Two Operational Systems of Strength $\psi(\Gamma_{\Omega+1})$

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Outline

Predicative closure seems ubiquitous

- $|\mathsf{RA}_{<\Gamma_0}| = \Gamma_0$.
- $\mathfrak{U}(\mathsf{NFA}) \equiv \mathsf{RA}_{<\Gamma_0} \equiv \mathsf{ATR}_0 \equiv \mathsf{IR} \equiv \Delta^1_1\text{-}\mathsf{DC} + (\mathsf{BR}) \equiv \mathsf{ML}_{<\omega} \equiv \cdots$.

Analogs for other systems?

Sol proposed to look at $\mathfrak{U}(\mathsf{ID}_1)$, as another example of predicative closure. That has strength $\psi(\Gamma_{\Omega+1})$. But are there other natural systems of this strength?

Today

Today I'll talk about two systems of operational or constructive nature with strength $\psi(\Gamma_{\Omega+1})$. One is a Martin-Löf type theory, the other is a system of Explicit Mathematics.
Both incorporate a type/class of constructive tree ordinals as a hierarchy of universes.

Outline

Type theory derives judgments $\mathcal{J}$, of the forms:

- Γ ctx
- $\Gamma \vdash a : A$
- $\Gamma \vdash a \equiv b : A$

Γ is of the form $x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$, where the free variables of A_j are among the x_i for $i < j$.

Type theory, structural rules and universes

$$\frac{}{\cdot \ \mathsf{ctx}} \qquad \frac{\Gamma \vdash A : \mathfrak{U}_i}{\Gamma, x : A \ \mathsf{ctx}} \qquad \frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash x : A} \ (x : A \in \Gamma)$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B : \mathfrak{U}_i}{\Gamma \vdash a : B}$$

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash A \equiv B : \mathfrak{U}_i}{\Gamma \vdash a \equiv b : B}$$

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \mathfrak{U}_i : \mathfrak{U}_{i+1}} \qquad \frac{\Gamma \vdash A : \mathfrak{U}_i}{\Gamma \vdash A : \mathfrak{U}_{i+1}}$$

Type theory, dependent products

$$\frac{\Gamma \vdash A : \mathfrak{U}_i \qquad \Gamma, x : A \vdash B : \mathfrak{U}_i}{\Gamma \vdash \Pi_{x:A} B : \mathfrak{U}_i}$$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash (\lambda x : A .\, b) : \Pi_{x:A} B}$$

$$\frac{\Gamma \vdash f : \Pi_{x:A} B \qquad \Gamma \vdash a : A}{f(a) : B[a/x]}$$

$$\frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x : A .\, b)(a) \equiv b[a/x] : B[a/x]}$$

$$\frac{\Gamma \vdash A \colon \mathfrak{U}_i \qquad \Gamma, x \colon A \vdash B \colon \mathfrak{U}_i}{\Gamma \vdash \Sigma_{x:A} B \colon \mathfrak{U}_i}$$

$$\frac{\Gamma \vdash A \colon \mathfrak{U}_i \qquad \Gamma \vdash a \colon A \qquad \Gamma \vdash b \colon B[a/x]}{\Gamma \vdash (a, b) \colon \Sigma_{x:A} B}$$

$$\frac{\begin{array}{c}\Gamma, z \colon \Sigma_{x:A} B \vdash C \colon \mathfrak{U}_j \\ \Gamma, x \colon A, y \colon B \vdash c \colon C[(x, y)/z] \qquad \Gamma \vdash p \colon \Sigma_{x:A} B\end{array}}{\Gamma \vdash \mathrm{ind}_{\Sigma_{x:A} B}(z.\, C, xy.\, c, p) \colon C[p/z]}$$

$$\frac{\begin{array}{cc}\Gamma, z \colon \Sigma_{x:A} B \vdash C \colon \mathfrak{U}_j & \Gamma \vdash a \colon A \\ \Gamma, x \colon A, y \colon B \vdash c \colon C[(x, y)/z] & \Gamma \vdash b \colon B[a/x]\end{array}}{\Gamma \vdash \mathrm{ind}_{\Sigma_{x:A} B}(z.\, C, xy.\, c, (a, b)) \equiv c[a, b/x, y] \colon C[(a, b)/z]}$$

$$\frac{\Gamma \vdash A : \mathfrak{U}_i \qquad \Gamma \vdash B : \mathfrak{U}_i}{\Gamma \vdash A + B : \mathfrak{U}_i}$$

$$\frac{\begin{array}{c}\Gamma \vdash A : \mathfrak{U}_i \\ \Gamma \vdash B : \mathfrak{U}_i \qquad \Gamma \vdash a : A\end{array}}{\mathrm{inl}(a) : A + B} \qquad \frac{\begin{array}{c}\Gamma \vdash A : \mathfrak{U}_i \\ \Gamma \vdash B : \mathfrak{U}_i \qquad \Gamma \vdash b : B\end{array}}{\mathrm{inr}(b) : A + B}$$

$$\frac{\Gamma, z : A + B \vdash C : U_i \qquad \begin{array}{c}\Gamma, x : A \vdash c : C[\mathrm{inl}(x)/z] \\ \Gamma, y : B \vdash d : C[\mathrm{inr}(y)/z]\end{array} \qquad \Gamma \vdash e : A + B}{\Gamma \vdash \mathrm{ind}_{A+B}(z.\,C, x.\,c, y.\,d, e) : C[e/z]}$$

$$\frac{\Gamma, z : A + B \vdash C : U_i \qquad \begin{array}{c}\Gamma, x : A \vdash c : C[\mathrm{inl}(x)/z] \\ \Gamma, y : B \vdash d : C[\mathrm{inr}(y)/z]\end{array} \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathrm{ind}_{A+B}(z.\,C, x.\,c, y.\,d, \mathrm{inl}(a)) \equiv c[a/x] : C[\mathrm{inl}(a)/z]}$$

(similar computation rule for inr)

$$\frac{\Gamma \vdash A\colon \mathfrak{U}_i \qquad \Gamma \vdash a\colon A \qquad \Gamma \vdash b\colon A}{\Gamma \vdash a =_A b\colon \mathfrak{U}_i}$$

$$\frac{\Gamma \vdash A\colon \mathfrak{U}_i \qquad \Gamma \vdash a\colon A}{\Gamma \vdash \mathsf{refl}_a\colon a =_A a}$$

opt.: $$\frac{\Gamma \vdash p\colon a =_A b}{\Gamma \vdash a \equiv b\colon A}$$

$$\frac{\begin{array}{cc} & \Gamma \vdash a\colon A \\ \Gamma, x\colon A, y\colon A, p\colon x =_A y \vdash C\colon \mathfrak{U}_i & \Gamma \vdash b\colon A \\ \Gamma, z\colon A \vdash c\colon C[z, z, \mathsf{refl}_z / x, y, p] & \Gamma \vdash q\colon a =_A b \end{array}}{\Gamma \vdash \mathsf{ind}_{=_A}(xyp.\, C, z.\, c, a, b, q)\colon C[a, b, q / x, y, p]}$$

$$\frac{\begin{array}{cc} \Gamma, x\colon A, y\colon A, p\colon x =_A y \vdash C\colon \mathfrak{U}_i & \\ \Gamma, z\colon A \vdash c\colon C[z, z, \mathsf{refl}_z / x, y, p] & \Gamma \vdash a\colon A \end{array}}{\Gamma \vdash \mathsf{ind}_{=_A}(xyp.\, C, z.\, c, a, a, \mathsf{refl}_a) \equiv c[a/z]\colon C[a, a, \mathsf{refl}_a / x, y, p]}$$

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \mathbb{N}_k : \mathcal{U}_i}$$

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \star_j : \mathbb{N}_k} \ (j < k)$$

$$\frac{\Gamma, x : \mathbb{N}_k \vdash C : \mathcal{U}_i \qquad \left\{ \Gamma \vdash c_j : C[\star_j / x] \right\}_{j<k} \qquad \Gamma \vdash a : \mathbb{N}_k}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}_k}(x.\, C, c_0, \ldots, c_{k-1}, a) : C[a/x]}$$

$$\frac{\Gamma, x : \mathbb{N}_k \vdash C : \mathcal{U}_i \qquad \left\{ \Gamma \vdash c_j : C[\star_j / x] \right\}_{j<k}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}_k}(x.\, C, c_0, \ldots, c_{k-1}, \star_j) \equiv c_j : C[\star_j/x]}$$

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathbb{N} : \mathcal{U}_i}$$

$$\frac{\Gamma\ \mathsf{ctx}}{0 : \mathbb{N}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{suc}(n) : \mathbb{N}}$$

$$\frac{\Gamma, z : \mathbb{N} \vdash C : \mathcal{U}_i \qquad \begin{array}{c}\Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathbb{N}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z]\end{array} \qquad \Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(z.\,C, c, xy.\,d, n) : C[n/z]}$$

$$\frac{\Gamma, z : \mathbb{N} \vdash C : \mathcal{U}_i \qquad \begin{array}{c}\Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathbb{N}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z]\end{array}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(z.\,C, c, xy.\,d, 0) \equiv c : C[0/z]}$$

$$\frac{\Gamma, z : \mathbb{N} \vdash C : \mathcal{U}_i \qquad \begin{array}{c}\Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathbb{N}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z]\end{array} \qquad \Gamma \vdash n : \mathbb{N}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(z.\,C, c, xy.\,d, \mathrm{suc}(n)) \equiv d[n, \mathrm{ind}_{\mathbb{N}}(z.\,C, c, xy.\,d, n)/x, y] \\ : C[\mathrm{suc}(n)/z]\end{array}}$$

How has type theory been interpreted?

- Feferman '82: embed ML_n in $\widehat{\mathrm{ID}}_n$ following Aczel '76 (unpublished).
- Beeson '82 in more detail for ML_1 in $\widehat{\mathrm{ID}}_1$. (Mentions that using Explicit Mathematics would be more convenient.)
- Setzer '93 (and '96) for $\mathrm{ML}_1 W$ in extensions of KP.
- Stronger systems modelled in realizability toposes.

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathfrak{T} : \mathfrak{U}_i}$$

$$\frac{\Gamma \text{ ctx}}{0 : \mathfrak{T}} \qquad \frac{\Gamma \vdash a : \mathfrak{T}}{\Gamma \vdash \mathrm{suc}(a) : \mathfrak{T}} \qquad \frac{\Gamma \vdash f : \mathbb{N} \to \mathfrak{T}}{\Gamma \vdash \lim(f) : \mathfrak{T}}$$

$$\frac{\begin{array}{c} \Gamma \vdash a : \mathfrak{T} \\ \Gamma, z : \mathfrak{T} \vdash C : \mathfrak{U}_i \end{array} \qquad \begin{array}{c} \Gamma \vdash c : C[0/z] \\ \Gamma, x : \mathfrak{T}, y : C[x/z] \vdash d : C[\mathrm{suc}(x)/z] \\ \Gamma, x : \mathbb{N} \to \mathfrak{T}, y : \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e : C[\lim(x)/z] \end{array}}{\Gamma \vdash \mathrm{ind}_{\mathfrak{T}}(z.\, C, c, xy.\, d, xy.\, e, a) : C[a/z]}$$

Type theory, tree ordinals, computation rules

$$\frac{\Gamma, z\colon \mathfrak{T} \vdash C\colon \mathfrak{U}_i \qquad \begin{array}{c}\Gamma \vdash c\colon C[0/z] \\ \Gamma, x\colon \mathfrak{T}, y\colon C[x/z] \vdash d\colon C[\mathrm{suc}(x)/z] \\ \Gamma, x\colon \mathbb{N} \to \mathfrak{T}, y\colon \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e\colon C[\lim(x)/z]\end{array}}{\Gamma \vdash \mathrm{ind}_{\mathfrak{T}}(z.\, C, c, xy.\, d, xy.\, e, 0) \equiv c\colon C[0/z]}$$

$$\frac{\begin{array}{c}\Gamma \vdash a\colon \mathfrak{T} \\ \Gamma, z\colon \mathfrak{T} \vdash C\colon \mathfrak{U}_i\end{array} \qquad \begin{array}{c}\Gamma \vdash c\colon C[0/z] \\ \Gamma, x\colon \mathfrak{T}, y\colon C[x/z] \vdash d\colon C[\mathrm{suc}(x)/z] \\ \Gamma, x\colon \mathbb{N} \to \mathfrak{T}, y\colon \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e\colon C[\lim(x)/z]\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathfrak{T}}(z.\, C, c, xy.\, d, xy.\, e, \mathrm{suc}(a)) \\ \equiv d[a, \mathrm{ind}_{\mathfrak{T}}(z.\, C, c, xy.\, d, xy.\, e, a)/x, y]\colon C[\mathrm{suc}(a)/z]\end{array}}$$

$$\frac{\begin{array}{c}\Gamma \vdash f\colon \mathbb{N} \to \mathfrak{T} \\ \Gamma, z\colon \mathfrak{T} \vdash C\colon \mathfrak{U}_i\end{array} \qquad \begin{array}{c}\Gamma \vdash c\colon C[0/z] \\ \Gamma, x\colon \mathfrak{T}, y\colon C[x/z] \vdash d\colon C[\mathrm{suc}(x)/z] \\ \Gamma, x\colon \mathbb{N} \to \mathfrak{T}, y\colon \Pi_{n:\mathbb{N}} C[x(n)/z] \vdash e\colon C[\lim(x)/z]\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathfrak{T}}(z.\, C, c, xy.\, d, xy.\, e, \lim(f)) \\ \equiv e[f, \lambda n\colon \mathbb{N}.\, \mathrm{ind}_{\mathfrak{T}}(z.\, C, c, xy.\, d, xy.\, e, f(n))/x, y]\colon C[\lim(f)/z]\end{array}}$$

Outline

UCN$\mathfrak{T}$

Here we introduce a system, UCN$\mathfrak{T}$, in the style of Explicit Mathematics with a type of tree ordinals, $\mathfrak{T}$, and a hierarchy of universes.
UCN$\mathfrak{T}$ is, as usually, formulated in classical two-sorted (operations and classes) first-order logic, where for the operation we have *partial terms*.
The operational sort includes

- variables $a, b, c, f, g, h, u, v, w, x, y, z, \ldots$
- constants k, s (combinators), $\mathrm{p}, \mathrm{p}_0, \mathrm{p}_1$ (pairing and projections), 0 (zero), s_{N} (numerical successor), p_{N} (numerical predecessor), d_{N} (definition by numerical cases), r_{N} (primitive recursion), ... (we will introduce more later)

UCNT, continued

UCN$\mathfrak{T}$, continued

The individual terms $(r, s, t, \ldots)$ are inductively generated from variables and constants by application terms $\cdot(s, t)$, also written st (left-associative). We write $s(t_1, \ldots, t_n)$ for $st_1 \ldots t_n$, and $\langle s, t \rangle$ for $\mathrm{p}(s, t)$.

$$(s \simeq t) := (s{\downarrow} \vee t{\downarrow}) \to (s = t)$$

- Partial combinatory algebra.
  1. $\mathrm{k}\,a\,b = a$.
  2. $\mathrm{s}\,a\,b\!\downarrow \wedge\, \mathrm{s}\,a\,b\,c \simeq a\,c\,(b\,c)$.
- Pairing and projection.
  1. $\mathrm{p}_0\langle a, b\rangle = a \wedge \mathrm{p}_1\langle a, b\rangle = b$.
- Natural numbers.
  1. $0 \in \mathsf{N} \wedge (a \in \mathsf{N} \to a' \in \mathsf{N})$.
  2. $a \in \mathsf{N} \to a' \neq 0 \wedge \mathrm{p}_{\mathsf{N}}(a') = a$.
  3. $a \in \mathsf{N} \wedge a \neq 0 \to \mathrm{p}_{\mathsf{N}}a \in \mathsf{N} \wedge (\mathrm{p}_{\mathsf{N}}a)' = a$.
- Definition by cases on N.
  1. $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a = b \to \mathrm{d}_{\mathsf{N}}\,u\,v\,a\,b = u$.
  2. $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a \neq b \to \mathrm{d}_{\mathsf{N}}\,u\,v\,a\,b = v$.
- Primitive recursion on N.
  1. $f \in (\mathsf{N}^2 \to \mathsf{N}) \wedge a \in \mathsf{N} \to \mathrm{r}_{\mathsf{N}}\,f\,a \in (\mathsf{N} \to \mathsf{N})$.
  2. $f \in (\mathsf{N}^2 \to \mathsf{N}) \wedge a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge h = \mathrm{r}_{\mathsf{N}}\,f\,a \to h\,0 = a \wedge h(b') = f\,b\,(h\,b)$.

We have additional operational constants: nat (natural numbers), id (identity), co (complement), un (union), dom (domain), inv (inverse image). There are two new relation symbols $\in$ (element) and $\Re$ (naming).

- Explicit representation and extensionality.
  1. $\exists x.\, \Re(x, U)$.
  2. $\Re(s, U) \wedge \Re(s, v) \rightarrow U = V$.
  3. $(\forall x.\, x \in U \leftrightarrow x \in V) \rightarrow U = V$.
- Class existence.
  1. $\Re(\mathrm{nat}) \wedge \forall x.\, x \mathrel{\dot{\in}} \mathrm{nat} \leftrightarrow x \in \mathsf{N}$.
  2. $\Re(\mathrm{id}) \wedge \forall x.\, x \mathrel{\dot{\in}} \mathrm{id} \leftrightarrow \exists y.\, x = \langle y, y \rangle$.
  3. $\Re(s) \rightarrow \Re(\mathrm{co}(s)) \wedge \forall x.\, x \mathrel{\dot{\in}} \mathrm{co}(s) \leftrightarrow x \mathrel{\dot{\notin}} s$.
  4. $\Re(s) \wedge \Re(t) \rightarrow \Re(\mathrm{un}(s, t)) \wedge \forall x.\, x \mathrel{\dot{\in}} \mathrm{un}(s, t) \leftrightarrow x \mathrel{\dot{\in}} s \vee x \mathrel{\dot{\in}} t$.
  5. $\Re(s) \rightarrow \Re(\mathrm{dom}(s)) \wedge \forall x.\, x \mathrel{\dot{\in}} \mathrm{dom}(s) \leftrightarrow \exists y.\, \langle x, y \rangle \mathrel{\dot{\in}} s$.
  6. $\Re(s) \rightarrow \Re(\mathrm{inv}(s, f)) \wedge \forall x.\, x \mathrel{\dot{\in}} \mathrm{inv}(s, f) \leftrightarrow f\, x \mathrel{\dot{\in}} s$.
- Class induction.

$$\forall X.\, 0 \in X \wedge (\forall x \in \mathsf{N}.\, x \in X \rightarrow x' \in X) \rightarrow \forall x \in \mathsf{N}.\, x \in X.$$

- Join adds a constant j such that

$$\Re(a) \land (\forall x \mathrel{\dot\in} a.\, \Re(f\ x)) \to \Re(\mathsf{j}(a, f) \land \\ \forall z.\, z \mathrel{\dot\in} \mathsf{j}(a, f) \leftrightarrow \exists xy.\, z = \langle x, y\rangle \land x \mathrel{\dot\in} a \land y \mathrel{\dot\in} f\ x.$$

- Universes add a relation symbol U such that
  1. $a \in \mathsf{U} \land b \mathrel{\dot\in} a \to \Re(a)$.
  2. $a \in \mathsf{U} \to \mathrm{nat} \mathrel{\dot\in} a$.
  3. Universes are closed under elementary comprehension and join.
  4. $\Re(a) \to \exists u \in \mathsf{U}.\, a \mathrel{\dot\in} u$.
  5. Universes are linearly ordered and cumulative.

Some earlier results

- ECJ $+$ n universes $+$ $(\text{T-I}_{\text{N}}) \equiv \widehat{\text{ID}}_{n+1}$ (Feferman '82).
- $|\text{UCN}| = \Gamma_0$ (Marzetta '93)

Tree ordinals

To obtain $\mathrm{UCN}\mathfrak{T}$ we add a name tree for a class $\mathfrak{T}$, and constants s_T (tree successor), ℓ_T (tree limit), r_T (tree recursion). We have induction on $\mathfrak{T}$ for classes.
All universes are now required to contain tree.

Theorem

The systems $\mathsf{ML}_{<\omega}\mathfrak{T}$, $\mathsf{UCN}\mathfrak{T}^{\mathsf{i}}$, $\mathsf{UCN}\mathfrak{T}$ *all have proof-theoretic strength* $\psi(\Gamma_{\Omega+1})$.

Outline

Lower bound

The lower bound is established with a well-ordering proof as in my thesis, but adapted to the type-theoretic setting using techniques of Setzer.

Outline

The Interpretation

The interpretation of $\mathrm{ML}_{<\omega}\mathfrak{T}$ into $\mathrm{UCN}\mathfrak{T}$ would be via extensional Kleene realizability.
Each closed type $A\colon \mathfrak{U}_i$ should be interpreted as a class of pairs, $\langle a, b\rangle$, such that $a \equiv b\colon A$.

Outline

Upper bound

The upper bound is established by interpreting $\mathrm{UCN}\mathfrak{T}$ in a system $\widehat{\mathrm{ID}}^{\bullet}_{<\omega}$.

Questions or Comments?