The Unfolding of Schematic Theories of Inductive Definitions

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Outline

1. Motivation of the Unfolding Program
2. Definition of the Unfoldings
3. Previous results in the Unfolding Program
4. Schematic Systems for Inductive Definitions
    - Lower bound
    - Upper bound
    - Further work

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- From new intuitions about the mathematical universe; or
- By expanding on what is already implicitly present in current axiomatizations.

Local reflection principle

For A a closed formula in the language of S,

(Rfn$_S$) $$\mathrm{Prov}_S(\ulcorner A \urcorner) \to A.$$

Remark: (Rfn$_S$) allows us to derive $\mathrm{Con}(S)$.

Uniform reflection principle

For $A(x)$ a formula in the language of S with only x free,

(RFN$_S$) $$\forall x. \mathrm{Prov}_S(\ulcorner A(\dot{x}) \urcorner) \to A(x).$$

Remark: (RFN$_S$) is equivalent to formalized ω-rule for number-theoretic systems.

Iterated reflection principles

We can iterate the reflections along constructive ordinals as in Turing '39 (local reflection) and Feferman '62 (uniform reflection). In this way, we obtain completeness results:

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Answer 1: Restrict to autonomous progressions, trying to capture *all principles of proof and ordinals which are implicit in given concepts* (Kreisel '70).
Answer 2: Try to explicate the implicit mathematical content of the concepts of S without any prima facie use of the notions of ordinal or well-ordering (Feferman '79).

Feferman, 1991, *Reflecting on Incompleteness*

The reflective closure $\mathrm{Ref}(\mathsf{S})$ is formulated in terms of partial truth and falsity predicates T and F and axioms for *self-reflecting truth* (after Kripke).
The reflective closure $\mathrm{Ref}^*(\mathsf{S})$ goes beyond that and allows reasoning about *schematic truth* as well.

$$\mathrm{Ref}(\mathsf{PA}) \equiv \mathsf{RA}_{<\varepsilon_0}$$
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$$\mathrm{Ref}^*(\mathsf{PA}(U)) \equiv \mathsf{RA}_{<\Gamma_0}$$

Problem: Theories of partial self-applicative truth are perhaps not convincingly an implicit part of S.

Genesis of the Unfolding Program

Feferman, Gödel '96, *Gödel's program for new axioms: why, where, how and what?*

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My concern in the rest of this paper is to concentrate on the consideration of axioms which are supposed to be "exactly as evident" (Gödel 1946) as those already accepted.

...

Given a schematic system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S?

Schematic systems, S, are formulated with a free predicate variable U accompanied by a rule of formula substitution

(Subst)

$$\frac{A(U)}{A(\{ x \mid B(x) \})}$$

Examples of schematic systems

- Non-finitist arithmetic, NFA with

$$U(0) \wedge (\forall x.\, U(x) \to U(x')) \to \forall x.\, U(x)$$

- Zermelo's set theory with

$$\forall a.\, \exists b.\, \forall x.\, x \in b \leftrightarrow x \in a \wedge U(x).$$

Outline

- $\mathcal{U}_0(\mathsf{S})$: operational unfolding
- $\mathcal{U}_1(\mathsf{S})$: intermediate unfolding
- $\mathcal{U}(\mathsf{S})$: full unfolding

Operational unfolding

- The universe of the operational unfolding consists of the original sorts of S, embedding into a single sort of operations by means of predicates V_s.
- Each n-ary function symbol f of S determines an total n-ary operation f^* on the corresponding sorts.
- Machinery to define new operations by recursion and explicit definition.
- Each n-ary predicate symbol R of S determines a predicate R^*.
- The axioms of S are included, relativized to the V_s.
- The logic of $\mathcal{U}_0(\mathrm{S})$ is the *logic of partial terms*.

We include in $\mathcal{U}_{\mathbf{0}}(\mathsf{S})$ the substitution rule:

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})} \ (\textsc{Subst})$$

- The language of $\mathcal{U}(\mathsf{S})$ extends the language of $\mathcal{U}_0(\mathsf{S})$ by additional constants for the predicate symbols of S plus $\mathrm{Eq}, \mathrm{Pr}_U, \mathrm{Inv}, \mathrm{Neg}, \mathrm{Conj}, \mathrm{Un}, \mathrm{Join}$.

Remark

Our formulation here allows us to focus on the rôle of the join operation. We get the intermediate unfolding by leaving out the join axioms.

- $\mathrm{Eq}\downarrow \wedge \forall x, y.\, (x, y) \in \mathrm{Eq} \leftrightarrow x = y.$
- $\mathrm{Pr}_U\downarrow \wedge \forall x.\, x \in \mathrm{Pr}_U \leftrightarrow U(x).$
- $\mathrm{Inv}(X, f_1, \ldots, f_m)\downarrow \wedge$
  $\forall \vec{x}.\, \vec{x} \in \mathrm{Inv}(X, f_1, \ldots, f_m) \leftrightarrow (f_1(\vec{x}), \ldots, f_m(\vec{x})) \in X.$
- $\mathrm{Neg}(X)\downarrow \wedge \forall \vec{x}.\, \vec{x} \in \mathrm{Neg}(X) \leftrightarrow \vec{x} \notin X.$
- $\mathrm{Conj}(X, Y)\downarrow \wedge \forall \vec{x}.\, \vec{x} \in \mathrm{Conj}(X, Y) \leftrightarrow \vec{x} \in X \wedge \vec{x} \in Y.$
- $\mathrm{Un}(X)\downarrow \wedge \forall \vec{x}.\, \vec{x} \in \mathrm{Un}(X) \leftrightarrow \forall y.\, (\vec{x}, y) \in X.$

Full unfolding, axioms

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- $\mathrm{Un}(X)\downarrow \wedge \forall \vec{x}.\, \vec{x} \in \mathrm{Un}(X) \leftrightarrow \forall y.\,(\vec{x}, y) \in X.$
- For $f : \iota \rightharpoonup \pi_n$ and $r : \pi_1$ we take

$$(\forall y.\, y \in r \to f(y)\downarrow) \to \mathrm{Join}(f, r)\downarrow \wedge$$
$$\forall \vec{x}, y.\,(\vec{x}, y) \in \mathrm{Join}(f, r) \leftrightarrow y \in r \wedge \vec{x} \in f(y).$$

For $\mathcal{U}(\mathrm{S})$ we restrict the substitution rule,

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})} \text{ (SUBST)} ,$$

by requiring A to be in the language of $\mathcal{U}_0(\mathrm{S})$. This is needed, because the full unfolding language reflects the free predicate U.

Outline

Theorem (Feferman and Strahm, 2000)

We have the following proof-theoretic equivalences

- $\mathcal{U}_0(\mathsf{NFA}) \equiv \mathsf{PA}$
- $\mathcal{U}_1(\mathsf{NFA}) \equiv \mathsf{RA}_{<\omega}$
- $\mathcal{U}(\mathsf{NFA}) \equiv \mathsf{RA}_{<\Gamma_0}$

In each case the systems prove the same arithmetical sentences.

Theorem (Feferman and Strahm, 2010)

All three unfolding systems for finitist arithmetic, $\mathcal{U}_0(\mathsf{FA}), \mathcal{U}_1(\mathsf{FA}), \mathcal{U}(\mathsf{FA})$, *are proof-theoretically equivalent to Primitive Recursive Arithmetic,* PRA.

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All three unfolding systems for finitist arithmetic with the bar rule, $\mathcal{U}_0(\mathsf{FA}+\mathsf{BR}), \mathcal{U}_1(\mathsf{FA}+\mathsf{BR}), \mathcal{U}(\mathsf{FA}+\mathsf{BR})$*, are proof-theoretically equivalent to Peano Arithmetic,* PA*.*

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Feasible arithmetic, FEA

A universe of binary words with concatenation and multiplication with logical operations $\wedge, \vee, \exists^{\leq}$ (bounded existential quantifier).

Theorem (Eberhard and Strahm, 2012)

The provably total functions of $\mathcal{U}_0(\mathsf{FEA})$ *and* $\mathcal{U}(\mathsf{FEA})$ *are exactly the polynomial time computable functions.*

Outline

Why study systems of Inductive Definitions

- Just as the unfolding of non-finitist arithmetic captures predicative (given the natural numbers) mathematics, unfolding of systems of inductive definitions could potentially capture *generalized* predicative mathematics.

Formal system for one arithmetical inductive definition, ID_1

The language of ID_1, $\mathcal{L}^1$, is that of first-order arithmetic, PA, (with a free predicative variable U) augmented by a predicate symbol $I_{\mathcal{A}}$ for an arithmetical positive operator form $\mathcal{A}(X, x)$ that does not contain U.

The universal case is that of Kleene's $\mathcal{O}$ consisting of canonical ordinal notations for recursive ordinals.

Schematic systems for Inductive Definitions

Number-theoretic axioms: The axioms of PA with exception of the induction scheme.

Schematic induction axiom on the natural numbers:

$$U(0) \wedge (\forall x.\, U(x) \to U(x')) \to \forall x.\, U(x).$$

Schematic inductive definition axioms:

$$\forall x.\, \mathcal{A}(I_{\mathcal{A}}, x) \to I_{\mathcal{A}}(x) \quad \text{and}$$
$$(\forall x.\, \mathcal{A}(U, x) \to U(x)) \to \forall x.\, I_{\mathcal{A}}(x) \to U(x).$$

Substitution rule: For A and $B(x)$ formulæ of ID_1:

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})}\ (\textsc{Subst})$$

Theorem

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Remark

$\psi(\Gamma_{\Omega+1}) = \theta_\tau(0)$, where $\tau = \theta_\Omega(\Omega + 1) + 1$, is also known as Bachmann's $H(1)$, or sometimes as *Aczel's ordinal* because Aczel '72 'proved' it was the closure ordinal of ordinals expressible by iteration in a certain transfinite type hierarchy over Ω.

Aczel '72 proposed an extension of the finite type ordinal iterations as follows:

$$\Omega^{(0)} := \Omega;$$
$$\overline{\Omega}^{(\alpha)} := \prod_{\xi<\alpha} \Omega^{(\xi)}$$
$$\Omega^{(\alpha)} := \{ f : \overline{\Omega}^{(\alpha)} \to \Omega^{(0)} \}$$

He proposed an iteration scheme, which however turned out to not quite work. Höwel '77 gave a better definition, making use of extra types, $\widetilde{\Omega}^{(\alpha)} = \alpha \to \Omega^{(\alpha)}$.

In Höwels framework we may define a sequence of functionals as follows:

$$\begin{aligned} \mathrm{H}^{\alpha}_{0} &:= \mathrm{Id}_{\alpha}, \\ \mathrm{H}^{\alpha}_{\gamma+1}\, x &:= \mathcal{E}\nu.\, (\tilde{\mathrm{H}}^{\alpha}_{\gamma})^{1+\nu}\, x, \\ \mathrm{H}^{\alpha}_{\lambda}\, x &:= \sup_{\xi<\lambda} \mathrm{H}^{\alpha}_{\xi}\, x. \end{aligned}$$

Here, the construction $\mathcal{E}\nu.\,\theta(\nu)$ extracts the first fixed point a normal function θ, and in general we may set it equal to $\theta^{\omega}(0)$.
Höwel conjectures that the H functionals will express ordinals up to $\mathrm{H}(1) = \psi(\Gamma_{\Omega+1})$.

Strategy

The strategy for the lower bound proof consists of combining two elements

- A lower bound proof for ID_1 (recall that $|\mathsf{ID}_1| = \psi(\varepsilon_{\Omega+1})$)
- The techniques for reaching a strongly critical ordinal using the predicate unfolding machinery from Feferman and Strahm '00.

There is a simultaneous finitary inductive definitions of terms, α, finite sets $\mathrm{K}(\alpha)$ and an ordering $\alpha < \beta$.

Definition of $\mathrm{SC} \subseteq \mathrm{H} \subseteq \mathrm{On}$:

- $\langle 0 \rangle \in \mathrm{On}$ (denoting 0),
- $\langle 1 \rangle \in \mathrm{SC}$ (denoting Ω),
- if $n > 1, \alpha_1, \ldots, \alpha_n \in \mathrm{H}$ and $\alpha_1 \geq \cdots \geq \alpha_n$, then $\langle 2, \alpha_1, \ldots \alpha_n \rangle \in \mathrm{On}$ (denoting $\alpha_1 + \cdots + \alpha_n$),
- $\alpha, \beta \in \mathrm{On}$, then $\langle 3, \alpha, \beta \rangle \in \mathrm{H}$ (denoting $\bar{\varphi}_\alpha \beta$).
- if $\alpha \in \mathrm{On}$ and $\mathrm{K}(\alpha) \subseteq \alpha$, then $\langle 4, \alpha \rangle \in \mathrm{SC}$ (denoting $\psi(\alpha)$).

Definition of $\mathrm{K}(\alpha)$:

$$\begin{aligned} \mathrm{K}(0) &:= \emptyset, \\ \mathrm{K}(\Omega) &:= \emptyset, \\ \mathrm{K}(\alpha_1 + \cdots + \alpha_n) &:= \mathrm{K}(\alpha_1) \cup \cdots \cup \mathrm{K}(\alpha_n), \\ \mathrm{K}(\bar{\varphi}_\alpha \beta) &:= \mathrm{K}(\alpha) \cup \mathrm{K}(\beta), \\ \mathrm{K}(\psi(\alpha)) &:= \{\, \alpha \,\} \cup \mathrm{K}(\alpha). \end{aligned}$$

For $\alpha, \beta \in \mathrm{On}$, put $\alpha < \beta$ if one of the following conditions obtains:

- $\alpha = 0$ and $\beta \neq 0$,
- $\alpha = \alpha_1 + \cdots + \alpha_m$, $\beta = \beta_1 + \cdots + \beta_n$, and either
  - $m \geq n$ and $\exists i \leq n . \alpha_i < \beta_i \wedge \forall j < i . \alpha_j = \beta_j$, or
  - $m < n$ and $\forall i \leq m . \alpha_i = \beta_i$.
- $\alpha = \alpha_1 + \cdots + \alpha_n$, $\beta \in \mathrm{H}$, and $\alpha_1 < \beta$.
- $\alpha \in \mathrm{H}$, $\beta = \beta_1 + \cdots + \beta_n$, and $\alpha \leq \beta_1$.
- $\alpha = \bar{\varphi}_{\alpha_1}\alpha_2$, $\beta = \bar{\varphi}_{\beta_1}\beta_2$ and one of the following obtains
  - $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$.
  - $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.
  - $\beta_1 > \alpha_1$ and $\beta_2 \leq \alpha$.
- $\alpha = \bar{\varphi}_{\alpha_1}\alpha_2$, $\beta \in \mathrm{SC}$, and $\alpha_1, \alpha_2 < \beta$.
- $\alpha \in \mathrm{SC}$, $\beta = \bar{\varphi}_{\beta_1}\beta_2$, and $\alpha \leq \beta_1$ or $\alpha \leq \beta_2$.
- $\alpha = \psi(\alpha_1)$, $\beta = \psi(\beta_1)$ and $\alpha_1 < \beta_1$.
- $\alpha = \psi(\alpha_1)$ and $\beta = \Omega$.

Definition

Let $\mathcal{A}cc$ be inductively defined as the accessible elements of the notation system below Ω, let $\mathrm{M} := \{ \alpha \mid \mathrm{SC}(\alpha) \cap \Omega \subseteq \mathcal{A}cc \}$ and define

$$\alpha <_1 \beta :\Leftrightarrow \alpha < \beta \wedge \alpha \in \mathrm{M} \wedge \beta \in \mathrm{M}.$$

Lemma

The class $\mathcal{A}cc$ is closed under all the parts of the notation system that are "from below", i.e., 0, $+$, $\bar{\varphi}$.

Lemma

- $\mathsf{ID}_1 \vdash \mathrm{TI}_1(\Omega+1, U) \wedge \mathrm{K}(\Omega+1) \subseteq \Omega+1 \wedge \Omega+1 \in \mathrm{M}$.
- *If* $\mathsf{ID}_1 \vdash \mathrm{TI}_1(\alpha, U) \wedge \mathrm{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathrm{M}$, *then* $\mathsf{ID}_1 \vdash \mathrm{TI}_1(\omega^\alpha, U) \wedge \mathrm{K}(\omega^\alpha) \subseteq \omega^\alpha \wedge \omega^\alpha \in \mathrm{M}$.

Lemma

If $\mathsf{ID}_1 \vdash \mathrm{TI}_1(\alpha, U) \wedge \mathrm{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathrm{M}$, *then* $\mathsf{ID}_1 \vdash \psi(\alpha) \in \mathcal{Acc}$.

Corollary

For any $\alpha < \psi(\varepsilon_{\Omega+1})$, $\mathsf{ID}_1 \vdash \mathrm{TI}(\alpha, U)$.

Let $A(X, \alpha, x)$ be a formula of ID_1 with at most X, α, x free. We wish to define segments (in terms of the $<_1$-relation) of the A jump hierarchy starting with U, given set-theoretically by the transfinite recursion

$$Y_0 := \{ x \mid U(x) \},$$
$$Y_\alpha := \{ x \mid A(Y^\alpha, \alpha, x) \}$$

where $Y^\alpha := \{ (\beta, m) \mid \beta <_1 \alpha \wedge m \in Y_\beta \}$.
Define a term $\mathrm{hier}_A : (\iota \rightharpoonup \pi_1)$ in $\mathcal{U}(\mathsf{ID}_1)$ by

$$\mathrm{hier}_A := \mathrm{LFP}\Big(\lambda f, \alpha. \{ \text{if } \alpha = 0 \text{ then } \mathrm{Pr}_U \text{ else } r_A\big(\mathrm{Join}(f, (<_1 \alpha)), \alpha\big) \}\Big).$$

Note that we really need the *dependent version* of the join operation.

Lemma

If $\mathcal{U}(\mathsf{ID}_1) \vdash \mathrm{TI}_1(\alpha, U)$, *then* $\mathcal{U}(\mathsf{ID}_1) \vdash \forall \beta <_1 \alpha.\, \mathrm{hier}_A(\beta){\downarrow}$.

By a clever choice of A (following Feferman and Schütte), we obtain:

Lemma

If $\mathcal{U}(\mathsf{ID}_1) \vdash \mathrm{TI}_1(\alpha, U) \wedge \mathrm{K}(\alpha) \subseteq \alpha \wedge \alpha \in \mathrm{M}$, *then*
$\mathcal{U}(\mathsf{ID}_1) \vdash \mathrm{TI}_1(\varphi_\alpha(0), U) \wedge \mathrm{K}(\varphi_\alpha(0)) \subseteq \varphi_\alpha(0) \wedge \varphi_\alpha(0) \in \mathrm{M}$.

Corollary

For any $\alpha < \psi(\Gamma_{\Omega+1})$, $\mathcal{U}(\mathsf{ID}_1) \vdash \mathrm{TI}(\alpha, U)$.

The strategy for the upper bound is:

- Embed $\mathcal{U}(\mathrm{ID}_1)$ in an intermediate system $(\mathrm{ID}_1)^+_\Omega + (\textsc{Subst})$ (in analogy with Feferman and Strahm '00).
- Interpret $(\mathrm{ID}_1)^+_\Omega + (\textsc{Subst})$ in infinitary calculus for ramified set theory with classes.
- Extract the upper bound using cut-elimination and asymmetric interpretation for the infinitary system.

We introduce a theory $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$, analogous to the system $\mathsf{PA}^+_\Omega + (\textsc{Subst})$ from Strahm '00.
$(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$ is formulated in the language $\mathscr{L}^1_\Omega$, which is obtained from the language of ID_1, $\mathscr{L}^1$, by

- adding a new sort for ordinal variables (with $<$ and $=$ relations), and
- an $(n+1)$-ary predicate symbol $P_{\mathfrak{A}}$ for each inductive operator form $\mathfrak{A}(X, \vec{x})$ over ID_1

(that is, $\mathfrak{A}$ is an $\mathscr{L}^1$-formula so it can contain U and both positive and negative occurrences of $I_{\mathcal{O}}$, but of course only positive occurrences of the fresh n-ary predicate variable X).
As a matter of notation we write $P^\alpha_{\mathfrak{A}}(\vec{x})$ instead of $P_{\mathfrak{A}}(\alpha, \vec{x})$, and we put $P^{<\alpha}_{\mathfrak{A}}(\vec{x}) := \exists \beta < \alpha.\, P^\beta_{\mathfrak{A}}(\vec{x})$.

We axiomatize $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$ by:

- Number-theoretic axioms
- Schematic induction on the natural numbers
- Schematic induction and closure of the arithmetical inductive definition.
- Inductive operator axioms:

$$P^{\sigma}_{\mathfrak{A}}(\vec{x}) \leftrightarrow \mathfrak{A}(P^{<\sigma}_{\mathfrak{A}}, \vec{x}).$$

- Linearity axioms for the ordinals.
- Σ-reflection scheme on the ordinal sort.
- Σ-induction scheme on the ordinal sort.
- Substitution rule: For A an $\mathcal{L}^1$-formula, and $B(x)$ an $\mathcal{L}^1_\Omega$-formula:

$$\frac{A(U)}{A(\{\, x \mid B(x) \,\})}\ (\textsc{Subst})$$

We embed $\mathcal{U}(\mathsf{ID}_1)$ into $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$ by

- Interpreting each partial operation by a code for a partial recursive function.
- Writing an inductive operator form that simultaneously defines:
  - A collection Π of (non-unique) codes of predicates of the unfolding.
  - A complimentary pair of relations $\in$ and $\bar{\in}$ that determine the extension for each such code.

  (The dependent join operator causes Π to depend on $\bar{\in}$, for example.)
- Note that the substitution rule of $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$ interprets the substitution rule of $\mathcal{U}(\mathsf{ID}_1)$.

Upper bound, roads not taken

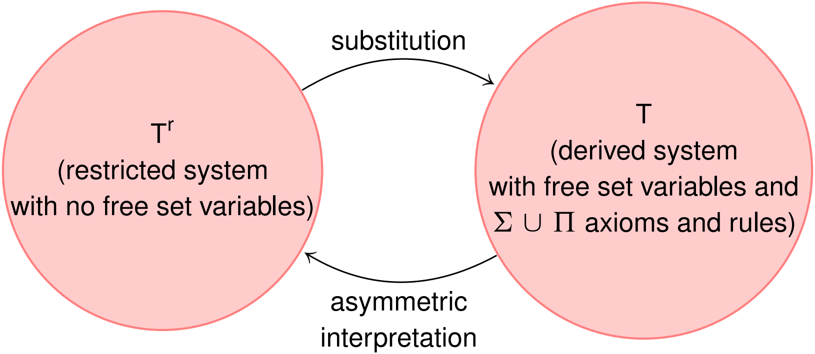
Alternatives for bounding the intermediate system?

- Work in $\mathscr{L}_\infty$: we would need to carve out a suitable fragment that would allow locally predicative cut-elimination, and it is not clear how to do this in the case of $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$ because of the need to do substitution.
- Work in systems of numbers and ordinals: we would need to simultaneously define $\mathcal{O}$ and the $P_{\mathfrak{A}}$ by one (non-monotone) operator, but then substitution would be hard to model: the stages before Ω are U-independent, while those after are not, but they need to interact during substitution.

Ramified set theory with class variables

- Introduce two infinitary calculi T^{r} and T of operator-controlled derivations for ramified set theory with class variables.
  - T^{r} enjoys the usual (locally) predicative cut elimation and collapsing theorems.
  - T enjoys partial cut elimination over the $\Sigma \cup \Pi$-fragment.
- We can substitute a general formula for a class variable X in a T^{r}-derivation to obtain a T-derivation.
- There is an asymmetric interpretation of the $\Sigma \cup \Pi$-fragment from T to T^{r}.
- This enables the upper bound proof for $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$ by interpreting the stages of $P_{\mathfrak{A}}$ using the constructible hierarchy.

Asymmetric Interpretation and Substitution



Lemma (Substitution lemma for T^{r} into T)

Let $\Gamma(X)$ be a finite set of $\mathcal{L}^{\mathsf{r}}_{\mathsf{RS}}$-formulæ and let $B(x)$ be a formula of $\mathcal{L}_{\mathsf{RS}}$. Assume $\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}]\frac{\alpha}{\rho}\ \Gamma(X)$ for some infinite ordinal α. Then $\mathsf{T}\ [\mathcal{H}[\mathrm{par}\ B]]\frac{\alpha}{<\omega}\ \Gamma(\{\, x \mid B(x)\,\})$.

Theorem (Asymmetric interpretation of T into T^{r})

Assume Γ is a finite set of $\Sigma \cup \Pi$-formulæ of T so that $\mathsf{T}\ [\mathcal{H}]\frac{\alpha}{1}\ \Gamma$. Then we have for all limit ordinals $\beta \geq \Omega$ and every $(\beta, \varphi\alpha(\beta+\beta))$-instance Λ of Γ for which $\mathrm{par}(\Lambda) \cup \{\beta\} \subseteq \mathcal{H}$ that

$$\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}]\frac{\varphi\alpha(\beta+\beta)}{\varphi\alpha(\beta+\beta)}\ \Lambda.$$

Theorem (Reduction of $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})$)

Let C be a formula of $\mathcal{L}^1_\Omega$, and let A be a closed formula of $\mathcal{L}^1$. Then we have for all natural numbers n, and all acceptable operators $\mathcal{H}$ closed under $\eta \mapsto \eta^+$:

1. $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})^{\leq n} \vdash C \quad \to \quad \mathsf{T}\,[\mathcal{H}] \frac{<\xi_{2n}}{1}\, C^\star.$
2. $(\mathsf{ID}_1)^+_\Omega + (\textsc{Subst})^{\leq n} \vdash A \quad \to \quad \mathsf{T}^{\mathsf{r}}\,[\mathcal{H}] \frac{<\xi_{2n+1}}{<\xi_{2n+1}}\, A^\star.$

Theorem (Boundedness theorem for T^{r})

Let $\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}]\frac{\alpha}{\rho}\ \Delta, F^{\mathsf{L}_\Omega}$ *with* F^{L_Ω} *a* Σ_1^{Ω}*-sentence. Then* $\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}]\frac{\alpha}{\rho}\ \Delta, F^{\mathsf{L}_\beta}$ *for all* $\beta \in [\alpha, \Omega) \cap \mathcal{H}$*.*

Theorem (Collapsing Theorem for T^{r})

Let Δ *be a set of* Σ_1^{Ω}*-sentences of* $\mathcal{L}_{\mathsf{RS}}^{\mathsf{r}}$*, and assume* $\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}_0]\frac{\alpha}{\Omega+1}\ \Delta$*. Then*

$$\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}_{\omega^{\Omega+1+\alpha}}]\frac{\psi(\omega^{\Omega+1+\alpha})}{\psi(\omega^{\Omega+1+\alpha})}\ \Delta.$$

Theorem (Soundness for T^{r})

If $\mathsf{T}^{\mathsf{r}}\ [\mathcal{H}]\frac{\alpha}{0}\ \Delta$, *then* $\mathbb{L} \models^{\alpha} \Delta$.

Corollary

$|(\mathsf{ID}_1)^{+}_{\Omega} + (\textsc{Subst})| \leq \psi(\Gamma_{\Omega+1})$.

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Questions or Comments?