The Unfolding of Schematic Theories of Inductive Definitions

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1. Motivation of the Unfolding Program
2. Definition of the Unfoldings
3. Previous results in the Unfolding Program
4. Schematic Systems for Inductive Definitions
   - Lower bound
   - Upper bound
   - Further work
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By Gödel’s First Incompleteness Theorem, any recursive and consistent formal system capable of formalizing Peano Arithmetic is incomplete.
Incompleteness and Gödel’s program for new axioms

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By Gödel’s First Incompleteness Theorem, any recursive and consistent formal system capable of formalizing Peano Arithmetic is incomplete. So we need new axioms. Where could these come from:

- From new intuitions about the mathematical universe; or
- By expanding on what is already implicitly present in current axiomatizations.
Local reflection principle
For \( A \) a closed formula in the language of \( S \),

\[(\text{Rfn}_S) \quad \text{Prov}_S(\neg A) \to A.\]

*Remark:* \((\text{Rfn}_S)\) allows us to derive \( \text{Con}(S) \).

Uniform reflection principle
For \( A(x) \) a formula in the language of \( S \) with only \( x \) free,

\[(\text{RFN}_S) \quad \forall x. \text{Prov}_S(\neg A(x)) \to A(x).\]

*Remark:* \((\text{RFN}_S)\) is equivalent to formalized \( \omega \)-rule for number-theoretic systems.
We can iterate the reflections along constructive ordinals as in Turing ’39 (local reflection) and Feferman ’62 (uniform reflection). In this way, we obtain completeness results:

- for $\Pi^0_1$-sentences for iterated local reflections, and
- for arithmetical sentences for iterated uniform reflections.

**Problem:**

Depends on the choice of notations on $\mathcal{O}$, the choice of which cannot be justified on the basis of the starting system $S$.

**Answer 1:**

Restrict to autonomous progressions, trying to capture all principles of proof and ordinals which are implicit in given concepts (Kreisel ’70).

**Answer 2:**

Try to explicate the implicit mathematical content of the concepts of $S$ without any prima facie use of the notions of ordinal or well-ordering (Feferman ’79).
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Feferman, 1991, *Reflecting on Incompleteness*

The reflective closure $\text{Ref}(S)$ is formulated in terms of partial truth and falsity predicates $T$ and $F$ and axioms for *self-reflecting truth* (after Kripke).

The reflective closure $\text{Ref}^*(S)$ goes beyond that and allows reasoning about *schematic truth* as well.

\[
\text{Ref}(\text{PA}) \equiv \text{RA}_{<\varepsilon_0} \\
\text{Ref}^*(\text{PA}(U)) \equiv \text{RA}_{<\Gamma_0}
\]
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\]

*Problem:* Theories of partial self-applicative truth are perhaps not convincingly an implicit part of S.
Feferman, Gödel ’96, Gödel’s program for new axioms: why, where, how and what?
Feferman, Gödel ’96, *Gödel’s program for new axioms: why, where, how and what?*

My concern in the rest of this paper is to concentrate on the consideration of axioms which are supposed to be “exactly as evident” (Gödel 1946) as those already accepted.

...Given a schematic system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S?
Schematic systems, S, are formulated with a free predicate variable \( U \) accompanied by a rule of formula substitution

\[
\text{Subst) } \quad \frac{A(U)}{A(\{ x \mid B(x) \})}
\]

Examples of schematic systems

- Non-finitist arithmetic, NFA with

\[
U(0) \land (\forall x. U(x) \rightarrow U(x')) \rightarrow \forall x. U(x)
\]

- Zermelo’s set theory with

\[
\forall a. \exists b. \forall x. x \in b \iff x \in a \land U(x).
\]
1 Motivation of the Unfolding Program

2 Definition of the Unfoldings

3 Previous results in the Unfolding Program

4 Schematic Systems for Inductive Definitions
   - Lower bound
   - Upper bound
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Definition of the unfoldings

- $\mathcal{U}_0(S)$: operational unfolding
- $\mathcal{U}_1(S)$: intermediate unfolding
- $\mathcal{U}(S)$: full unfolding
The universe of the operational unfolding consists of the original sorts of $S$, embedding into a single sort of operations by means of predicates $V_S$.

Each $n$-ary function symbol $f$ of $S$ determines an total $n$-ary operation $f^*$ on the corresponding sorts.

Machinery to define new operations by recursion and explicit definition.

Each $n$-ary predicate symbol $R$ of $S$ determines a predicate $R^*$.

The axioms of $S$ are included, relativized to the $V_S$.

The logic of $\mathcal{U}_0(S)$ is the logic of partial terms.
We include in \( \mathcal{U}_0(S) \) the substitution rule:

\[
\frac{A(U)}{A(\{ x \mid B(x) \})} \quad (\text{SUBST})
\]
The language of $\mathcal{U}(S)$ extends the language of $\mathcal{U}_0(S)$ by additional constants for the predicate symbols of $S$ plus $\text{Eq}$, $\text{Pr}_U$, $\text{Inv}$, $\text{Neg}$, $\text{Conj}$, $\text{Un}$, $\text{Join}$.

**Remark**

Our formulation here allows us to focus on the rôle of the join operation. We get the intermediate unfolding by leaving out the join axioms.
- Eq $\downarrow \land \forall x, y. (x, y) \in \text{Eq} \iff x = y$.
- Pr$_U$ $\downarrow \land \forall x. x \in \text{Pr}_U \iff U(x)$.
- Inv$(X, f_1, \ldots, f_m) \downarrow \land$
  $\forall \bar{x} . \bar{x} \in \text{Inv}(X, f_1, \ldots, f_m) \iff (f_1(\bar{x}), \ldots, f_m(\bar{x})) \in X$.
- Neg$(X) \downarrow \land \forall \bar{x} . \bar{x} \in \text{Neg}(X) \iff \bar{x} \notin X$.
- Conj$(X, Y) \downarrow \land \forall \bar{x} . \bar{x} \in \text{Conj}(X, Y) \iff \bar{x} \in X \land \bar{x} \in Y$.
- Un$(X) \downarrow \land \forall \bar{x} . \bar{x} \in \text{Un}(X) \iff \forall y. (\bar{x}, y) \in X$.
- $\text{Eq} \downarrow \land \forall x, y. (x, y) \in \text{Eq} \iff x = y$.
- $\text{Pr}_U \downarrow \land \forall x. x \in \text{Pr}_U \iff U(x)$.
- $\text{Inv}(X, f_1, \ldots, f_m) \downarrow \land \forall \bar{x}. \bar{x} \in \text{Inv}(X, f_1, \ldots, f_m) \iff (f_1(\bar{x}), \ldots, f_m(\bar{x})) \in X$.
- $\text{Neg}(X) \downarrow \land \forall \bar{x}. \bar{x} \in \text{Neg}(X) \iff \bar{x} \notin X$.
- $\text{Conj}(X, Y) \downarrow \land \forall \bar{x}. \bar{x} \in \text{Conj}(X, Y) \iff \bar{x} \in X \land \bar{x} \in Y$.
- $\text{Un}(X) \downarrow \land \forall \bar{x}. \bar{x} \in \text{Un}(X) \iff \forall y. (\bar{x}, y) \in X$.
- For $f : \iota \rightarrow \pi_n$ and $r : \pi_1$ we take

$$
(\forall y. y \in r \rightarrow f(y) \downarrow) \rightarrow \text{Join}(f, r) \downarrow \land
\forall \bar{x}, y. (\bar{x}, y) \in \text{Join}(f, r) \iff y \in r \land \bar{x} \in f(y).
$$
For $\mathcal{U}(S)$ we restrict the substitution rule,

$$
\frac{A(U)}{A(\{ x \mid B(x) \})} \quad (\text{SUBST})
$$

by requiring $A$ to be in the language of $\mathcal{U}_0(S)$. This is needed, because the full unfolding language reflects the free predicate $U$. 
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Unfolding of non-finitist arithmetic

Theorem (Feferman and Strahm, 2000)

We have the following proof-theoretic equivalences

- $\mathcal{U}_0(\text{NFA}) \equiv \text{PA}$
- $\mathcal{U}_1(\text{NFA}) \equiv \text{RA}_{<\omega}$
- $\mathcal{U}(\text{NFA}) \equiv \text{RA}_{<\Gamma_0}$

In each case the systems prove the same arithmetical sentences.
The unfolding of finitist arithmetic

Theorem (Feferman and Strahm, 2010)

All three unfolding systems for finitist arithmetic, $U_0(FA)$, $U_1(FA)$, $U(FA)$, are proof-theoretically equivalent to Primitive Recursive Arithmetic, PRA.

Remark: This supports Tait’s analysis of finitism.

Theorem

All three unfolding systems for finitist arithmetic with the bar rule, $U_0(FA) C BR$, $U_1(FA) C BR$, $U(FA) C BR$, are proof-theoretically equivalent to Peano Arithmetic, PA.

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Theorem (Feferman and Strahm, 2010)

All three unfolding systems for finitist arithmetic, \( \mathcal{U}_0(FA) \), \( \mathcal{U}_1(FA) \), \( \mathcal{U}(FA) \), are proof-theoretically equivalent to Primitive Recursive Arithmetic, PRA.

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Theorem

All three unfolding systems for finitist arithmetic with the bar rule, $\mathcal{U}_0(FA + BR)$, $\mathcal{U}_1(FA + BR)$, $\mathcal{U}(FA + BR)$, are proof-theoretically equivalent to Peano Arithmetic, PA.
The unfolding of finitist arithmetic

**Theorem (Feferman and Strahm, 2010)**

*All three unfolding systems for finitist arithmetic, $\mathcal{U}_0(FA)$, $\mathcal{U}_1(FA)$, $\mathcal{U}(FA)$, are proof-theoretically equivalent to Primitive Recursive Arithmetic, PRA.*

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**Theorem**

*All three unfolding systems for finitist arithmetic with the bar rule, $\mathcal{U}_0(FA + BR)$, $\mathcal{U}_1(FA + BR)$, $\mathcal{U}(FA + BR)$, are proof-theoretically equivalent to Peano Arithmetic, PA.*

*Remark:* This supports Kreisel’s analysis of finitism.
Feasible arithmetic, FEA

A universe of binary words with concatenation and multiplication with logical operations $\land, \lor, \exists^\leq$ (bounded existential quantifier).

Theorem (Eberhard and Strahm, 2012)

The provably total functions of $\mathcal{U}_0$(FEA) and $\mathcal{U}$(FEA) are exactly the polynomial time computable functions.
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Just as the unfolding of non-finitist arithmetic captures predicative (given the natural numbers) mathematics, unfolding of systems of inductive definitions could potentially capture generalized predicative mathematics.
Formal system for one arithmetical inductive definition, $\text{ID}_1$

The language of $\text{ID}_1$, $\mathcal{L}^1$, is that of first-order arithmetic, PA, (with a free predicative variable $U$) augmented by a predicate symbol $I_A$ for an arithmetical positive operator form $A(X, x)$ that does not contain $U$.

The universal case is that of Kleene’s $\mathcal{O}$ consisting of canonical ordinal notations for recursive ordinals.
Schematic systems for Inductive Definitions

Number-theoretic axioms: The axioms of PA with exception of the induction scheme.

Schematic induction axiom on the natural numbers:

\[ U(0) \land (\forall x. U(x) \rightarrow U(x')) \rightarrow \forall x. U(x). \]

Schematic inductive definition axioms:

\[ \forall x. A(I_A, x) \rightarrow I_A(x) \quad \text{and} \quad (\forall x. A(U, x) \rightarrow U(x)) \rightarrow \forall x. I_A(x) \rightarrow U(x). \]

Substitution rule: For \( A \) and \( B(x) \) formulæ of ID\(_1\):

\[
\frac{A(U)}{A(\{ x \mid B(x) \})} \quad \text{(SUBST)}
\]
Theorem

We have $|\mathcal{U}(\text{ID}_1)| = \psi(\Gamma_{\Omega+1})$. 
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Remark

$\psi(\Gamma_{\Omega+1}) = \theta_\tau(0)$, where $\tau = \theta_\Omega(\Omega + 1) + 1$, is also known as Bachmann’s $H(1)$, or sometimes as *Aczel’s ordinal* because Aczel ’72 ‘proved’ it was the closure ordinal of ordinals expressible by iteration in a certain transfinite type hierarchy over $\Omega$. 
Aczel ‘72 proposed an extension of the finite type ordinal iterations as follows:

\[
\begin{align*}
\Omega^{(0)} &:= \Omega; \\
\overline{\Omega}(\alpha) &:= \prod_{\xi < \alpha} \Omega(\xi) \\
\Omega(\alpha) &:= \{ f : \overline{\Omega}(\alpha) \rightarrow \Omega^{(0)} \}
\end{align*}
\]

He proposed an iteration scheme, which however turned out to not quite work. Höwel ‘77 gave a better definition, making use of extra types, \( \overline{\Omega}(\alpha) = \alpha \rightarrow \Omega(\alpha) \).
In Höwels framework we may define a sequence of functionals as follows:

\[
\begin{align*}
H^\alpha_0 & := \text{Id}_\alpha, \\
H^\alpha_{\gamma+1} x & := \mathcal{E}_\nu. (H^\alpha_\gamma)^{1+\nu} x, \\
H^\alpha_\lambda x & := \sup_{\xi<\lambda} H^\alpha_\xi x.
\end{align*}
\]

Here, the construction \(\mathcal{E}_\nu. \theta(\nu)\) extracts the first fixed point a normal function \(\theta\), and in general we may set it equal to \(\theta^\omega(0)\). Höwel conjectures that the \(H\) functionals will express ordinals up to \(H(1) = \psi(\Gamma_{\Omega+1})\).
Strategy

The strategy for the lower bound proof consists of combining two elements

- A lower bound proof for $\text{ID}_1$ (recall that $|\text{ID}_1| = \psi(\varepsilon_{\Omega+1})$)
- The techniques for reaching a strongly critical ordinal using the predicate unfolding machinery from Feferman and Strahm ’00.
There is a simultaneous finitary inductive definitions of terms, $\alpha$, finite sets $K(\alpha)$ and an ordering $\alpha < \beta$.

Definition of $\text{SC} \subseteq H \subseteq \text{On}$:

- $\langle 0 \rangle \in \text{On}$ (denoting 0),
- $\langle 1 \rangle \in \text{SC}$ (denoting $\Omega$),
- if $n > 1$, $\alpha_1, \ldots, \alpha_n \in H$ and $\alpha_1 \geq \cdots \geq \alpha_n$, then $\langle 2, \alpha_1, \ldots, \alpha_n \rangle \in \text{On}$ (denoting $\alpha_1 + \cdots + \alpha_n$),
- $\alpha, \beta \in \text{On}$, then $\langle 3, \alpha, \beta \rangle \in H$ (denoting $\bar{\varphi}_{\alpha\beta}$).
- if $\alpha \in \text{On}$ and $K(\alpha) \subseteq \alpha$, then $\langle 4, \alpha \rangle \in \text{SC}$ (denoting $\psi(\alpha)$).
Definition of $K(\alpha)$:

$$
K(0) := \emptyset,
K(\Omega) := \emptyset,
K(\alpha_1 + \cdots + \alpha_n) := K(\alpha_1) \cup \cdots \cup K(\alpha_n),
K(\varphi_{\alpha \beta}) := K(\alpha) \cup K(\beta),
K(\psi(\alpha)) := \{ \alpha \} \cup K(\alpha).
$$
For $\alpha, \beta \in \mathbb{On}$, put $\alpha < \beta$ if one of the following conditions obtains:

- $\alpha = 0$ and $\beta \neq 0$,
- $\alpha = \alpha_1 + \cdots + \alpha_m$, $\beta = \beta_1 + \cdots + \beta_n$, and either
  - $m \geq n$ and $\exists i \leq n. \alpha_i < \beta_i \land \forall j < i. \alpha_j = \beta_j$, or
  - $m < n$ and $\forall i \leq m. \alpha_i = \beta_i$.
- $\alpha = \alpha_1 + \cdots + \alpha_n$, $\beta \in \mathbb{H}$, and $\alpha_1 < \beta$.
- $\alpha \in \mathbb{H}$, $\beta = \beta_1 + \cdots + \beta_n$, and $\alpha \leq \beta_1$.
- $\alpha = \bar{\varphi}\alpha_1 \alpha_2$, $\beta = \bar{\varphi}\beta_1 \beta_2$ and one of the following obtains
  - $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$.
  - $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.
  - $\beta_1 > \alpha_1$ and $\beta_2 \leq \alpha$.
- $\alpha = \bar{\varphi}\alpha_1 \alpha_2$, $\beta \in \mathbb{SC}$, and $\alpha_1, \alpha_2 < \beta$.
- $\alpha \in \mathbb{SC}$, $\beta = \bar{\varphi}\beta_1 \beta_2$, and $\alpha \leq \beta_1$ or $\alpha \leq \beta_2$.
- $\alpha = \psi(\alpha_1)$, $\beta = \psi(\beta_1)$ and $\alpha_1 < \beta_1$.
- $\alpha = \psi(\alpha_1)$ and $\beta = \Omega$. 
Definition

Let $\mathcal{A}_{\text{cc}}$ be inductively defined as the accessible elements of the notation system below $\Omega$, let $M := \{ \alpha \mid \text{SC}(\alpha) \cap \Omega \subseteq \mathcal{A}_{\text{cc}} \}$ and define

$$\alpha <_{1} \beta : \iff \alpha < \beta \land \alpha \in M \land \beta \in M.$$  

Lemma

*The class $\mathcal{A}_{\text{cc}}$ is closed under all the parts of the notation system that are “from below”, i.e., 0, +, $\bar{\varphi}$.***
Lemma

- $\mathsf{ID}_1 \vdash \mathsf{TI}_1(\Omega + 1, U) \land \mathsf{K}(\Omega + 1) \subseteq \Omega + 1 \land \Omega + 1 \in M$.
- If $\mathsf{ID}_1 \vdash \mathsf{TI}_1(\alpha, U) \land \mathsf{K}(\alpha) \subseteq \alpha \land \alpha \in M$, then
  $\mathsf{ID}_1 \vdash \mathsf{TI}_1(\omega^\alpha, U) \land \mathsf{K}(\omega^\alpha) \subseteq \omega^\alpha \land \omega^\alpha \in M$.

Lemma

If $\mathsf{ID}_1 \vdash \mathsf{TI}_1(\alpha, U) \land \mathsf{K}(\alpha) \subseteq \alpha \land \alpha \in M$, then $\mathsf{ID}_1 \vdash \psi(\alpha) \in \mathcal{A}_\mathsf{cc}$.

Corollary

For any $\alpha < \psi(\varepsilon_{\Omega+1})$, $\mathsf{ID}_1 \vdash \mathsf{TI}(\alpha, U)$. 
Let $A(X, \alpha, x)$ be a formula of ID$_1$ with at most $X, \alpha, x$ free. We wish to define segments (in terms of the $<_1$-relation) of the $A$ jump hierarchy starting with $U$, given set-theoretically by the transfinite recursion

$$
Y_0 := \{ x \mid U(x) \}, \\
Y_\alpha := \{ x \mid A(Y^\alpha, \alpha, x) \}
$$

where $Y^\alpha := \{ (\beta, m) \mid \beta <_1 \alpha \wedge m \in Y_\beta \}$.

Define a term $\text{hier}_A : (\iota \to \pi_1)$ in $\mathcal{U}(\text{ID}_1)$ by

$$
\text{hier}_A := \text{LFP}\left( \lambda f, \alpha. \{ \text{if } \alpha = 0 \text{ then } \text{Pr}_U \text{ else } r_A(\text{Join}(f, (<_1 \alpha)), \alpha) \} \right).
$$

Note that we really need the dependent version of the join operation.
Lemma

If $\mathcal{U}(\text{ID}_1) \models \text{TI}_1(\alpha, U)$, then $\mathcal{U}(\text{ID}_1) \models \forall \beta <_1 \alpha. \text{hier}_A(\beta) \downarrow$.

By a clever choice of $A$ (following Feferman and Schütte), we obtain:

Lemma

If $\mathcal{U}(\text{ID}_1) \models \text{TI}_1(\alpha, U) \land K(\alpha) \subseteq \alpha \land \alpha \in M$, then $\mathcal{U}(\text{ID}_1) \models \text{TI}_1(\varphi_\alpha(0), U) \land K(\varphi_\alpha(0)) \subseteq \varphi_\alpha(0) \land \varphi_\alpha(0) \in M$.

Corollary

For any $\alpha < \psi(\Gamma_{\Omega+1})$, $\mathcal{U}(\text{ID}_1) \models \text{TI}(\alpha, U)$. 

The strategy for the upper bound is:

- Embed $\mathcal{U}(\text{ID}_1)$ in an intermediate system $(\text{ID}_1)_{\Omega}^+ + (\text{SUBST})$ (in analogy with Feferman and Strahm ’00).
- Interpret $(\text{ID}_1)_{\Omega}^+ + (\text{SUBST})$ in infinitary calculus for ramified set theory with classes.
- Extract the upper bound using cut-elimination and asymmetric interpretation for the infinitary system.
The intermediate system

We introduce a theory \((\text{ID}_1)_\Omega^+ + (\text{SUBST})\), analogous to the system \(\text{PA}_\Omega^+ + (\text{SUBST})\) from Strahm ’00. 

\((\text{ID}_1)_\Omega^+ + (\text{SUBST})\) is formulated in the language \(\mathcal{L}_\Omega^1\), which is obtained from the language of \(\text{ID}_1\), \(\mathcal{L}_1\), by

- adding a new sort for ordinal variables (with \(<\) and \(=\) relations), and
- an \((n + 1)\)-ary predicate symbol \(P_{\mathcal{L}}\) for each inductive operator form \(\mathcal{A}(X, \bar{x})\) over \(\text{ID}_1\)

(that is, \(\mathcal{A}\) is an \(\mathcal{L}_1\)-formula so it can contain \(U\) and both positive and negative occurrences of \(I_\emptyset\), but of course only positive occurrences of the fresh \(n\)-ary predicate variable \(X\)).

As a matter of notation we write \(P_{\mathcal{L}}^\alpha(\bar{x})\) instead of \(P_{\mathcal{L}}(\alpha, \bar{x})\), and we put \(P_{\mathcal{L}}^{<\alpha}(\bar{x}) := \exists \beta < \alpha. P_{\mathcal{L}}^\beta(\bar{x})\).
We axiomatize \((\text{ID}_1)_{\Omega}^{+} + (\text{SUBST})\) by:

- Number-theoretic axioms
- Schematic induction on the natural numbers
- Schematic induction and closure of the arithmetical inductive definition.
- Inductive operator axioms:

\[
P_{\Omega}^{\sigma}(\bar{x}) \leftrightarrow \exists(P_{\Omega}^{\sigma}, \bar{x}).
\]

- Linearity axioms for the ordinals.
- \(\Sigma\)-reflection scheme on the ordinal sort.
- \(\Sigma\)-induction scheme on the ordinal sort.
- Substitution rule: For \(A\) an \(\mathcal{L}^1\)-formula, and \(B(x)\) an \(\mathcal{L}^1_{\Omega}\)-formula:

\[
\frac{A(U)}{A(\{ x \mid B(x) \})} \quad \text{(SUBST)}
\]
We embed $\mathcal{U}(ID_1)$ into $(ID_1)^+_{\Omega} + (\text{SUBST})$ by

- Interpreting each partial operation by a code for a partial recursive function.
- Writing an inductive operator form that simultaneously defines:
  - A collection $\mathcal{P}$ of (non-unique) codes of predicates of the unfolding.
  - A complimentary pair of relations $\in$ and $\bar{\in}$ that determine the extension for each such code.

(The dependent join operator causes $\mathcal{P}$ to depend on $\bar{\in}$, for example.)

- Note that the substitution rule of $(ID_1)^+_{\Omega} + (\text{SUBST})$ interprets the substitution rule of $\mathcal{U}(ID_1)$. 

Alternatives for bounding the intermediate system?

- Work in $\mathcal{L}_\infty$: we would need to carve out a suitable fragment that would allow locally predicative cut-elimination, and it is not clear how to do this in the case of $(\text{ID}_1)^+_\Omega + (\text{SUBST})$ because of the need to do substitution.

- Work in systems of numbers and ordinals: we would need to simultaneously define $\Theta$ and the $P_{\Omega}$ by one (non-monotone) operator, but then substitution would be hard to model: the stages before $\Omega$ are $U$-independent, while those after are not, but they need to interact during substitution.
Introduce two infinitary calculi $T^r$ and $T$ of operator-controlled derivations for ramified set theory with class variables.

- $T^r$ enjoys the usual (locally) predicative cut elimination and collapsing theorems.
- $T$ enjoys partial cut elimination over the $\Sigma \cup \Pi$-fragment.

We can substitute a general formula for a class variable $X$ in a $T^r$-derivation to obtain a $T$-derivation.

There is an asymmetric interpretation of the $\Sigma \cup \Pi$-fragment from $T$ to $T^r$.

This enables the upper bound proof for $(ID_1)^+_{\Omega} + (\text{SUBST})$ by interpreting the stages of $P_{\aleph_1}$ using the constructible hierarchy.
Asymmetric Interpretation and Substitution

\( T' \)
(restricted system with no free set variables)

\( T \)
(derived system with free set variables and \( \Sigma \cup \Pi \) axioms and rules)

substitution

asymmetric interpretation
**Lemma (Substitution lemma for T' into T)**

Let $\Gamma(X)$ be a finite set of $\mathcal{L}_{RS}^r$-formulæ and let $B(x)$ be a formula of $\mathcal{L}_{RS}$. Assume $T^r \left[ \mathcal{H} \right]^{\alpha}_{\rho} \Gamma(X)$ for some infinite ordinal $\alpha$. Then

$$T \left[ \mathcal{H} [\text{par } B] \right]^{\alpha}_{<\omega} \Gamma(\{ x \mid B(x) \})$$

**Theorem (Asymmetric interpretation of T into T')**

Assume $\Gamma$ is a finite set of $\Sigma \cup \Pi$-formulæ of $T$ so that $T \left[ \mathcal{H} \right]^{\alpha}_{1} \Gamma$. Then we have for all limit ordinals $\beta \geq \Omega$ and every $(\beta, \varphi\alpha(\beta + \beta))$-instance $\Lambda$ of $\Gamma$ for which $\text{par}(\Lambda) \cup \{ \beta \} \subseteq \mathcal{H}$ that

$$T^r \left[ \mathcal{H} \right]^{\varphi\alpha(\beta + \beta)}_{\varphi\alpha(\beta + \beta)} \Lambda.$$
Theorem (Reduction of $(\text{ID}_1)_\Omega^+ + (\text{SUBST})$)

Let $C$ be a formula of $\mathcal{L}_\Omega^1$, and let $A$ be a closed formula of $\mathcal{L}_1$. Then we have for all natural numbers $n$, and all acceptable operators $\mathcal{H}$ closed under $\eta \mapsto \eta^+$:

1. $(\text{ID}_1)_\Omega^+ + (\text{SUBST})^{\leq n} \vdash C \quad \rightarrow \quad T\left[\mathcal{H}\right]^{<\xi_{2n+1}}_{\xi_{2n+1}} C^*$.

2. $(\text{ID}_1)_\Omega^+ + (\text{SUBST})^{\leq n} \vdash A \quad \rightarrow \quad T^r\left[\mathcal{H}\right]^{<\xi_{2n+1}}_{\xi_{2n+1}} A^*$.
Theorem (Boundedness theorem for $T^r$)

Let $T^r \left[ \mathcal{H} \right]_{\frac{\alpha}{\rho}} \Delta$, $F^{L\Omega}$ with $F^{L\Omega}$ a $\Sigma^\Omega_1$-sentence. Then

$T^r \left[ \mathcal{H} \right]_{\frac{\alpha}{\rho}} \Delta$, $F^{L\beta}$ for all $\beta \in [\alpha, \Omega) \cap \mathcal{H}$.

Theorem (Collapsing Theorem for $T^r$)

Let $\Delta$ be a set of $\Sigma^\Omega_1$-sentences of $L^r_{RS}$, and assume

$T^r \left[ \mathcal{H}_0 \right]_{\frac{\alpha}{\Omega+1}} \Delta$. Then

$T^r \left[ \mathcal{H}_{\omega^{\Omega+1+\alpha}} \right]_{\psi(\omega^{\Omega+1+\alpha})}^{\psi(\omega^{\Omega+1+\alpha})} \Delta$. 
Theorem (Soundness for $T^r$)

If $T^r \left[ \mathcal{H} \right] \frac{\alpha}{0} \Delta$, then $L \models \frac{\alpha}{\Delta}$.

Corollary

$|\left( ID_1 \right)_{\Omega}^+ + (\text{SUBST})| \leq \psi(\Gamma_{\Omega+1})$. 

Soundness and boundedness
Further work

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Give a general method for studying mixed iterations of fixed points and least fixed points (both involve asymmetric interpretation, but in different ways).
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- The unfolding of schematic Zermelo—and Zermelo-Fraenkel—set theory.
Questions or Comments?