-module M and any abelian group G the set of $\mathbb{Z}$-module homomorphisms (i.e. group homomorphisms) $\mathrm{Hom}_{\mathbb{Z}}(M, G)$ has a natural A-module structure defined by:

$$(a\phi)(m) = \phi(am) \text{ for all } a \in A, m \in M, \phi \in \mathrm{Hom}_{\mathbb{Z}}(M, G).$$

By an argument similar to the one preceding 2.18 we have a canonical $\mathbb{Z}$-module isomorphism (i.e. canonical group isomorphism)

$$\pi_M : \mathrm{Hom}_{\mathbb{Z}}(M \otimes_A A, G) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G)),$$

i.e.

$$\pi_M : \mathrm{Hom}_{\mathbb{Z}}(M, G) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G)).$$

Category-theoretically both $\mathrm{Hom}_{\mathbb{Z}}(-, G)$ and $\mathrm{Hom}_A(-, \mathrm{Hom}_{\mathbb{Z}}(A, G))$ are functors from the category $\mathfrak{A}$ of A-modules to the category $\mathfrak{Z}$ of $\mathbb{Z}$-modules. Moreover these two functors are *naturally equivalent* in the sense that for any arrow $f : M \longrightarrow N$ in $\mathfrak{A}$ the diagram (in $\mathfrak{Z}$)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(M, G) \arrow[r, "\pi_M"] \arrow[d, "f'"'] & \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, G)) \arrow[d, "f''"] \\
\mathrm{Hom}_{\mathbb{Z}}(N, G) \arrow[r, "\pi_N"'] & \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbb{Z}}(A, G))
\end{tikzcd} \tag{1.2}$$

commutes, where f' and f'' are the corresponding arrows in $\mathfrak{Z}$ of f under the two functors. Now if G is divisible then using the natural equivalence we may conclude that $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ is an injective A-module.

Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is divisible as a $\mathbb{Z}$-module, $\mathbb{Q}/\mathbb{Z}$ is injective. This implies that for any nontrivial abelian group G there is a nontrivial homomorphism from G into $\mathbb{Q}/\mathbb{Z}$. Hence if M is nontrivial then $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is nontrivial, consequently $\mathrm{Hom}_A(M, A^*)$ is nontrivial.

Fact 1.6. *The following are equivalent:*

(1) I is injective;

(2) for every short exact sequence $M \rightarrowtail N \twoheadrightarrow Q$ the induced sequence $\mathrm{Hom}(Q, I) \rightarrowtail \mathrm{Hom}(N, I) \twoheadrightarrow \mathrm{Hom}(M, I)$ is exact;

(3) every short exact sequence $0 \longrightarrow I \xrightarrow{f} N \longrightarrow M \longrightarrow 0$ cosplits, i.e. there is a $g : N \longrightarrow I$ such that $g \circ f = \mathrm{id}$;

(4) I is a direct factor in every module which contains it as a submodule;

(5) I is a direct factor in a cofree module.

The notion of a *direct factor* is defined in the obvious way. Of course in the present context the use of this notion is redundant as direct product and direct sum of modules are the same thing. This phenomenon also occurs in a more general situation, namely, in the so-called *abelian categories*, where finite product and finite coproduct are “self-dual”. In fact this is the proper context in which the story of homological algebra should be told.

Fact 1.7. *(1) Every cofree module is injective.*

(2) Every module is a submodule of a cofree module.

(3) A direct product $\prod_{j \in J} I_j$ *is injective if and only if each* I_j *is.*

(4) If A *is a principal ideal domain then every quotient of an injective module is injective.*

Note that Fact 1.7 (4) follows from Fact 1.4 and an easy observation that every quotient of a divisible module is divisible. On the other hand Fact 1.2 (4) is actually a corollary of the fact that over a principal ideal domain every submodule of a free module is free. However, the dual form of this fact is false: consider $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$, if we let H be the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all the elements whose order is a power of 2 then the quotient $\mathbb{Q}/\mathbb{Z}/H$ is clearly not a direct product of $\mathbb{Q}/\mathbb{Z}$.

Definition 1.8. A short exact sequence $M \rightarrowtail I \twoheadrightarrow N$ with I injective is called an *injective presentation* of M.

2 The Primitive Version of Ext

We still have a long way to go before we can define the Ext and the Tor functors. But we are ready to discuss the primitive versions of them with respect to short exact sequences.

Let $N \overset{g}{\rightarrowtail} P \overset{f}{\twoheadrightarrow} M$ be a projective presentation of M. For any module T, by 2.9, the sequence

$$\mathrm{Hom}(M, T) \overset{\bar{f}}{\rightarrowtail} \mathrm{Hom}(P, T) \xrightarrow{\bar{g}} \mathrm{Hom}(N, T)$$

is exact. Since $\bar{g}$ is not guaranteed to be surjective, $\mathrm{coker}(\bar{g})$ can be used to measure to what extent $\bar{g}$ fails to be surjective. Define

$$\mathrm{Ext}^f(M, T) = \mathrm{coker}(\bar{g}),$$

where the superscript f is a part of the operator because it is defined via a particular projective presentation of M. For any $\alpha, \beta \in \mathrm{Hom}(N, T)$ clearly $\alpha + \mathrm{im}(\bar{g}) = \beta + \mathrm{im}(\bar{g})$ if and only if $\alpha - \beta$ extends to P. It is not hard to see that $\mathrm{Ext}^f(M,-)$ is a functor from $\mathfrak{A}$ to $\mathfrak{A}$.[1]

Now we reveal the trade secret as to why, instead of arbitrary exact sequences, only projective presentations are considered. In a nutshell, since it is really desirable to make the operator $\mathrm{Ext}(-,-)$ into a bifunctor from $\mathfrak{A}$ to $\mathfrak{A}$ (contravariant in the first and covariant in the second), it should better be independent of the choice of projective presentations. This can be made precise by once again invoking the category-theoretical concept of a natural equivalence as expressed in the diagram 1.2.

First consider two projective presentations

$$N \xrightarrow{g} P \xrightarrow{f} M \text{ and } N' \xrightarrow{g'} P' \xrightarrow{f'} M'.$$

Given $h : M \longrightarrow M'$, an arrow in $\mathfrak{A}$, we can construct a commutative diagram

$$\begin{tikzcd}
N \arrow[r, tail, "g"] \arrow[d, "\bar{\bar{h}}"'] & P \arrow[r, two heads, "f"] \arrow[d, "\bar{h}"] & M \arrow[d, "h"] \\
N' \arrow[r, tail, "g'"'] & P' \arrow[r, two heads, "f'"'] & M'
\end{tikzcd} \tag{2.1}$$

where $\bar{h}$ is induced by P being projective and the composition $h \circ f$, and $\bar{\bar{h}}$ is in turn induced by $\bar{h}$. Now given any modules T_1, T_2 and an arrow $t : T_1 \longrightarrow T_2$ clearly the above diagram induces the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^{f'}(M', T_1) \arrow[r, "\bar{h}_1^*"] \arrow[d, "t_{f'}"'] & \mathrm{Ext}^f(M, T_1) \arrow[d, "t_f"] \\
\mathrm{Ext}^{f'}(M', T_2) \arrow[r, "\bar{h}_2^*"'] & \mathrm{Ext}^f(M, T_2)
\end{tikzcd} \tag{2.2}$$

where notice that the morphisms $\bar{h}_1^*$, $\bar{h}_2^*$ go in the opposite direction to h. Since $\bar{h}_1^*$, $\bar{h}_2^*$ are in general not isomorphisms, the transformation from the

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[1] Almost every abelian group that arises in the present note carries an A-module structure, this is why we have restricted $\mathrm{Ext}^f(M,-)$ to be a functor from $\mathfrak{A}$ to $\mathfrak{A}$. In the sequel other functors will be treated in this way as well. As mentioned above, the standard context for a discussion of homological algebra is abelian category. In this broader context all the functors in question need to take values in the category $\mathfrak{Ab}$ of abelian groups.

functor $\mathrm{Ext}^{f'}(M', -)$ to the functor $\mathrm{Ext}^{f}(M, -)$ induced by the choice of $\bar{h}$ is in general not a natural equivalence, but only a *natural*, well, *transformation.* Obviously here we are concerned with the case $M = M'$ and $h = \mathrm{id}$, in which if the transformation between the functors is a natural equivalence then $\mathrm{Ext}(-, -)$ can be made into a bifunctor by choosing an arbitrary projective presentation for each module M.

To that end, suppose in the diagram 2.1 we choose another morphism $\bar{\eta} : P \longrightarrow P'$ and its induced morphism $\bar{\bar{\eta}} : N \longrightarrow N'$. For any $\alpha + \mathrm{im}(\bar{g'}) \in \mathrm{Ext}^{f'}(M', T)$ we have $\bar{h}^*(\alpha + \mathrm{im}(\bar{g'})) = \alpha \circ \bar{\bar{h}} + \mathrm{im}(\bar{g})$ and $\bar{\eta}^*(\alpha + \mathrm{im}(\bar{g'})) = \alpha \circ \bar{\bar{\eta}} + \mathrm{im}(\bar{g})$. The situation is illustrated as follows:

$$\begin{tikzcd}
N \arrow[r, tail, "g"] \arrow[d, shift right, "\bar{\bar{\eta}}"'] \arrow[d, shift left, "\bar{\bar{h}}"] & P \arrow[r, two heads, "f"] \arrow[d, shift right, "\bar{\eta}"'] \arrow[d, shift left, "\bar{h}"] & M \arrow[d, "h"] \\
N' \arrow[r, tail, "g'"'] \arrow[d, "\alpha"'] & P' \arrow[r, two heads, "f'"'] & M' \\
T & &
\end{tikzcd}$$

Since $f' \circ \bar{h} - f' \circ \bar{\eta} = h \circ f - h \circ f = 0$, we deduce $\mathrm{im}(\bar{h} - \bar{\eta}) \subseteq \ker(f') = \mathrm{im}(g')$. This means that the morphism $\bar{\alpha} : \mathrm{im}(g') \longrightarrow T$ induced by α and g' is such that $\bar{\alpha} \circ (\bar{h} - \bar{\eta})$ is the extension of $\alpha \circ (\bar{\bar{h}} - \bar{\bar{\eta}})$ to P. Hence $\alpha \circ \bar{\bar{h}} + \mathrm{im}(\bar{g}) = \alpha \circ \bar{\bar{\eta}} + \mathrm{im}(\bar{g})$, i.e. $\bar{h}^* = \bar{\eta}^*$. So we really should write $h^* : \mathrm{Ext}^{f'}(M', T) \longrightarrow \mathrm{Ext}^{f}(M, T)$ for both $\bar{h}^*$ and $\bar{\eta}^*$ as this map only depends on the choice of h. Now it is easy to see that if $M = M'$ and $h = \mathrm{id}$ then $(\mathrm{id} \circ \mathrm{id})^* = \mathrm{id}^* \circ \mathrm{id}^*$ must be the identity map from $\mathrm{Ext}^{f}(M, T)$ to itself. This shows that id^* is an isomorphism. Hence we have proved that the functor $\mathrm{Ext}^{f}(M, -)$ is unique up to natural equivalence. So from now on we may drop the superscript f and simply write $\mathrm{Ext}(M, T)$ for each module T.

3 The Primitive Version of Tor

The definition of the functor $\mathrm{Tor}^{f}(M, -)$ for a given projective presentation $N \overset{g}{\rightarrowtail} P \overset{f}{\twoheadrightarrow} M$ is similar, only the $\mathrm{Hom}(-, -)$ operation is replaced by the tensor operation $\otimes$. For any module T, by 2.18, we have

$$\begin{tikzcd} N \otimes T \arrow[r, "g\otimes 1_T"] & P \otimes T \arrow[r, two heads, "f\otimes 1_T"] & M \otimes T \end{tikzcd}$$

where $1_T = \mathrm{id}_T$. Since $g \otimes 1_T$ is not guaranteed to be injective, $\ker(g \otimes 1_T)$ can be used to measure to what extent $g \otimes 1_T$ fails to be injective. Define

$$\mathrm{Tor}^f(M, T) = \ker(g \otimes 1_T).$$

As above we want to show that the functor $\mathrm{Tor}^f(M, -)$ really does not depend on which projective presentation of M is used. For this we need a lemma, due to J. Lambek. But before we state the lemma we shall first introduce some terminology.

Definition 3.1. Let Σ be a commutative square of modules:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Sigma"] & N \arrow[d, "h"] \\ M' \arrow[r, "k"'] & N' \end{tikzcd}$$

We define the kernel $\ker(\Sigma)$ to be $\ker(h \circ f)/(ker(f) + ker(g))$, the image $\mathrm{im}(\Sigma)$ to be $(\mathrm{im}(k) \cap \mathrm{im}(h))/\mathrm{im}(h \circ f)$.

Lemma 3.2. *Let*

$$\begin{tikzcd} N \arrow[r, tail, "g"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Sigma_1"] & P \arrow[r, two heads, "f"] \arrow[d, "\beta"] \arrow[dr, phantom, "\Sigma_2"] & M \arrow[d, "\alpha"] \\ N' \arrow[r, tail, "g'"'] & P' \arrow[r, two heads, "f'"'] & M' \end{tikzcd}$$

be a commutative diagram with the rows exact. Then β induces an isomorphism $\bar{\beta} : \ker(\Sigma_2) \longrightarrow \mathrm{im}(\Sigma_1)$.

The proof is just diagram chasing, much less complicated than the diagram 0.4. Now let $\Lambda : N \overset{g}{\rightarrowtail} P \overset{f}{\twoheadrightarrow} M$ be a projective presentation of M and suppose V is a projective module. Since V is a direct summand of a free module U, say $U = V \oplus V'$, and $N \otimes U \xrightarrow{g\otimes 1_U} P \otimes U$ is of the form $N \otimes A^{(I)} \xrightarrow{g\otimes 1_U} P \otimes A^{(I)}$, by 2.14 clearly $g \otimes 1_U$ is injective and hence $N \otimes V \xrightarrow{g\otimes 1_V} P \otimes V$ is injective. So $\mathrm{Tor}^f(M, V) = 0$. Now let $\Xi : W \overset{k}{\rightarrowtail} V \overset{h}{\twoheadrightarrow} T$ be a projective presentation of T. We define $\Lambda \otimes \Xi$ to be the diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_5"] & \mathrm{Tor}^h(T,M) \arrow[d, tail] \\
 & N\otimes W \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_3"] & P\otimes W \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\Sigma_4"] & M\otimes W \arrow[d] \\
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Sigma_1"] & N\otimes V \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\Sigma_2"] & P\otimes V \arrow[r, two heads] \arrow[d, two heads] & M\otimes V \arrow[d, two heads] \\
\mathrm{Tor}^f(M,T) \arrow[r, tail] & N\otimes T \arrow[r] & P\otimes T \arrow[r, two heads] & M\otimes T
\end{tikzcd} \tag{3.1}$$

where all arrows are canonical. Applying the above lemma repeatedly we get $\mathrm{Tor}^f(M,T) \cong \mathrm{im}(\Sigma_1) \cong \ker(\Sigma_2) \cong \mathrm{im}(\Sigma_3) \cong \ker(\Sigma_4) \cong \mathrm{im}(\Sigma_5) \cong \mathrm{Tor}^h(T,M)$. Also, by 2.10, there is a boundary map d such that the sequence

$$\mathrm{Tor}^h(T,N) \longrightarrow \mathrm{Tor}^h(T,P) \longrightarrow \mathrm{Tor}^h(T,M)$$
$$\xrightarrow{d} N\otimes T \longrightarrow P\otimes T \longrightarrow M\otimes T \longrightarrow 0$$

is exact, where of course $\mathrm{Tor}^h(T,P) = 0$.

That the two wing tips in the diagram 3.1 are isomorphic is the key to showing that the functor $\mathrm{Tor}^f(M, -)$ is unique up to natural equivalence. Let $\Lambda' : N' \overset{g'}{\rightarrowtail} P' \overset{f'}{\twoheadrightarrow} M'$ be a projective presentation of M'. Given $l : M \longrightarrow M'$, we can again choose two morphisms $\bar{l}, \bar{\bar{l}}$ to construct the diagram 2.1. Examining the two tensor products $\Lambda' \otimes \Xi$ and $\Lambda \otimes \Xi$ one immediately sees

$$\mathrm{Tor}^f(M,T) \cong \mathrm{Tor}^h(T,M) \xrightarrow{l^*} \mathrm{Tor}^h(T,M') \cong \mathrm{Tor}^{f'}(M',T),$$

which depends on every bit of information but the choice of $\bar{l}, \bar{\bar{l}}$. Therefore, if $l = 1_M$ then there is a canonical isomorphism that witnesses the natural equivalence of the two functors $\mathrm{Tor}^f(M, -)$ and $\mathrm{Tor}^{f'}(M, -)$. Hence we are justified to drop the superscript and simply write $\mathrm{Tor}(M, -)$ for each module M.

4 Homology

The functors $\mathrm{Ext}^n(-, M)$ and $\mathrm{Tor}_n(M, -)$ are "derived" from the functors $\mathrm{Ext}(-, -)$ and $\mathrm{Tor}(-, -)$. Before we discuss these we need to go through the basic concepts of homological algebra.

A *chain complex* is a sequence of pairs $\langle \mathbf{M}, \mathbf{d} \rangle = \langle (M_n, d_n) : n \in \mathbb{Z} \rangle$ such that $\mathbf{d}^2 = 0$. That is, it is a diagram of the shape

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r, "d_{n+1}"] & M_n \arrow[r, "d_n"] & M_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

such that $d_n d_{n+1} = 0$ for all n. The morphism $\mathbf{d}$ is called the *boundary operator*. Given two chain complexes $\mathbf{M}$ and $\mathbf{N}$, a *chain map* $\varphi : \mathbf{M} \longrightarrow \mathbf{N}$ is a sequence $\langle \varphi_n : n \in \mathbb{Z} \rangle$ of maps such that for all n the following diagram commutes:

$$\begin{tikzcd}
M_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"'] & M_{n-1} \arrow[d, "\varphi_{n-1}"] \\
N_n \arrow[r, "d_n"'] & N_{n-1}
\end{tikzcd}$$

Note that usually we do not distinguish in notation the boundary operators of chain complexes. It is easily seen that chain complexes form a category.

Let $\mathbf{M}$ be a chain complex. Since $d_n d_{n+1} = 0$, i.e. $\mathrm{im}(d_{n+1}) \subseteq \ker(d_n)$, we can associate with $\mathbf{M}$ a sequence $H(\mathbf{M}) = \langle H_n(\mathbf{M}) : n \in \mathbb{Z} \rangle$ of modules where

$$H_n(\mathbf{M}) = \ker(d_n) / \mathrm{im}(d_{n+1}).$$

$H(\mathbf{M})$ is then called the *homology module* of $\mathbf{M}$ and $H_n(\mathbf{M})$ the n-th *homology module* of $\mathbf{M}$. Clearly any chain map $\varphi : \mathbf{M} \longrightarrow \mathbf{N}$ induces a map $H_n(\varphi) : H_n(\mathbf{M}) \longrightarrow H_n(\mathbf{N})$ for each n. Hence each $H_n(-)$ is a functor from the category of chain complexes into $\mathfrak{A}$. Usually we just write φ_* for every $H_n(\varphi)$.

Common practice in category theory dictates that one dualizes notions by reversing arrows and adding the prefix "co" to every noun. If you do this to what we have done in the last couple of paragraphs you will get a rant about *cohomology*. But the theory of cohomology can also be viewed as the dual of the theory of homology. So there is no need to treat them separately.

The homology modules can be computed as follows. Consider the induced sequence

$$\mathrm{im}(d_{n+1}) \xrightarrow{\mathrm{id}_n} M_n \xrightarrow{d_n} \mathrm{im}(d_n) \xrightarrow{\mathrm{id}_{n-1}} \ker(d_{n-1}) \xrightarrow{0} M_{n-2}.$$

Since $d_n \circ \mathrm{id}_n = 0$, there is an induced map $d_n^* : \mathrm{coker}(d_{n+1}) \longrightarrow \ker(d_{n-1})$. It is easy to see that $H_n(\mathbf{M}) = \ker(d_n^*)$ and $H_{n-1}(\mathbf{M}) = \mathrm{coker}(d_n^*)$.

Now we may think of $\mathbf{O} \overset{\varphi}{\rightarrowtail} \mathbf{M} \overset{\psi}{\twoheadrightarrow} \mathbf{N}$ as an exact sequence of complexes if it is exact componentwise. Hence for each n we have a commutative diagram:

$$\begin{tikzcd}
O_n \arrow[r, tail, "\varphi_n"] \arrow[d, "d_n"'] & M_n \arrow[r, two heads, "\psi_n"] \arrow[d, "d_n"] & N_n \arrow[d, "d_n"] \\
O_{n-1} \arrow[r, tail, "\varphi_{n-1}"'] & M_{n-1} \arrow[r, two heads, "\psi_{n-1}"'] & N_{n-1}
\end{tikzcd}$$

This gives rise to the commutative diagram:

$$\begin{tikzcd}
 & O_n \arrow[r, tail, "\varphi_n"] \arrow[d] & M_n \arrow[r, two heads, "\psi_n"] \arrow[d] & N_n \arrow[d] & \\
 & \mathrm{coker}(d_{n+1}) \arrow[r] \arrow[d, "d_n^*"'] & \mathrm{coker}(d_{n+1}) \arrow[r] \arrow[d, "d_n^*"] & \mathrm{coker}(d_{n+1}) \arrow[r] \arrow[d, "d_n^*"] & 0 \\
0 \arrow[r] & \ker(d_{n-1}) \arrow[r] \arrow[d, tail] & \ker(d_{n-1}) \arrow[r] \arrow[d, tail] & \ker(d_{n-1}) \arrow[d, tail] & \\
 & O_{n-1} \arrow[r, tail, "\varphi_{n-1}"'] & M_{n-1} \arrow[r, two heads, "\psi_{n-1}"'] & N_{n-1} &
\end{tikzcd}$$

where d_n^* is the map described in the last paragraph. Therefore using 2.10 we get a boundary map

$$\omega_n : H_n(\mathbf{N}) \longrightarrow H_{n-1}(\mathbf{O}). \tag{4.1}$$

This means that every short exact sequence of complexes gives rise to a chain complex that integrate the three homologies together.

4.1 Homotopy

The motivation for this notion of a homotopy is roughly this. Given two maps $\varphi, \psi : \mathbf{M} \longrightarrow \mathbf{N}$, when do they induce the same homomorphism between $H(\mathbf{M})$ and $H(\mathbf{N})$?

Definition 4.1. A *homotopy* $\Sigma : \varphi \longrightarrow \psi$ between two chain maps $\varphi, \psi : \mathbf{M} \longrightarrow \mathbf{N}$ is a morphism of degree $+1$ such that in the diagram

$$\begin{tikzcd}
M_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "\psi_{n+1}"'] \arrow[d, shift left, "\varphi_{n+1}"] & M_n \arrow[r, "d_n"] \arrow[d, shift right, "\psi_n"'] \arrow[d, shift left, "\varphi_n"] \arrow[dl, "\Sigma_n"'] & M_{n-1} \arrow[d, shift right, "\psi_{n-1}"'] \arrow[d, shift left, "\varphi_{n-1}"] \arrow[dl, "\Sigma_{n-1}"'] \\
N_{n+1} \arrow[r, "d_{n+1}"'] & N_n \arrow[r, "d_n"'] & N_{n-1}
\end{tikzcd}$$

we have

$$\psi_n - \varphi_n = d_{n+1}\Sigma_n + \Sigma_{n-1}d_n.$$

We say that φ, ψ are *homotopic*, denoted by $\varphi \simeq \psi$, if there exists a homotopy $\Sigma : \varphi \longrightarrow \psi$.

The following basic facts about homotopy can be easily checked. Firstly, the homotopy relation is an equivalence relation. So we can associate a *homotopy category* to every category of chain complexes by identifying homotopic chain maps. Two chain complexes are of *the same homotopy type* if they are isomorphic in the associated homotopy category (warning: this does not mean that they are isomorphic in their category of chain complexes). Secondly, it is closed under composition; that is, let $\varphi \simeq \psi : \mathbf{M} \longrightarrow \mathbf{N}$ and $\varphi' \simeq \psi' : \mathbf{N} \longrightarrow \mathbf{O}$ be homotopic chain maps, then $\varphi'\varphi \simeq \psi'\psi : \mathbf{M} \longrightarrow \mathbf{O}$. Thirdly, homotopy is preserved by additive functors (from one category of modules to another category of modules). Now a little bit of diagram chasing shows that if $\varphi \simeq \psi : \mathbf{M} \longrightarrow \mathbf{N}$, then $H(\varphi) = H(\psi) : H(\mathbf{M}) \longrightarrow H(\mathbf{N})$ (warning: the converse of this is not true; that is, existence of a homotopy is not a necessary condition for two chain complex maps to induce the same homology homomorphism). So immediately we get $H(F\varphi) \simeq H(F\psi) : H(F\mathbf{M}) \longrightarrow H(F\mathbf{N})$, where F is an additive functor between categories of modules.

We are advancing towards a definition of derived functors, which is the key to the definitions of the functors $\mathrm{Ext}^n(-, M)$ and $\mathrm{Tor}_n(M, -)$. For a given module N, we first need to associate a canonical chain complex to it in order to use all this machinery presented above. It will be seen that "canonical" means "homotopic". Recall that by Fact 1.2 we have enough projectives in any module category. Let $M_0 \rightarrowtail P_0 \twoheadrightarrow N$ be a projective presentation of N, $M_1 \rightarrowtail P_1 \twoheadrightarrow M_0$ be a projective presentation of M_0, etc. We therefore obtain a chain complex:

$$\mathbf{P} : \ldots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

with every P_n projective (such a thing is called a *projective chain complex*), $H_n(\mathbf{P}) = 0$ for every $n > 0$ (such a thing is called *acyclic*), and $H_0(\mathbf{P}) = N$. $\mathbf{P}$ is called a *projective resolution* of N. It is a canonical object in the following sense:

Theorem 4.2. *Let $\mathbf{P}$, $\mathbf{P}'$ be projective resolutions of M, N respectively. Let $f : M \longrightarrow N$ be a module homomorphism. Then there is a chain map $\varphi : \mathbf{P} \longrightarrow \mathbf{P}'$ inducing f, that is, the induced homology map $H_0(\mathbf{P}) \longrightarrow H_0(\mathbf{P}')$ is f. Moreover, any two chain maps that induce f are homotopic; in particular, if $f = \mathrm{id}$ then $\mathbf{P}$ and $\mathbf{P}'$ are of the same homotopy type.*

The proof of this theorem relies on $\mathbf{P}$ being projective and $\mathbf{P}'$ being acyclic. The rest of it is just routine induction.

4.2 Derived Functors

It is just easier to talk in abstract nonsense. Hence we will discuss left and right derived functors of arbitrary (covariant and contravariant) additive functors, even though we only care about the contravariant functor $\mathrm{Hom}(-, T)$ and the covariant functor $T \otimes -$ (or $- \otimes T$, which is the same thing in the present context as we are only considering module categories of commutative rings).

Let $F : \mathfrak{A} \longrightarrow \mathfrak{A}$ be an additive covariant functor. Given a module N and a projective resolution $\mathbf{P}$ of N, we define the nth *left derived functor* of F by

$$L_nF(N) = H_n(F\mathbf{P}), \quad n = 0, 1, \ldots.$$

This definition looks suspicious because it seems that the righthand side depends on the particular projective resolution $\mathbf{P}$ while the lefthand side only operates on the module N. Solutions to situations like this are always similar. For a homomorphism $\alpha : M \longrightarrow N$ and two projective resolutions $\mathbf{P}, \mathbf{Q}$ of M, N, there is an induced homomorphism $\alpha_n : H_n(F\mathbf{P}) \longrightarrow H_n(F\mathbf{Q})$ which is, by Theorem 4.2, unique up to homotopy. Since compositions of homomorphisms between modules are naturally carried over to their projective resolutions, if $\alpha = \mathrm{id}$ in the previous sentence, we easily get a canonical isomorphism

$$\eta_{\mathbf{P},\mathbf{Q}} : H_n(F\mathbf{P}) \xrightarrow{\sim} H_n(F\mathbf{Q}), \quad n = 0, 1, \ldots.$$

This is why we may suppress mentioning any projective resolution in the definition of left derive functor.

Definition 4.3. The covariant functor $F : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called *right exact* if for every exact sequence $M \longrightarrow N \longrightarrow O \longrightarrow 0$, the sequence

$$FM \longrightarrow FN \longrightarrow FO \longrightarrow 0$$

is exact.

Right exact functors are additive. Our prime example here is of course $T \otimes -$.

We now perform dualization. Let $G : \mathfrak{A} \longrightarrow \mathfrak{A}$ be an additive contravariant functor. So $G : \mathfrak{A}^{op} \longrightarrow \mathfrak{A}$ is an additive covariant functor. Given a module N and a projective resolution $\mathbf{P}$ of N in $\mathfrak{A}$ (i.e. an injective resolution of N in $\mathfrak{A}^{op}$), we define the nth *right derived functor* of G by

$$R^nG(N) = H^n(G\mathbf{P}), \quad n = 0, 1, \ldots.$$

Note that $G\mathbf{P}$ is a cochain complex in $\mathfrak{A}$. All the little facts about right derived functors are completely dual to those about left derived functors. We only highlight the following concept:

Definition 4.4. The contravariant functor $G : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called *left exact* if the covariant functor $G : \mathfrak{A}^{op} \longrightarrow \mathfrak{A}$ is left exact; that is, for every exact sequence $M \longrightarrow N \longrightarrow O \longrightarrow 0$ in $\mathfrak{A}$, the sequence

$$0 \longrightarrow GO \longrightarrow GN \longrightarrow GM$$

is exact.

Left exact functors are additive. Our prime example here is of course $\mathrm{Hom}(-, T)$.

From this point forth the diagrams in homological algebra start to become really complicated, even though the results represented by them are very often quite simple. It really takes time to draw them, for example, the two basic long exact sequences associated with the derived functors, see Section 6, Chapter IV, [2]. (Theorem 6.1 there is more pertinent to our discussion below. It is basically a specialized version of the derivation of 4.1, only now the three chain complexes are projective resolutions.) Except a few, it does not serve our purpose to reproduce them here as we have covered enough ground for our exercises (recall that we are just trying to do a couple of exercises here!) and advanced results beyond this point are not necessary for this purpose. All we need now is to formulate the definitions of the derived functors $\mathrm{Ext}^n(-, T)$ and $\mathrm{Tor}_n(T, -)$ and state a few basic facts about them.

4.3 The Functors $\mathrm{Ext}^n(-, T)$ and $\mathrm{Tor}_n(T, -)$

Let T be a module.

Definition 4.5. $\mathrm{Ext}^n(-, T) = R^n(\mathrm{Hom}(-, T))$, $n = 0, 1, \ldots$.

It is easily checked that $\mathrm{Ext}_1(M, T) \cong \mathrm{Ext}(M, T)$ for any module M, hence the notation. Obviously $\mathrm{Ext}^n(-, T)$ is a contravariant functor. Of course we can vary the second variable and make $\mathrm{Ext}^n(T, -)$ into a covariant functor. By the basic facts about derived functors in Section 6, Chapter IV, [2], $\mathrm{Ext}^n(-, -)$ is, not surprisingly, a bifunctor. Now by Theorem IV.6.1, [2], given a short exact sequence $M \rightarrowtail N \twoheadrightarrow O$, we obtain a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Ext}^0(O, T) \longrightarrow \mathrm{Ext}^0(N, T) \longrightarrow \mathrm{Ext}^0(M, T) \xrightarrow{\omega_0} \mathrm{Ext}^1(O, T) \longrightarrow \ldots \\ \ldots \longrightarrow \mathrm{Ext}^n(O, T) \longrightarrow \mathrm{Ext}^n(N, T) \longrightarrow \mathrm{Ext}^n(M, T) \xrightarrow{\omega_n} \mathrm{Ext}^{n+1}(O, T) \longrightarrow \ldots . \end{array} \tag{4.2}$$

This is called the *long exact* Ext-*sequence in the first variable*. Similarly we can obtain *long exact* Ext-*sequence in the second variable* with respect to the covariant functor $\mathrm{Ext}^n(T,-)$. These long exact Ext-sequences are natural in every possible way (for details see Section 7, Chapter IV, [2]).

Definition 4.6. $\mathrm{Tor}_n(T,-) = L_n(T \otimes -)$, $n = 0, 1, \ldots$.

As an understanding of this definition is the main purpose of this epitome of the fundamentals of homological algebra, we shall repeat here how the abelian groups $\mathrm{Tor}_n(T, M)$ is calculated for a module M. Choose a projective resolution $\mathbf{P}$ of M, form the chain complex $T \otimes \mathbf{P}$ and then take homology

$$\mathrm{Tor}_n(T, M) = H_n(T \otimes \mathbf{P}).$$

As $T \otimes -$ is right exact, clearly $\mathrm{Tor}_0(T, M) = T \otimes M$. Also it is easily checked that $\mathrm{Tor}_1(T, M) = \mathrm{Tor}(T \otimes M)$, hence the notation. The *long exact* Tor-*sequences* are, in a fashion completely dual to the case of the long exact Ext-sequences, natural in every possible way. Now $\mathrm{Tor}_n(-,-)$ is not only a bifunctor, it is also balanced; that is, for two modules, projective resolutions of the first and projective resolutions of the second yield the same results for each n.

Finally we mention the following theorem that relate the two functors, which is, in essence, just the Yoneda Lemma (see any textbook of category theory for a general discussion of this lemma). For two functors F and G, let $[F, G]$ be the collection of all natural transformations from F to G.

Theorem 4.7. *Let T, S be two modules. There are natural isomorphisms*

$$\Gamma : [\mathrm{Ext}^n(T,-), S \otimes -] \xrightarrow{\sim} \mathrm{Tor}_n(S, T), \quad n = 0, 1, \ldots.$$

5 The Answer, Finally!

After the lengthy review above, the answer to this problem appears completely trivial.

(i) $\Rightarrow$ (ii): Well, since M is flat, the chain complex $M \otimes \mathbf{P}$ is exact for any projective resolution $\mathbf{P}$ of N, hence for every $n > 0$ the homology group is trivial.

(ii) $\Rightarrow$ (iii): Manifestly trivial.

(iii) $\Rightarrow$ (i): The only question is that how the exact sequence in the hint is obtained. Well, it is an initial chunk of the long exact Tor-sequence in the second variable!

25. Recall that $\operatorname{Tor}_n(-,-)$ is a balanced functor. For any module M, just look at the following exact segment of the long exact Tor-sequence in the first variable

$$\begin{tikzcd} \operatorname{Tor}_2(N'', M) \arrow[r] & \operatorname{Tor}_1(N', M) \arrow[r] & \operatorname{Tor}_1(N, M) \arrow[r] & \operatorname{Tor}_1(N'', M) \end{tikzcd}$$

and substitute 0 in accordingly.

26. The "$\Rightarrow$" direction is by Exercise 24. For the other direction, we just follow the hint. Suppose $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for every finitely generated ideal $\mathfrak{a}$. This means that the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \otimes N \arrow[r] & A \otimes N \arrow[r] & A/\mathfrak{a} \otimes N \arrow[r] & 0 \end{tikzcd}$$

is exact for every finitely generated ideal $\mathfrak{a}$. Running the proof of (iv) $\Rightarrow$ (iii) in 2.19 for *any* ideal $\mathfrak{a}$ we deduce that actually $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for any ideal $\mathfrak{a}$. That is, $\operatorname{Tor}_1(M, N) = 0$ for all cyclic modules M. Then by Exercise 25 and induction on the number of generators we easily get that $\operatorname{Tor}_1(M, N) = 0$ for all finitely generated modules M. This, by 2.19 and Exercise 24 again, implies that N is flat.

27. (i) $\Rightarrow$ (ii): The diagram commutes because composition and tensor interact in a natural way (p. 27). We have that

$$\beta(ax \otimes (b + (x))) = a \otimes (bx + (x^2)) = 0,$$

i.e. $A \otimes ((x)/(x^2)) = 0$, i.e. $(x) = (x^2)$.

(ii) $\Rightarrow$ (iii): Only note that the last sentence of the hint is by Exercise 1.22.

(iii) $\Rightarrow$ (i): Since $\operatorname{Tor}_1(A, N) = 0$ for any module N, clearly any direct summand A' of A gives $\operatorname{Tor}_1(A', N) = 0$ for any module N. Now use Exercise 26.

28. See Exercise 1.11 for Boolean rings. For the ring in Exercise 1.7 only note that for any element x with $x^n = x$ we have $x^{3n-4}x^2 = x$. If $f : A \twoheadrightarrow B$ is a ring homomorphism, then (ii) of Exercise 27 is preserved in B. For a local ring, if it is absolutely flat then for any nonzero x there is a nonzero a such that ax is nonzero and idempotent, then by Exercise 1.12 $ax = 1$.

Again, if A is absolutely flat, then for every nonunit nonzero x there is a nonzero a such that ax is idempotent, i.e. $ax - (ax)^2 = 0$. Since $ax \neq 1$, x is a zero-divisor.

References

[1] C. C. Chang and H. J. Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland, Amsterdam, 1990.

[2] P. J. Hilton and U. Stammbach, *A course in homological algebra*, Springer-Verlag, New York, 1971.

[3] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, New York, 1994.