Carnegie Mellon Tepper School of Business

Global Constraints

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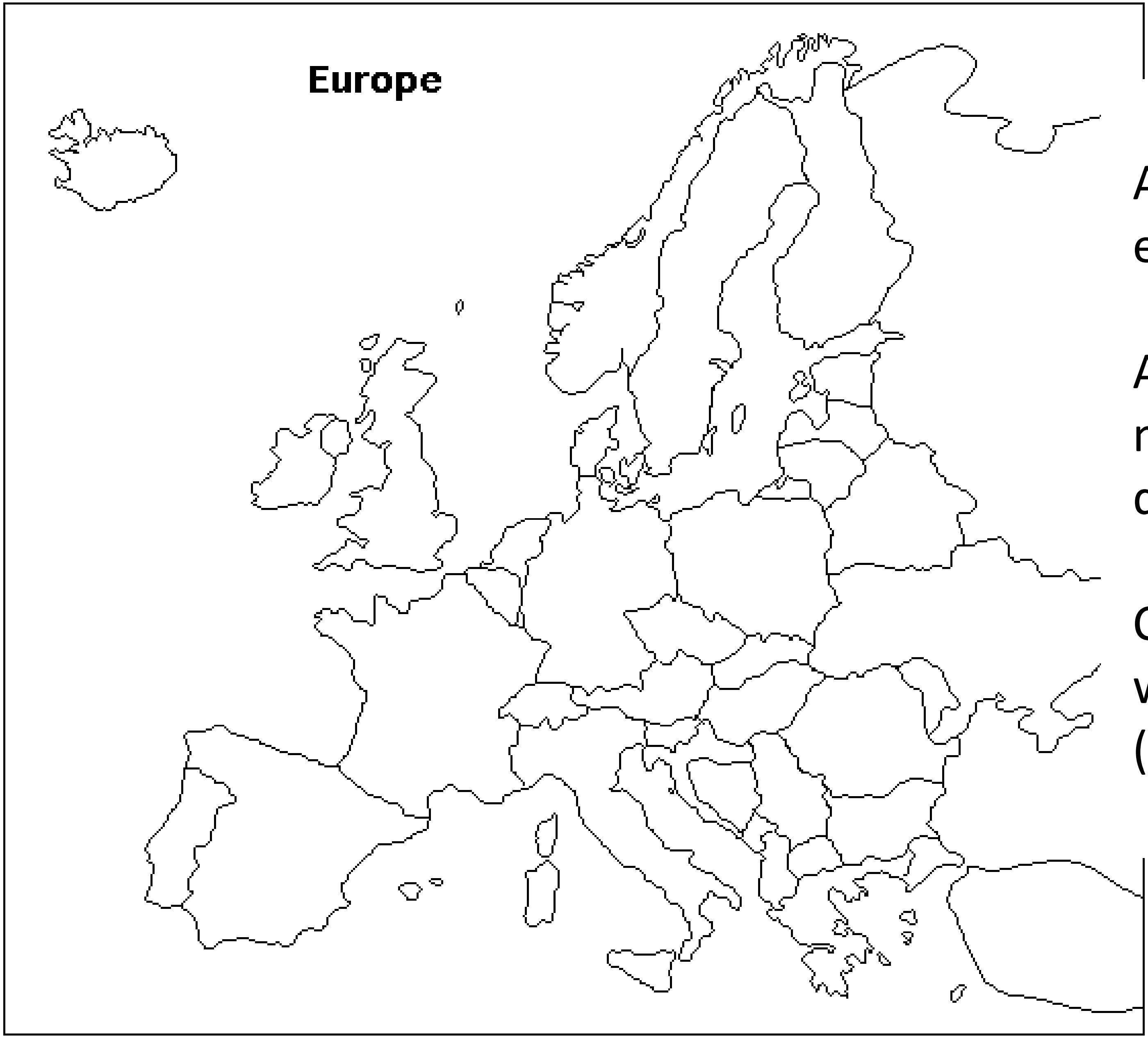
Outline

- Introduction
- Alldifferent constraint
- Knapsack constraint
- Regular constraint
- Research directions

Example: Graph coloring

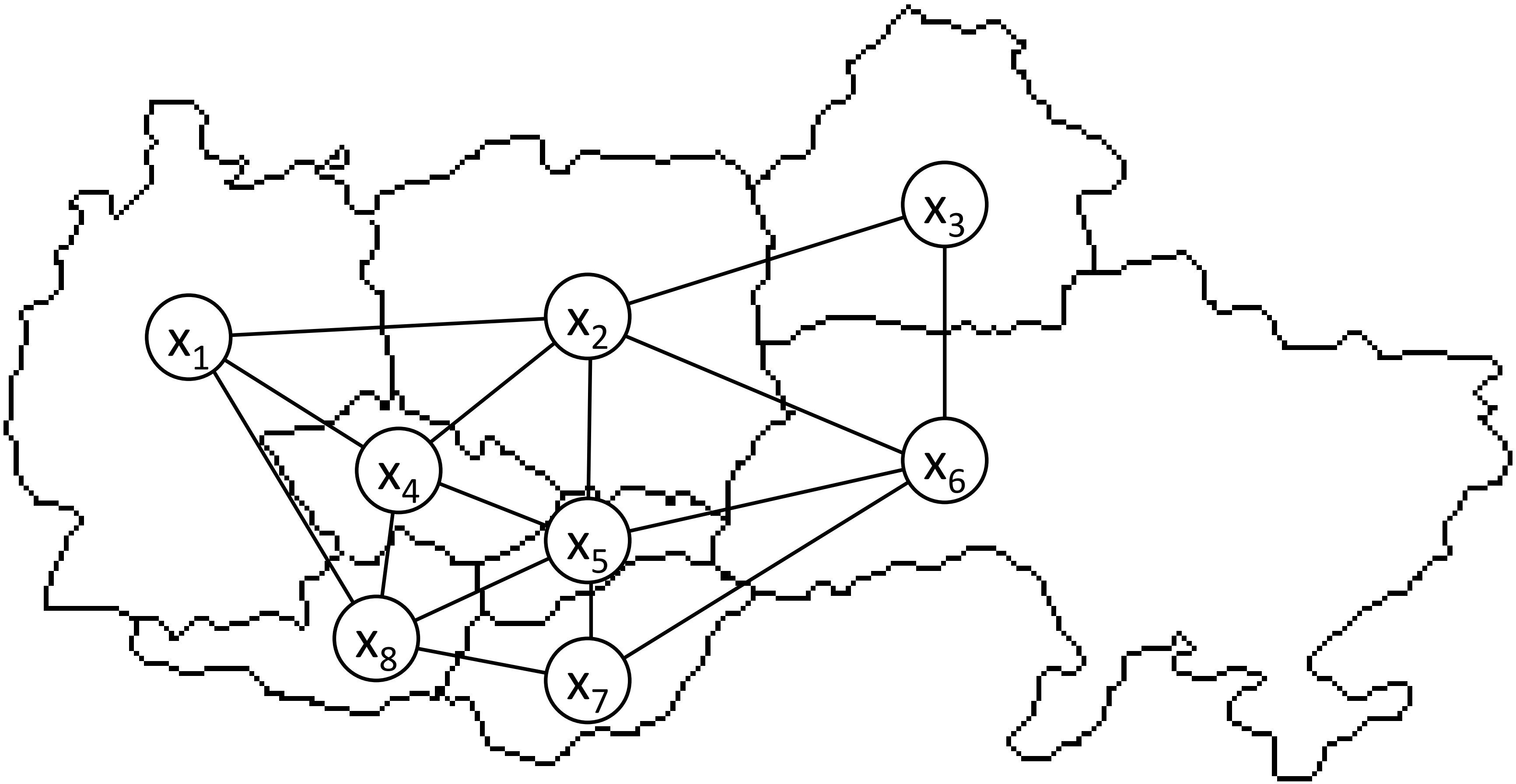


Assign a color to each country

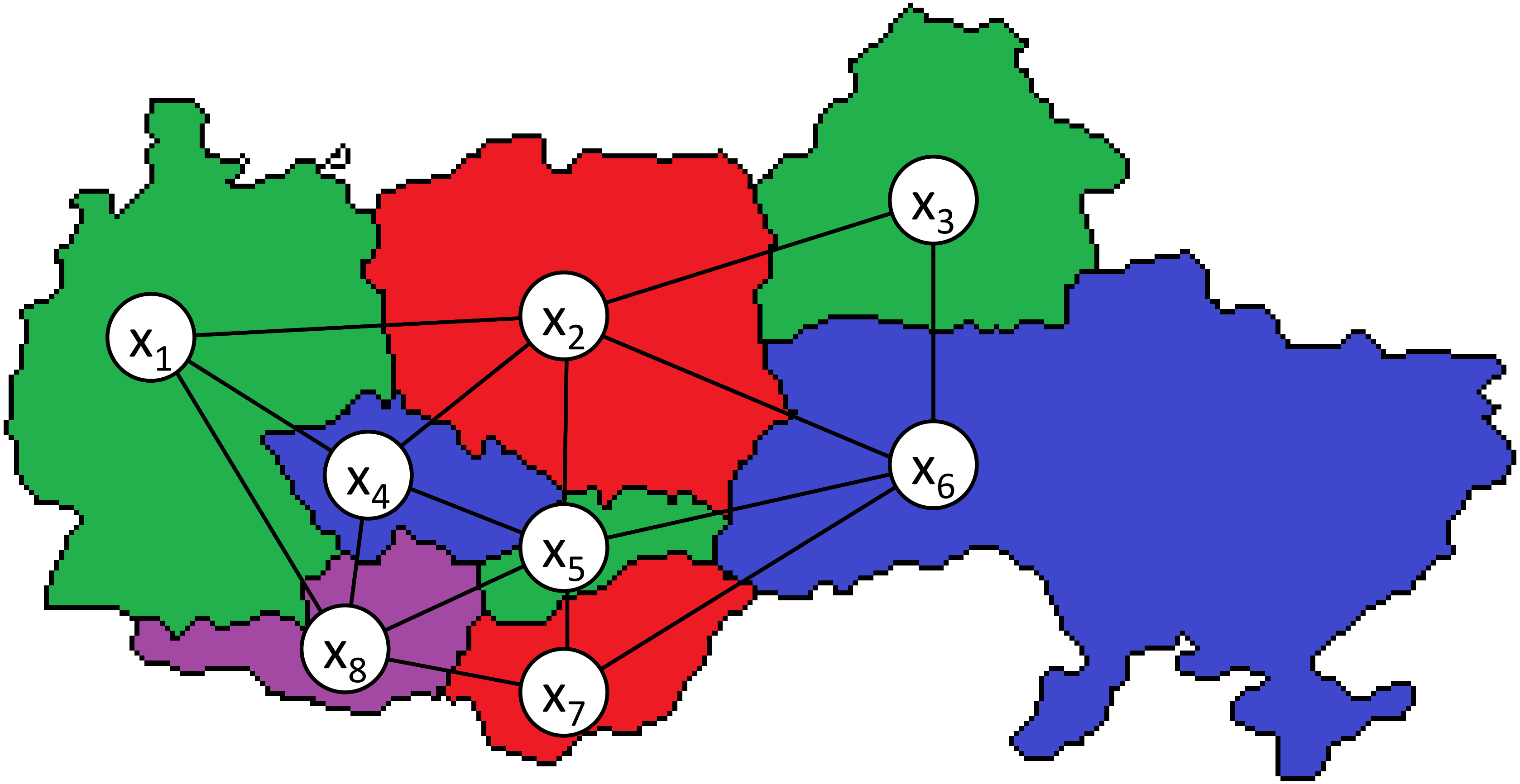
Adjacent countries must have different colors

Can this be done with *k* colors? (minimize *k*)

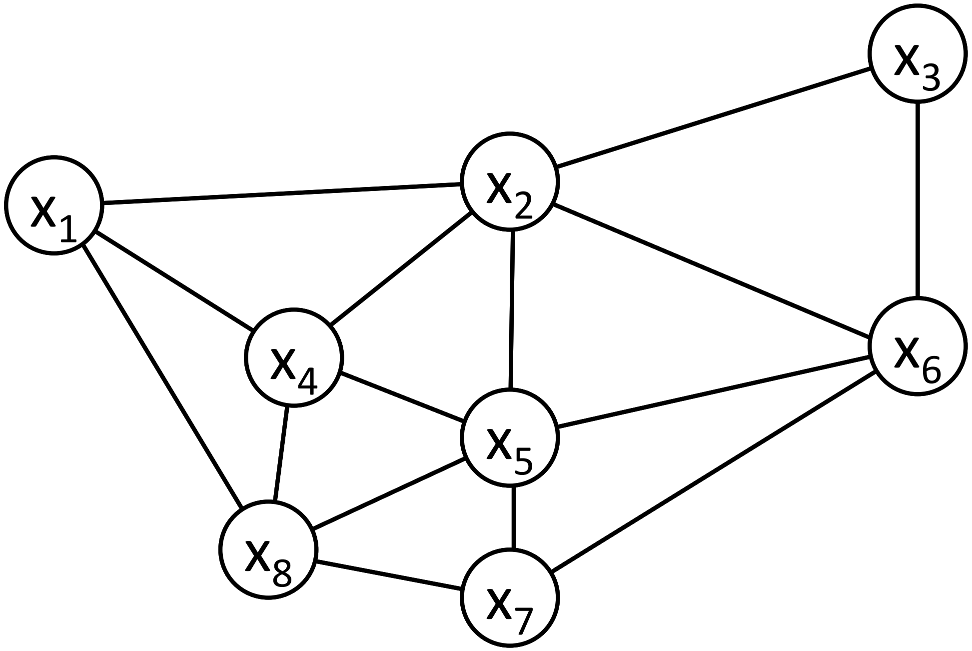
Smaller (8 variable) instance



Solution with four colors



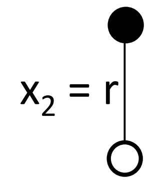
CP Model for k=3 colors

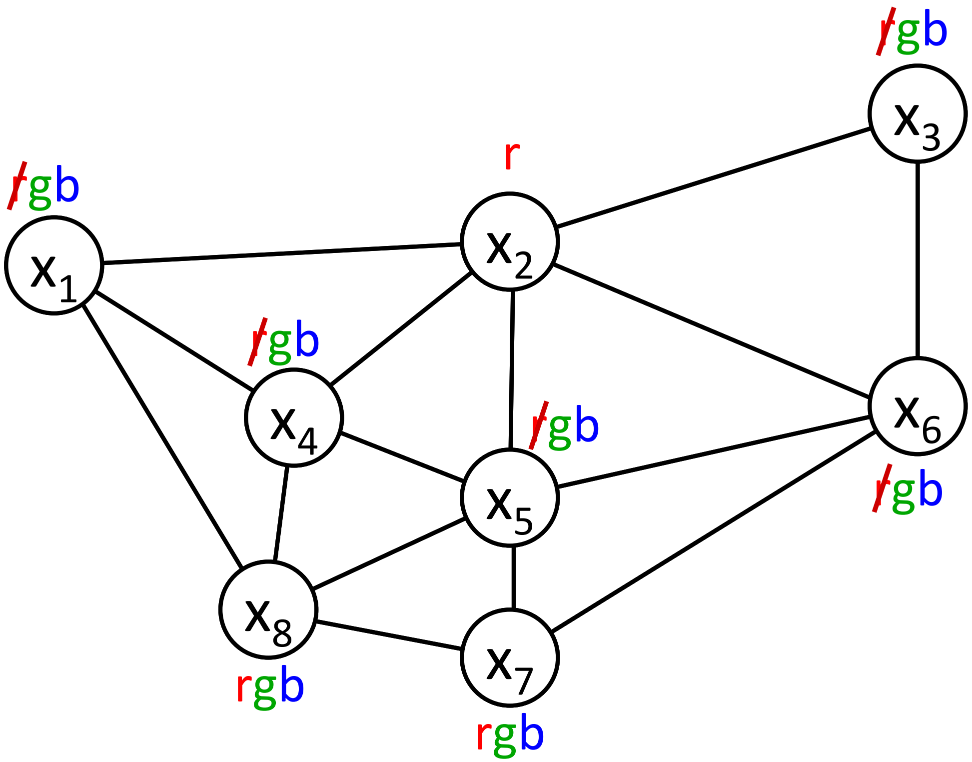


Variables and domains: x_i in {r,g,b} for all i

Constraints: $x_i \neq x_j$ for all edges (i,j)

Search & propagate

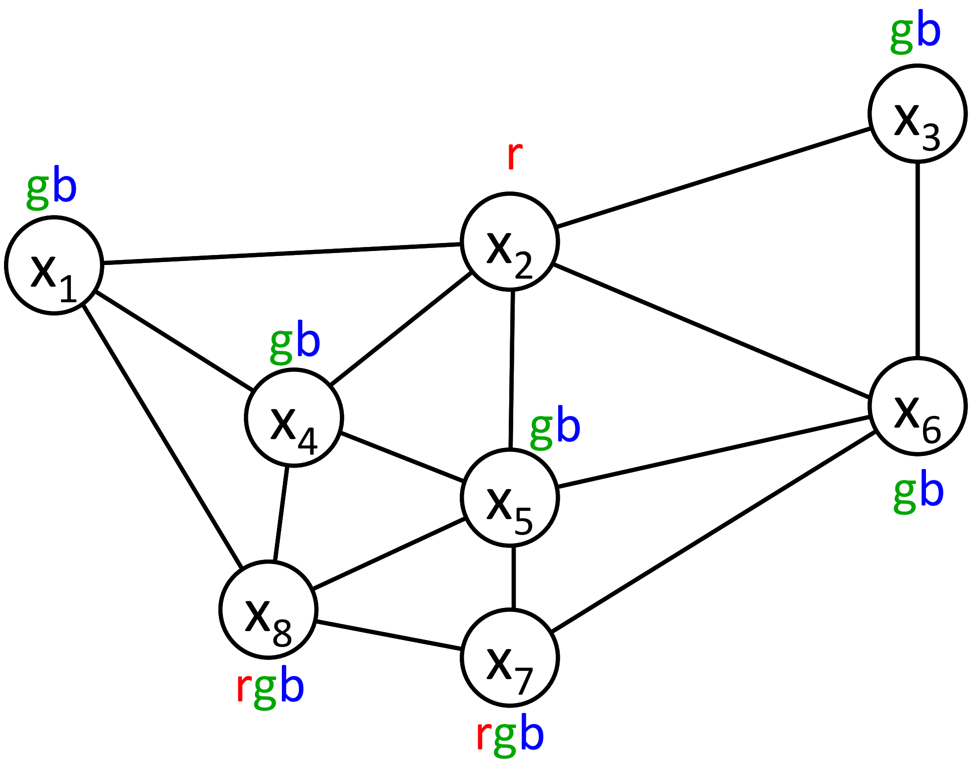


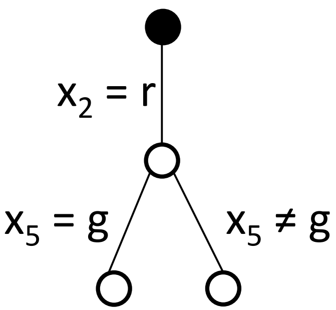


Search choice: $x_2 = r$

(by symmetry, no need to consider $x_2 = g, b$)

Search & propagate

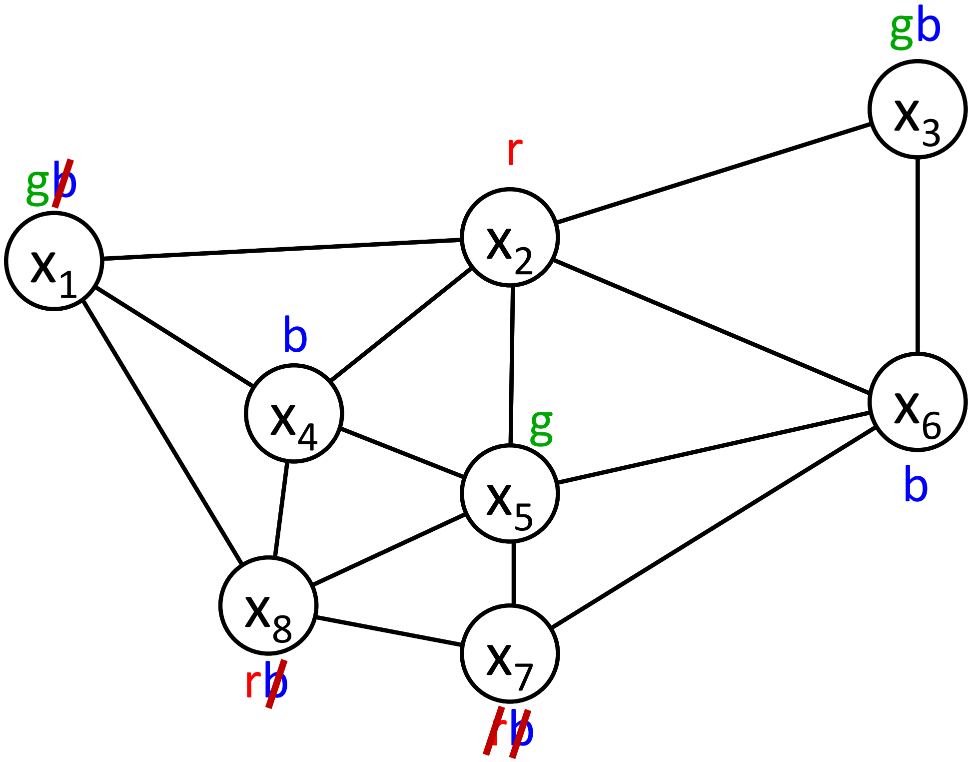


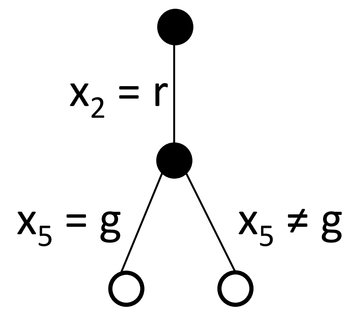


Search choice: x_5 = g

(be prepared to backtrack)

Search & propagate

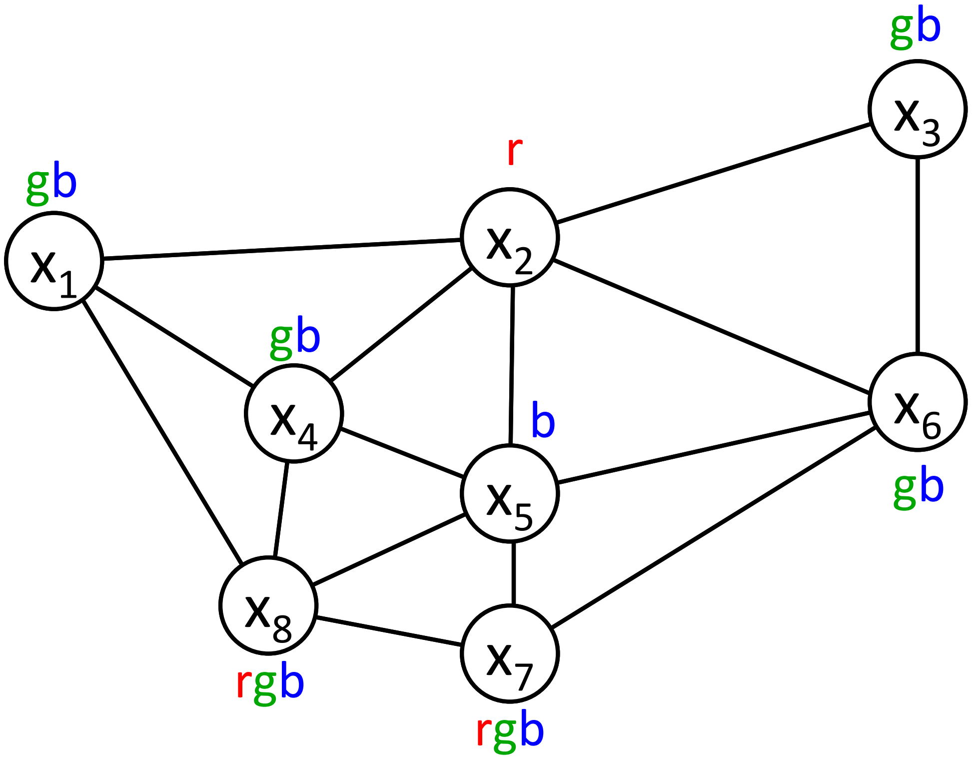


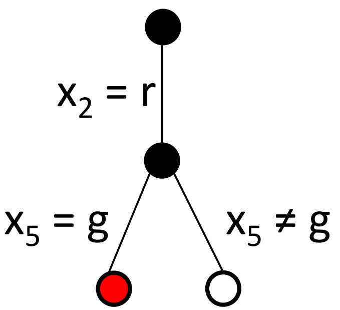


… and propagate

x_7 has an empty domain: we need to backtrack

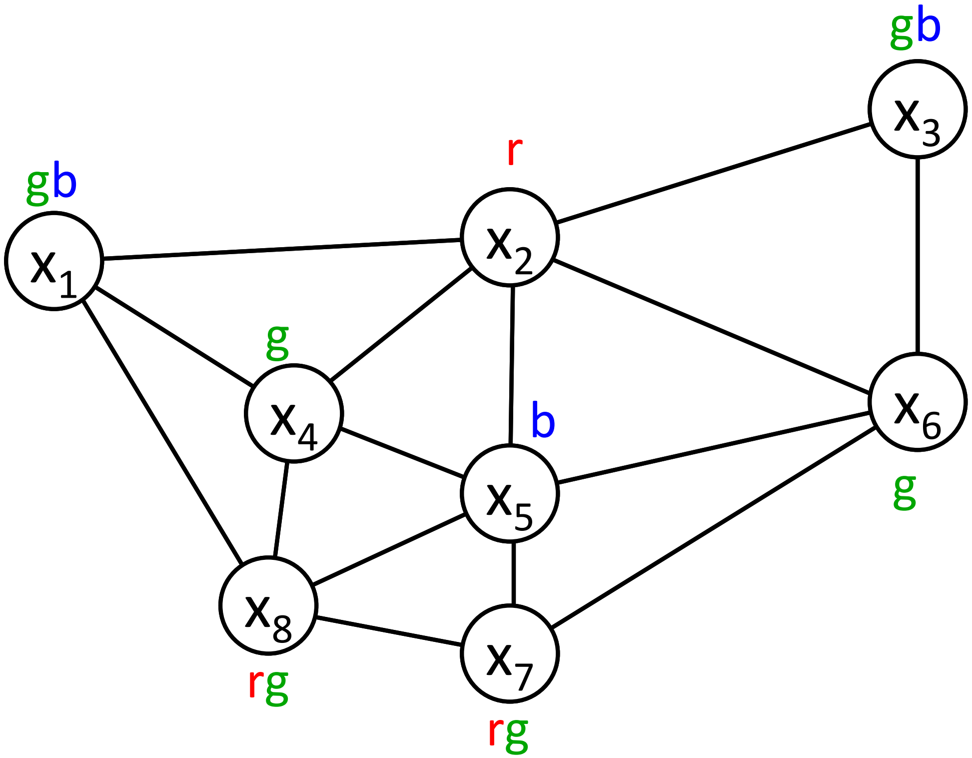
Search & propagate

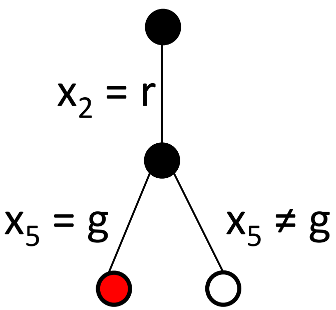




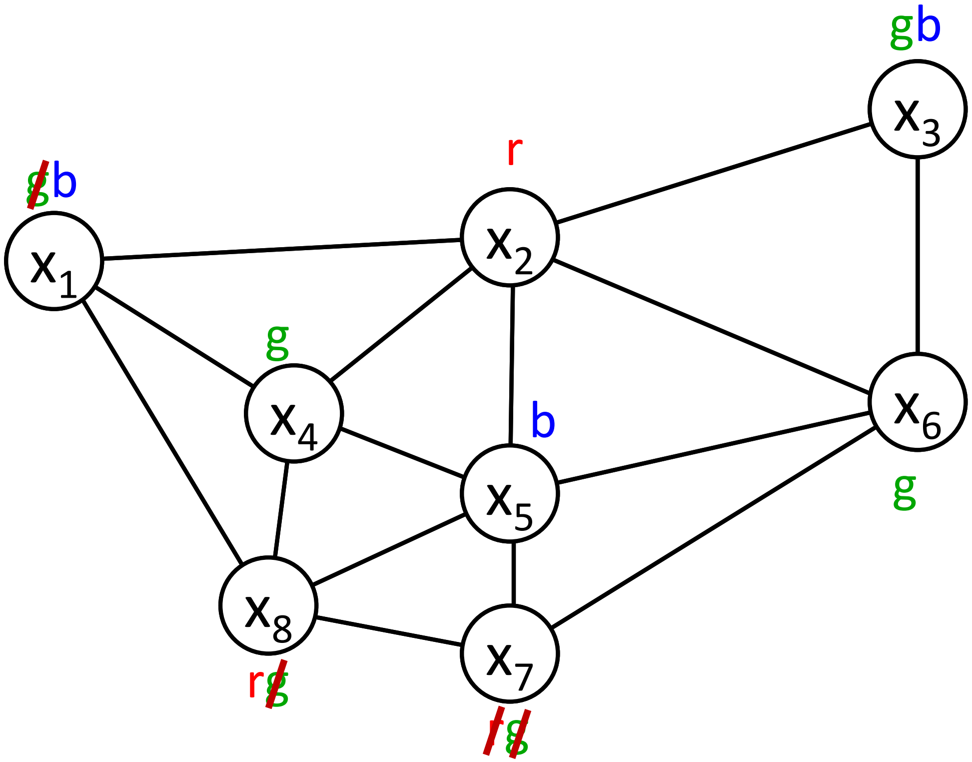
Propagate…

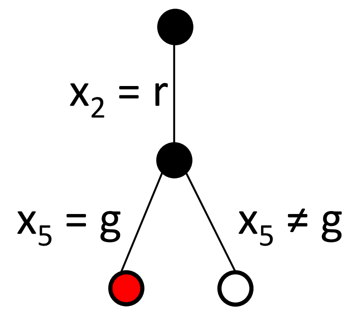
Search & propagate





Search & propagate

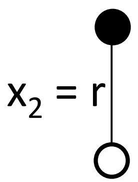


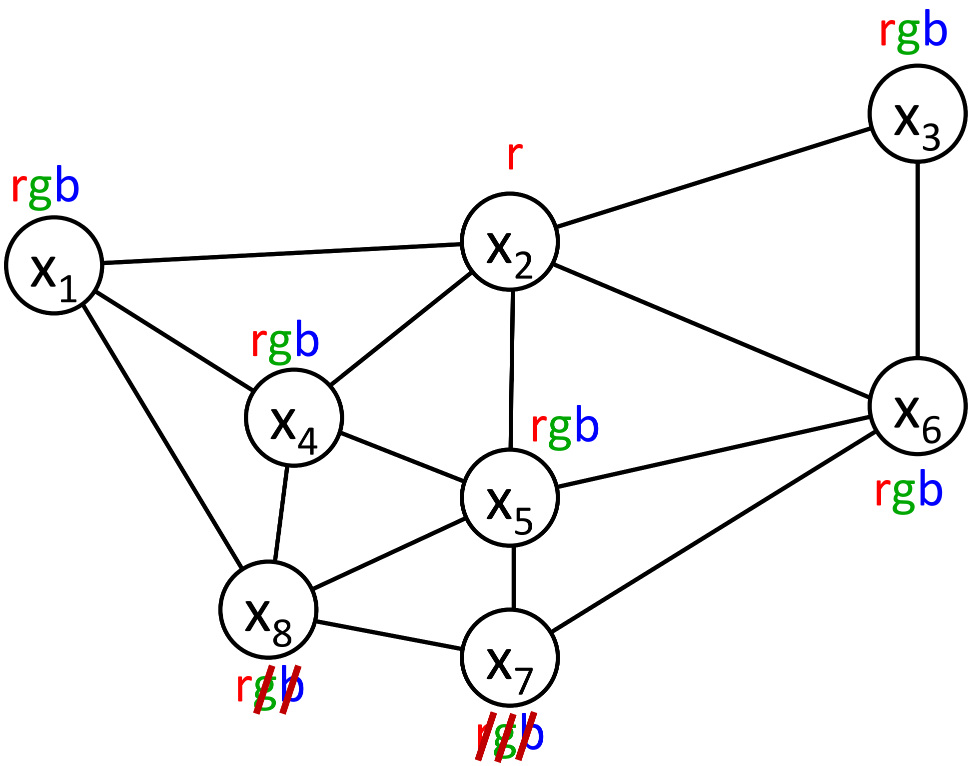


…and propagate

x_7 has an empty domain: we are done

Recall example: first propagation





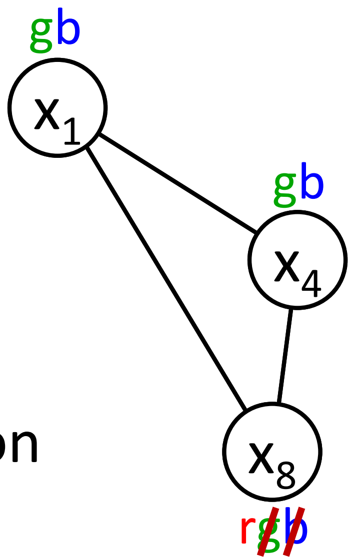
Can we do more propagation?

After x_2 = r we are done.

Introduce global constraints

- We can increase the inference by adding more knowledge to the solver
  - in this case, group not-equal constraints that form a clique
  - use *alldifferent* constraints

$$\textit{alldifferent}(x_1,x_2,\ldots,x_n) := \bigwedge_{i<j} \; x_i \neq x_j$$



Model 1: $x_1 \in \{g,b\}$, $x_4 \in \{g,b\}$, $x_8 \in \{r,g,b\}$

$x_1 \neq x_4$, $x_1 \neq x_8$, $x_4 \neq x_8$ — no propagation

Model 2: $x_1 \in \{g,b\}$, $x_4 \in \{g,b\}$, $x_8 \in$ {r,~~g~~,~~b~~}

alldifferent(x_1,x_4,x_8) — $x_8 = r$

Impact of global constraint propagation

- Graph coloring problem; random instances

- Can set *alldifferent* propagation level from ‘low’ to ‘extended’
  - ‘low’: pairwise not-equal constraints
  - ‘extended’: best possible propagation
  - notice the difference in search tree size (search choices or failures) and solving time

Global constraints overview

- Examples
  - *Alldifferent, Cardinality, Circuit, BinPacking*, …
- Global constraints represent combinatorial structure
  - can be viewed as the combination of elementary constraints
  - expressive building blocks for modeling applications
  - embed powerful algorithms from OR, Graph Theory, AI, CS, …
- Essential for the successful application of CP
  - User can identify global constraints to be used in model
  - Automated detection for certain constraints (ILOG CPO)

Embedded Algorithms

Constraint	Structure/technique
alldifferent	bipartite matching [Régin, 1994]
cardinality	network flow [Régin, 1996]
knapsack	dynamic programming [Trick, 2003]
regular	directed acyclic graph [Pesant, 2004]
sequence	various [vH et al., 2006,09] [Brand et al., 2007] [Maher et al., 2008]
BinPacking	various [Shaw, 2004] [Cambazard et al., 2010] [Schaus et al., 2010-13]
N-value	various [Beldiceanu et al., 2001] [Bessiere et al., 2005, 10]
circuit	network flow [Genc Kaya & Hooker, 2006]
weighted circuit	AP [Focacci et al., 1999], 1-Tree [Benchimol et al., 2012]
disjunctive/cumulative	dedicated algorithm [Nuijten 1994, Carlier et al., 1994] [Vilim, 2009]
. . .	. . .

The 'global constraint catalog' currently contains 364 constraints
http://sofdem.github.io/gccat/

Role of global constraints

Global constraints can typically play three roles

1. Convenient modeling
   – Global constraints are the building blocks of a complex problem
2. More effective constraint propagation
   – Identify more inconsistent domain values; reduce the search space
3. Help guide the search
   – Provide variable and value ordering heuristics

Consistency notions for global constraints

- Hyperarc consistency
  - (a global constraint defines a hyperarc in the constraint network)
  - ensure that *all* domain values are consistent w.r.t. the constraint
  - a.k.a. generalized arc consistency or <u>domain consistency</u>
- Bounds consistency
  - treat the domains as intervals, and ensure that all domain bounds are consistent
- Ad-hoc consistencies
  - constraint dependent; can be based on relaxations of the constraint

Global constraints during propagation

- Algorithms that enforce a local consistency are referred to as *domain filtering* algorithms, or *propagation* algorithms
- General tasks for a propagation algorithm:
  1. Determine whether the constraint is satisfiable (consistency check)
  2. Remove some or all inconsistent domain values (the actual domain filtering)
- The consistency check and the filtering are typically done separately for efficiency reasons

Propagation algorithm for *alldifferent*

J.-C. Régin. A filtering algorithm for constraints of difference in CSPs. In *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, pp. 362-367, 1994.

Alldifferent Propagation

- Goal: establish domain consistency on *alldifferent*
  - Guarantee that each remaining domain value participates in at least one solution
  - Can we do this in polynomial time?

- We already saw that the decomposition is not sufficient to establish domain consistency

$x_1 \in \{a,b\}$, $x_2 \in \{a,b\}$, $x_3 \in \{a,b,c\}$

$x_1 \neq x_2$, $x_1 \neq x_3$, $x_2 \neq x_3$ versus *alldifferent*(x_1,x_2,x_3)

First attempt: Hall's Theorem

Hall's Marriage Theorem [1935]:

> If a group of men and women marry only if they have been introduced to each other previously, then a complete set of marriages is possible if and only if every subset of men has collectively been introduced to at least as many women, and vice versa.

For *alldifferent*(X) this means that a solution exists iff

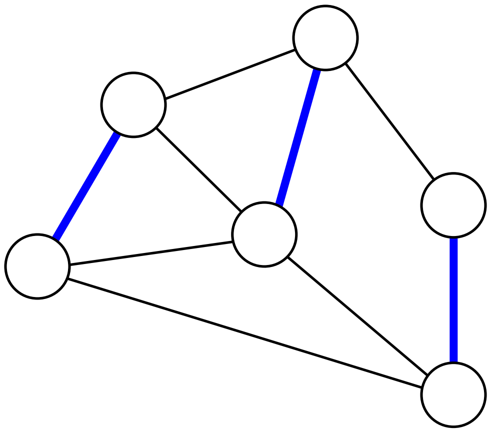
$$|K| \leq |\cup_{x \in K} D(x)| \qquad \forall K \subseteq X$$

Example: $x_1 \in \{b,c\}$, $x_2 \in \{b,c\}$, $x_3 \in \{a,b,c\}$, $x_4 \in \{a,b,c\}$

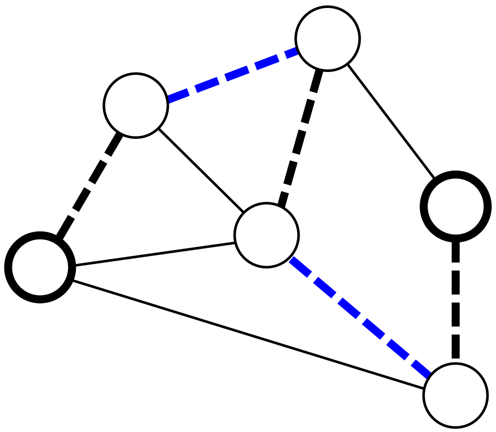
- solution exists for any subset of 3 variables
- no solution when $K = \{x_1, x_2, x_3, x_4\}$

Matchings in graphs

- Definition: Let G = (V,E) be a graph with vertex set V and edge set E. A *matching* in G is a subset of edges M such that no two edges in M share a vertex.



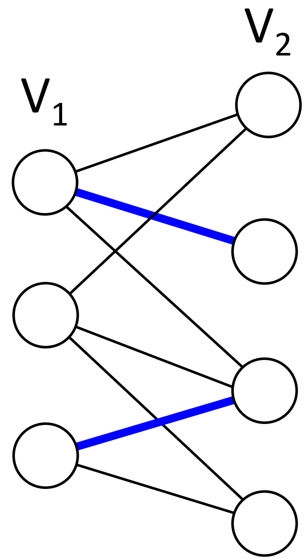
- A *maximum matching* is a matching of maximum size
- Definition: An *M-augmenting path* is a vertex-disjoint path with an odd number of edges whose endpoints are M-free



- Theorem: Either M is a maximum-size matching, or there exists an M-augmenting path [Petersen, 1891]

Finding a maximum matching

- The augmenting path theorem can be used to iteratively find a maximum matching in a graph G:
  - given M, find an M-augmenting path P
  - if P exists, augment M along P and repeat
  - otherwise, M is maximum
- For a bipartite graph $G = (V_1,V_2,E)$, an M-augmenting path can be found in $O(|E|)$ time
  - finding a maximum matching can then be done in $O(|V_1|\cdot|E|)$, as we need to compute at most $|V_1|$ paths (assume $|V_1| \leq |V_2|$)
  - this can be improved to $O(\sqrt{|V_1|}\cdot|E|)$ time [Hopcroft & Karp, 1973]



- For general graphs this is more complex, but still tractable
  - can be done in $O(\sqrt{|V|}\cdot|E|)$ time [Micali & Vazirani, 1980]

Value Graph Representation

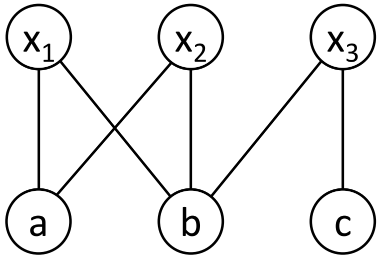
- Definition: The *value graph* of a set of variables X is a bipartite graph (X, D, E) where
  - node set X represents the variables
  - node set D represents the union of the variable domains
  - edge set E is $\{ (x,d) \mid x \in X, d \in D(x) \}$

- Example:

  $x_1 \in \{a,b\}$

  $x_2 \in \{a,b\}$

  $x_3 \in \{b,c\}$



From alldifferent to matchings

Lemma [Régin, 1994]:

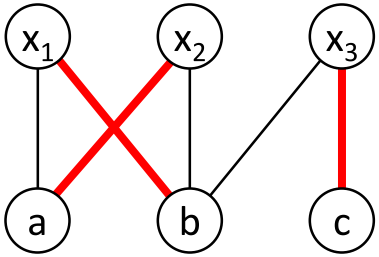
solution to *alldifferent*(X) $\Leftrightarrow$

matching in value graph covering X

Example:

$x_1 \in \{a,b\}$, $x_2 \in \{a,b\}$, $x_3 \in \{b,c\}$

alldifferent(x_1,x_2,x_3)



Theorem: Domain consistency for *alldifferent*:

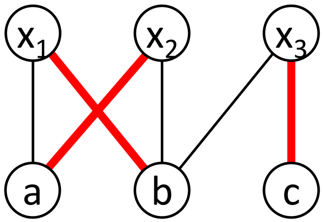
remove all edges (and corresponding domain values) that are not in any maximum matching

Propagation Algorithm

1. Verify consistency of the constraint
   - find maximum matching M in value graph $O(\sqrt{|X|} \cdot |E|)$
   - if M does not cover all variables: inconsistent
2. Verify consistency of each edge $O(\sqrt{|X|} \cdot |E|^2)$
   - for each edge e in value graph:

     fix e in M, and extend M to maximum matching

     if M does not cover all variables: remove e from graph

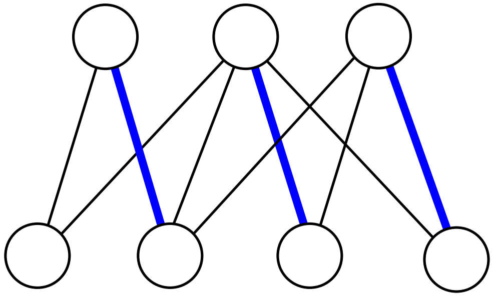


What is the time complexity?

- Establishes domain consistency in polynomial time
- But not very efficient in practice… can we do better?

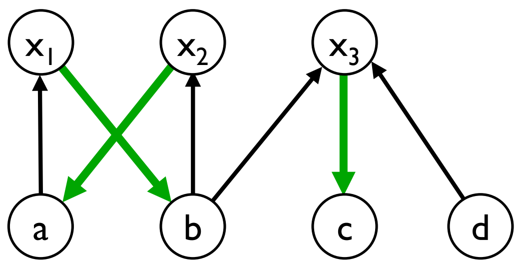
A useful theorem

- Theorem [Petersen, 1891] [Berge, 1970]: Let G be graph and M a maximum matching in G. An edge e belongs to a maximum-size matching if and only if
  - it either belongs to M
  - or to an even M-alternating path starting at an M-free vertex
  - or to an M-alternating circuit



A Better Filtering Algorithm

1. compute a maximum matching *M*: covering all variables *X* ?
2. direct edges in *M* from X to D, and edges not in *M* from D to X
3. compute the strongly connected components (SCCs)
4. edges in *M*, edges within SCCs and edges on path starting from M-free vertices are all consistent
5. all other edges are not consistent and can be removed



- SCCs can be computed in O(|E|+|V|) time [Tarjan, 1972]
- consistent edges can be identified in O(|E|) time
- filtering in O(|E|) time

Note: SCCs correspond to 'tight' Hall sets *K*: $|K| = |\cup_{x \in K} D(x)|$

Important aspects

- Separation of consistency check ($O(\sqrt{|X|} \cdot |E|)$) and domain filtering ($O(|E|)$)
- Incremental algorithm
  - Maintain the graph structure during search
  - When k domain values have been removed, we can repair the matching in $O(km)$ time
  - Note that these algorithms are typically invoked many times during search / constraint propagation, so being incremental is very important in practice

Propagation algorithm for *knapsack*

M. A. Trick. A dynamic programming approach for consistency and propagation for knapsack constraints. *Annals of Operations Research* 118 (1-4):73-84, 2003.

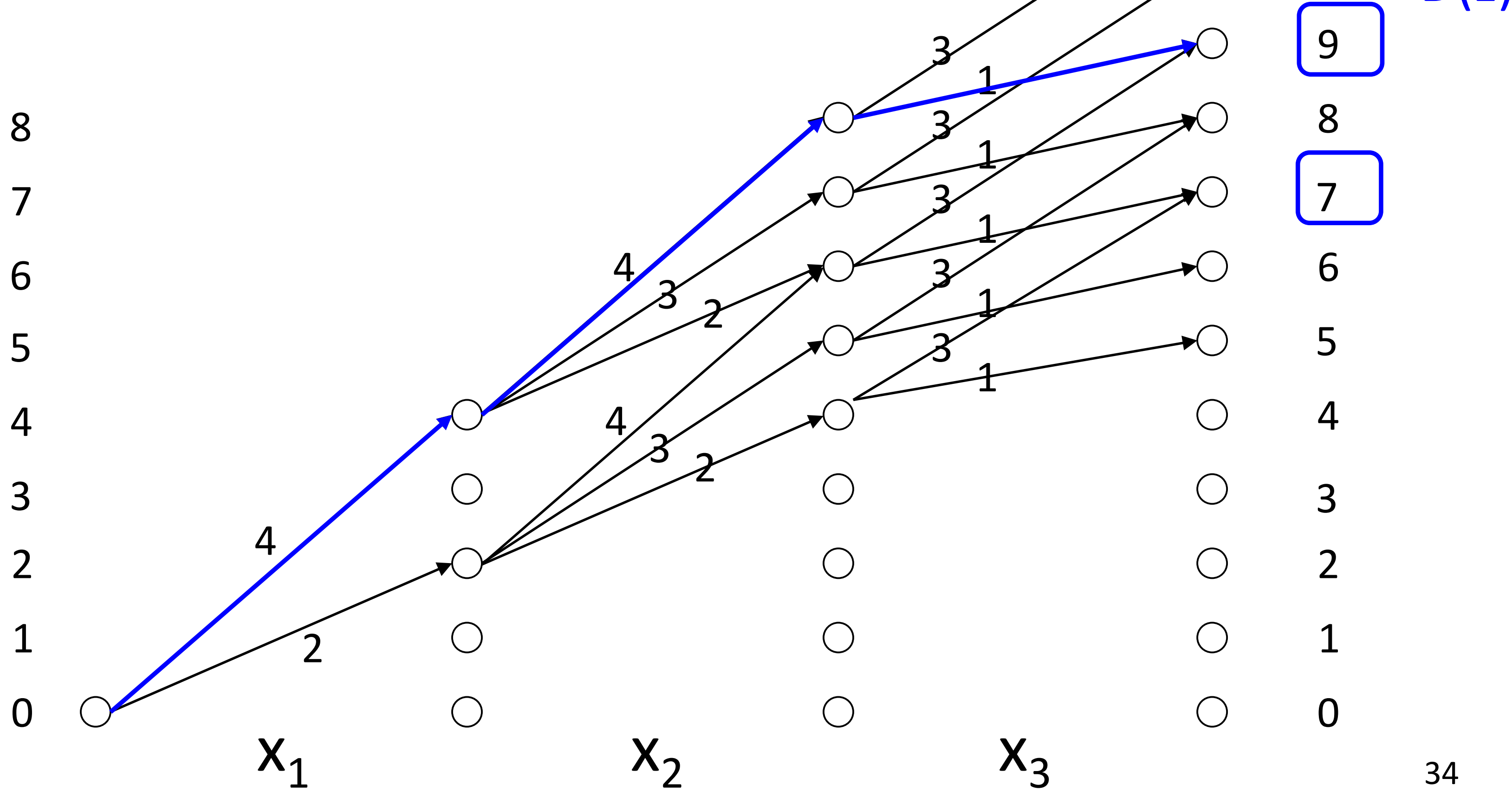
Knapsack constraints

- Knapsack constraints restrict a weighted linear sum to be no more than a given maximum:
  - Variables $X = \{x_1,\ldots,x_n\}$ with finite integer domains
  - Integer weights w_i ($i=1..n$)
  - Integer variable z representing the capacity
  - *Knapsack*(X, z, w) := $\sum_i w_i x_i \leq z$

Questions:

1. Can we determine in polynomial time whether the *knapsack* constraint is consistent (satisfiable)? NP-hard [Garey&Johnson, 1979]
2. Can we establish domain consistency (remove all inconsistent domain values) in polynomial time?

- Example:
  - $x_1 \in \{2,4\}$, $x_2 \in \{2,3,4\}$, $x_3 \in \{1, 3\}$, $z \in \{7,9,12\}$
  - unit weights ($w_i = 1$)
  - $\sum_i x_i \leq z$



Characterization *[Trick, 2003]*

Lemma: Any path in the graph from the origin to a goal state corresponds to a feasible solution to the knapsack constraint

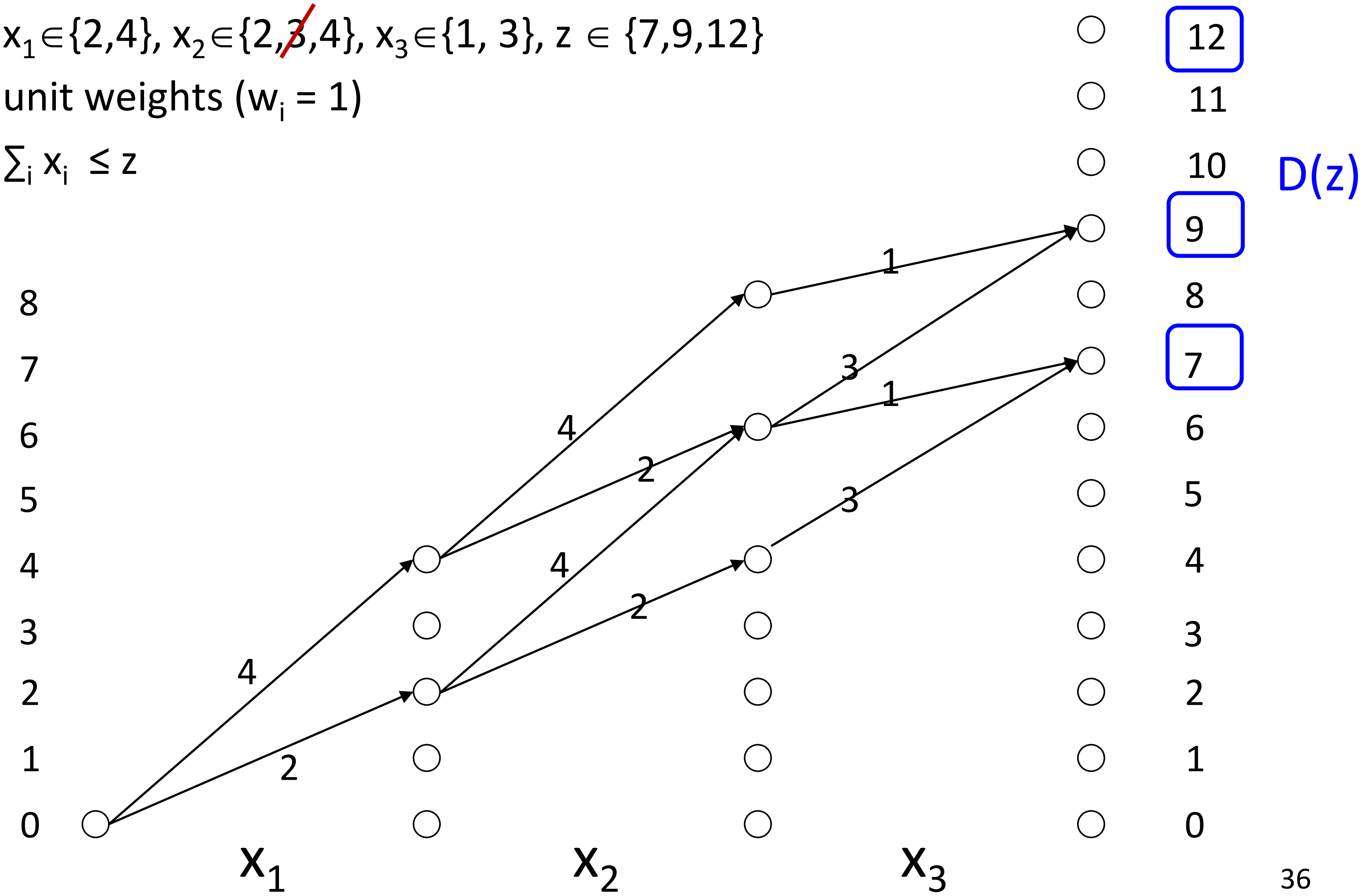
Lemma: If a variable x_i has no edge with label d in the graph, then d can be removed from $D(x_i)$ without affecting the set of solutions

Theorem: Domain consistency for *knapsack*:

remove all edges (and corresponding domain values) that are not in any path to a goal state

Filtering the graph and domains

- Example:
  - $x_1 \in \{2,4\}$, $x_2 \in \{2,\cancel{3},4\}$, $x_3 \in \{1, 3\}$, $z \in \{7,9,12\}$
  - unit weights ($w_i = 1$)
  - $\sum_i x_i \leq z$



Time complexity

- Filtering the graph takes linear time
  - one forward and one backward pass suffices to establish domain consistency
  - but size of graph depends on domain size: pseudo-polynomial time
  - no need to re-compute from scratch each time; we can maintain the graph incrementally

Propagation algorithm for *regular*

N. Beldiceanu, M. Carlsson, T. Petit. Deriving Filtering Algorithms from Constraint Checkers. In *Proceedings of CP*, pp. 107-122, 2004

G. Pesant. A Regular Language Membership Constraint for Finite Sequences of Variables. In *Proceedings of CP*, pp. 482-495, 2004.

Regular constraint

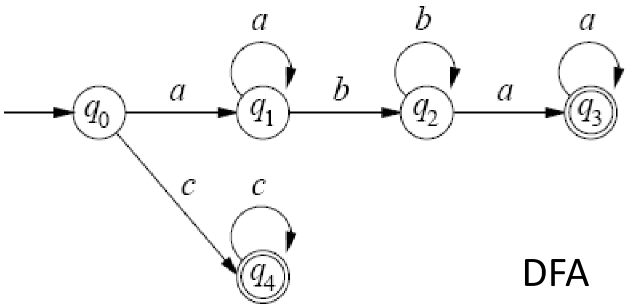
A regular language can be represented by a deterministic finite automaton (DFA):

automaton accepts string ⇔ string belongs to regular language

Example:

start state: q_0, end states: q_3 and q_4

each transition between states has a label



DFA

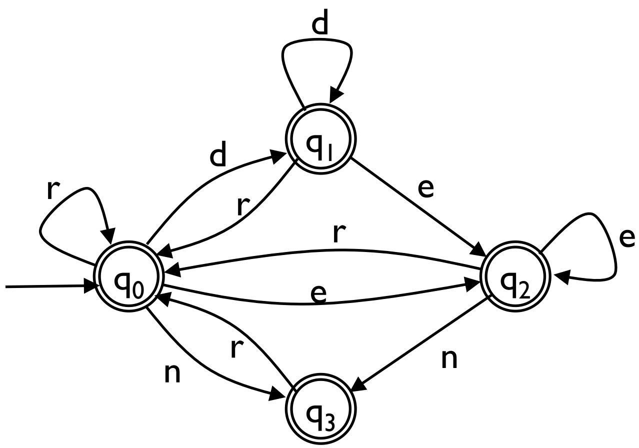
e.g. strings ‘aabbaa’ and ‘ccc’ accepted
string ‘caabbac’ not accepted

Given a DFA, the constraint regular(x_1,x_2,...,x_n, DFA) imposes that the ‘string’ $x_1x_2\cdots x_n$ is accepted by DFA (actually; NFA is also fine)

Regular constraint - application

Nurse rostering problem

- each nurse works at most one shift a day
- each shift contains 8 consecutive hours
  - day shift: 8am-4pm
  - evening shift: 4pm-12am
  - night shift: 12am-8am
- after a night shift, nurse needs to take one day rest
- after an evening shift, nurse may not work a day shift



Feasible (7-day) schedule: day - day - evening - night - rest - day - day

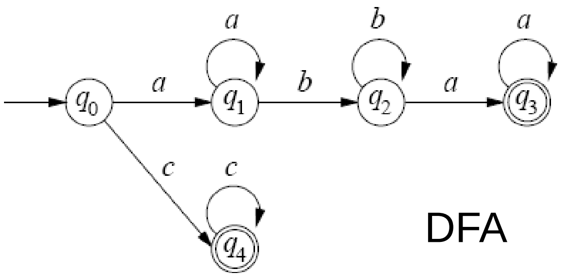
- For each nurse, introduce variables $X = \{x_1, x_2, \ldots, x_7\}$ representing shift on day 1,2,...,7 with domains $D(x) = \{r,d,e,n\}$ for all $x \in X$
- Model the requirements as regular(X, DFA) for each nurse

Propagation for the regular constraint

Theorem:

solution to regular ⇔ path from q_0 to 'goal state' in layered graph

Example:

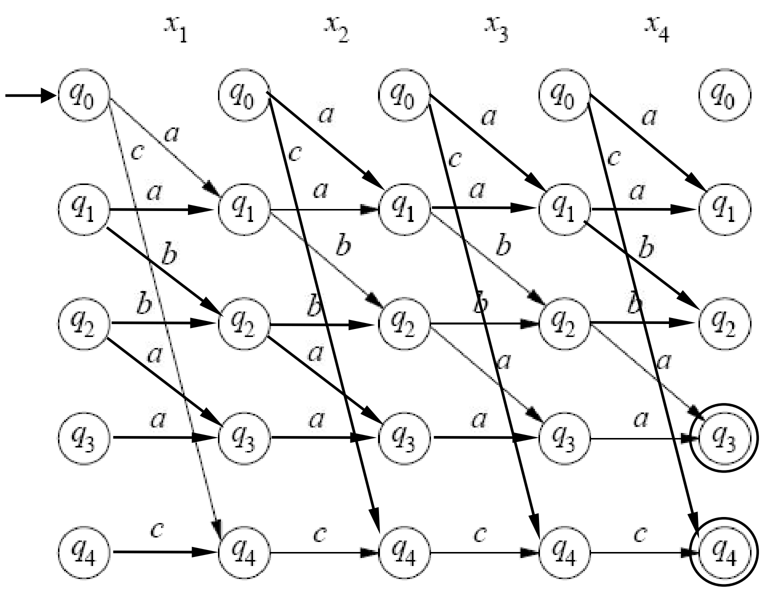


DFA

$x_1 \in \{a,b,c\}$, $x_2 \in \{a,b,c\}$,

$x_3 \in \{a,b,c\}$, $x_4 \in \{a,b,c\}$

regular(x_1,x_2,x_3,x_4,DFA)



Domain consistency: remove all arcs whose label is not supported by domain value and vice versa (linear time in size of graph)

Propagation for the regular constraint

Theorem:

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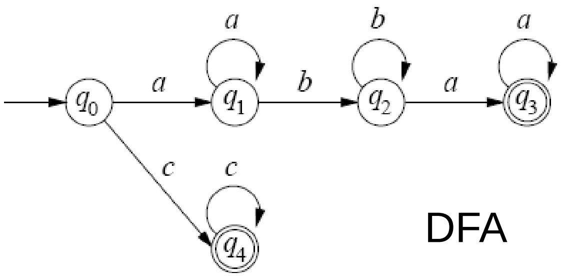
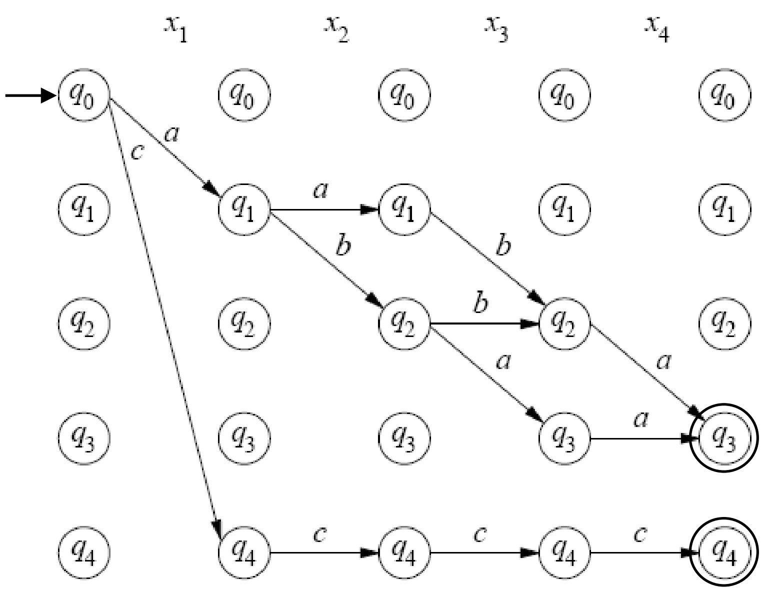


DFA

$x_1 \in \{a,\cancel{b},c\}$, $x_2 \in \{a,b,c\}$,

$x_3 \in \{a,b,c\}$, $x_4 \in \{a,\cancel{b},c\}$

regular(x_1,x_2,x_3,x_4,DFA)



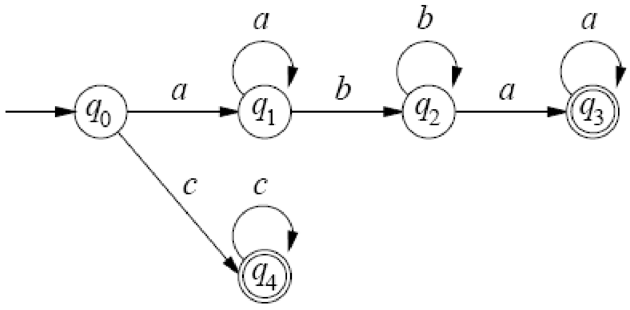
Domain consistency: remove all arcs whose label is not supported by domain value and vice versa (linear time in size of graph)

Reformulation [Beldiceanu et al., 2004]

- We can 'decompose' *regular* into separate transitions:

1. create a 'table' representing all possible transitions (edges)

$$T: \{ (q_0, a, q_1), (q_2, a, q_3), (q_1, a, q_1), (q_3, a, q_3), (q_1, b, q_2), (q_0, c, q_4), (q_2, b, q_2), (q_4, c, q_4) \}$$



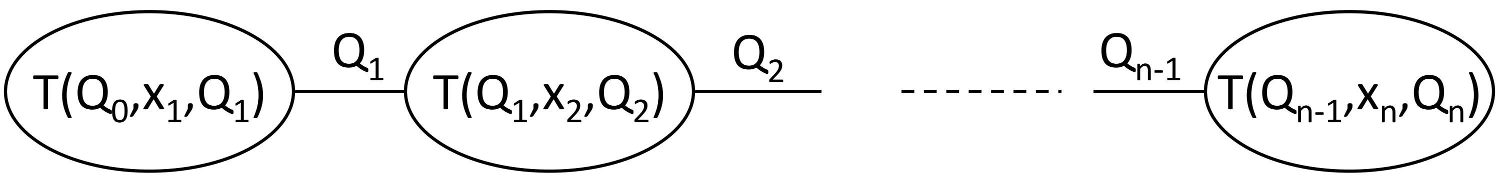
2. define 'state' variables Q_0, Q_1, Q_2, Q_3, Q_4, with
   $Q_0 \in \{q_0\}$, $Q_1, Q_2, Q_3 \in \{q_0, q_1, q_2, q_3, q_4\}$, $Q_4 \in \{q_3, q_4\}$

3. define transition constraint $T(Q_i, x_{i+1}, Q_{i+1})$ for i=0,1,2,3

Domain consistency via reformulation

- Theorem [Beldiceanu et al. 2004, 2005]:
  - establishing domain consistency on the reformulation is equivalent to establishing domain consistency on regular
  - the reformulation can be made domain consistent in O(n|T|) time (here |T| is number of transitions), which is the same as regular

  Proof: dual constraint graph is acyclic



- The reformulation is easier to implement, and can be more efficient than Pesant's algorithm in practice [Quimper&Walsh, 2006]

Research directions

- Not all 364 constraints in the catalog are equally useful
  - Most solvers only support a handful of constraints: alldifferent, cardinality, table constraints, constraints for scheduling
  - Unsupported global constraints are simply reformulated or decomposed
- Challenge seems not to be in creating new constraints, but into handling/utilizing existing constraints better

Direction 0: optimization

- By design, pure CP solvers are based on feasibility reasoning
  - relatively weak support for optimization (compared to e.g., MIP)

- Adapt global constraints for optimization
- Utilize known relaxations (linear programming, Lagrangian relaxations, …)
  - progress over last 10~15 years
  - this will be covered in other lectures (incl. Hybrid Methods on Thursday)

Direction 1: learn and automate

- Automate the process of identifying the ‘right’ global constraint to apply
  - ModelSeeker does this by learning constraints from example solutions [Beldiceanu&Simonis, 2012]
  - IBM ILOG CPO does this by grouping together specific constraints
- Learn no-goods during search
  - Record the implications from the propagation process
  - Explain search failure by identifying a minimal conflict set to be added as ‘no-good’ (e.g., Lazy Clause Generation)
  - Need to derive explanations from global constraints [Rochart et al., 2003-2005], [Downing et al., 2012]

Direction 2: guide search

- Use global constraints to dynamically define a good variable and value selection heuristic
  - Counting-based search: for each variable/value pair, count the number of solutions in which it appears
    [Pesant, Zanarini, et al., 2007-2013]
  - Two strategies: highest solution density first, or lowest solution density first
- Global constraints can also be used to guide local search methods
  - automatic definition of neighborhood or penalty function
    [Galinier & Hao, 2000, 2005], [Nareyek, 2001], [Michel & Van Hentenryck, 2002, 2005]

Direction 3: improve communication

- Current CP solvers are centered around *domain* propagation
  - In effect, very limited information is communicated between (global) constraints
- One approach is to study pairs (or more) of constraints
- Another approach is to propagate more structured information
  - precedence constraints in scheduling applications
  - constraints over structured domains such as set variables
  - for general CP: propagate *approximate decision diagrams* [Andersen et al., 2007], [Hadzic et al., 2007-2009], [Hoda et al., 2010], …

Summary

- Global constraints provide convenient building blocks for modeling and solving practical applications of optimization
- Constraint propagation is usually divided in two parts
  - consistency check
  - domain filtering

  (in some cases, domain consistency can be established in polynomial time)
- Global constraints embed efficient algorithms
  - some are adapted from known techniques: matchings, networks, dynamic programming, …
  - others are new, dedicated, algorithms

References

- J-C. Régin. Global Constraints: a survey. In *Hybrid Optimization*, M. Milano and P. Van Hentenryck (eds.), pp. 63-134. Springer, 2011.

- v.H. and I. Katriel. Global Constraints. Chapter 6 of F. Rossi, P. van Beek and T. Walsh (eds.), *Handbook of Constraint Programming*. Elsevier, 2006.