Survey of Systems of Strength $\psi(\Gamma_{\Omega+1})$

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Russell and Poincaré worried about *vicious circles* as a source of paradoxes. Thus *predicativity* emerged as a prescription to avoid paradoxes by avoiding vicious circles.

No universal agreement on what exactly that means, though!

Standard analysis takes the set of natural numbers as given. On the Feferman-Schütte analysis, the ordinal $\Gamma_0$ arises as the limit of predicatively acceptable ordinals, and as the proof-theoretic ordinal of various limits of predicatively acceptable systems.

$$\Gamma_0 := \lim_n \xi_n$$

where $\xi_0 := \varepsilon_0 := \lim_n \omega \cdots \omega^0$ (tower with $n$ $\omega$s), and $\xi_{n+1} := \varphi(\xi_n, 0)$ where $\varphi$ is the binary Veblen function obtained by enumerating fixed points starting with the normal function $\lambda \alpha. \omega^\alpha$. 

$$\varphi(\xi_n, 0)$$
Predicative closure seems ubiquitous

All the following systems have proof-theoretic ordinal $\Gamma_0$:

- $\text{Aut}(\text{RA})$ (autonomous closure of ramified analysis),
- $\mathcal{U}(\text{NFA})$ (Feferman’s unfolding of non-finitist arithmetic),
- $\text{ATR}_0$ (H. Friedman’s system of arithmetic transfinite recursion),
- $\text{FP}_0$ (Avigad’s fixed point theory),
- $\text{IR}$ (Feferman’s system of Induction-Recusion),
- $\Delta^1_1\text{-DC}_0 + \text{(SUB)} (\Delta^1_1$ dependent choices plus substitution in second order arithmetic),
- $\widehat{\text{ID}}_{<\omega}$ (finitely many generalized arithmetic fixed points),
- $\text{ML}_{<\omega}$ (predicative Martin-Löf type theory with a hierarchy of universes).

A system of Explicit Mathematics with a hierarchy of universes with induction restricted to types (otherwise the strength jumps to that of $\widehat{\text{ID}}_{<\varepsilon_0}$),

...
Analogous systems for other notions of predicativity?

- If we do not have the natural numbers as a given, get a notion of predicative closure of finitism, $\mathcal{U}(FA)$, equivalent to primitive recursive arithmetic (Feferman and Strahm).
- With a stricter notion of quantification, we obtain a predicative closure of feasible arithmetic, $\mathcal{U}(FEA)$, with provably total functions exactly the polynomial time computable ones (Eberhard and Strahm).
- How about stronger notions? What happens if we take one generalized (arithmetic) inductive definition as given?
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Unfolding of \( \text{ID}_1 \)

Feferman proposed to look at \( \mathcal{U}(\text{ID}_1) \), as another example of predicative closure.

Formal system for one arithmetical inductive definition, \( \text{ID}_1 \)

The language of \( \text{ID}_1 \), \( \mathcal{L}^1 \), is that of first-order arithmetic, \( \text{PA} \), (with a free predicative variable \( U \)) augmented by a predicate symbol \( I_A \) for an arithmetical positive operator form \( A(X,x) \) that does not contain \( U \).

The universal case is that of Kleene’s \( \mathcal{O} \) consisting of ordinal notations for recursive ordinals.
What is unfolding, again?

The unfoldings are defined for *schematic systems* in first-order logic.

Schematic systems, $S$, are formulated with a free predicate variable $U$ accompanied by a rule of formula substitution

$\text{(Subst)} \quad \frac{A(U)}{A(\{ x \mid B(x) \})}$

**Examples of schematic systems**

- Non-finitist arithmetic, NFA with

  $U(0) \land (\forall x. U(x) \rightarrow U(x')) \rightarrow \forall x. U(x)$

- Zermelo’s set theory with

  $\forall a. \exists b. \forall x. x \in b \leftrightarrow x \in a \land U(x)$. 
The universe of the operational unfolding consists of the original sorts of \( S \), embedding into a single sort of operations by means of predicates \( V_s \).

Each \( n \)-ary function symbol \( f \) of \( S \) determines an total \( n \)-ary operation \( f^* \) on the corresponding sorts.

Machinery to define new operations by recursion and explicit definition.

Each \( n \)-ary predicate symbol \( R \) of \( S \) determines a predicate \( R^* \).

The axioms of \( S \) are included, relativized to the \( V_s \).

The logic of \( \mathcal{U}_0(S) \) is the logic of partial terms.
We include in $\mathcal{U}_0(S)$ the substitution rule:

$$\frac{A(U)}{A(\{x \mid B(x)\})} \quad \text{(Subst)}$$
The language of $\mathcal{U}(S)$ extends the language of $\mathcal{U}_0(S)$ by additional constants for the predicate symbols of $S$ plus $\text{Eq}, \text{Pr}_U, \text{Inv}, \text{Neg}, \text{Conj}, \text{Un}, \text{Join}$.
Full unfolding, axioms

- $\text{Eq} \downarrow \land \forall x, y. (x, y) \in \text{Eq} \iff x = y.$
- $\text{Pr}_U \downarrow \land \forall x. x \in \text{Pr}_U \iff U(x).$
- $\text{Inv}(X, f_1, \ldots, f_m) \downarrow \land$
  $\forall \vec{x}. \vec{x} \in \text{Inv}(X, f_1, \ldots, f_m) \iff (f_1(\vec{x}), \ldots, f_m(\vec{x})) \in X.$
- $\text{Neg}(X) \downarrow \land \forall \vec{x}. \vec{x} \in \text{Neg}(X) \iff \vec{x} \notin X.$
- $\text{Conj}(X, Y) \downarrow \land \forall \vec{x}. \vec{x} \in \text{Conj}(X, Y) \iff \vec{x} \in X \land \vec{x} \in Y.$
- $\text{Un}(X) \downarrow \land \forall \vec{x}. \vec{x} \in \text{Un}(X) \iff \forall y. (\vec{x}, y) \in X.$
- For $f : \iota \rightarrow \pi_n$ and $r : \pi_1$ we take

$$(\forall y. y \in r \rightarrow f(y)) \downarrow \rightarrow \text{Join}(f, r) \downarrow \land$$

$${\forall \vec{x}, y. (\vec{x}, y) \in \text{Join}(f, r) \iff y \in r \land \vec{x} \in f(y).}$$
For $\mathcal{U}(S)$ we restrict the substitution rule,

$$\frac{A(U)}{A(x \mid B(x))} \quad \text{(Subst)},$$

by requiring $A$ to be in the language of $\mathcal{U}_0(S)$. This is needed, because the full unfolding language reflects the free predicate $U$. 
Theorem

$\mathcal{U}(\text{ID}_1)$ has proof-theoretic strength $\psi(\Gamma_{\Omega+1})$.

Remark

$\psi(\Gamma_{\Omega+1})$ figured in the original 1950 paper of Bachmann that inspired Howard’s work on ID$_1$. 
Definition

By recursion on $\alpha$ we define simultaneously

$$\text{Cl}(\alpha, \beta) := \text{the least set containing } \beta \cup \{0, \Omega\}$$
and closed under $+$, the Veblen function $\lambda \xi \eta \cdot \varphi_{\xi}(\eta)$,
and the restricted function $\psi \upharpoonright \alpha := \lambda \xi < \alpha \cdot \Psi(\xi)$,

$$\psi(\alpha) := \min\{\beta \mid \text{Cl}(\alpha, \beta) \cap \Omega \subseteq \beta\}.$$

Thus, $\psi(\Gamma_{\Omega+1})$ is the collapse of the first strongly-critical ordinal greater than $\Omega$. 

Outline

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We can model one generalized inductive definition in second order arithmetic as follows:

- Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the languages of first and second order arithmetic, respectively.
- Let $\mathcal{A}[P, u]$ range over $P$-positive formulas of $\mathcal{L}_1(P)$ that contains at most $u$ free (these are the inductive operator forms).
- Extend $\mathcal{L}_2$ to $\mathcal{L}_2^\bullet$ by added a fresh unary relation symbol $P_{2\mathcal{A}}$ for each inductive operator form $\mathcal{A}[P, u]$.
- An $\mathcal{L}_2^\bullet$ formula is called elementary in case it does not contain bounded set variables.
Recall the schema of *arithmetical comprehension*:

$$\exists X. \forall a. a \in X \leftrightarrow A[a],$$

for arithmetic formulas $A[u]$ of $\mathcal{L}_2$.

Recall also the *induction axiom*:

$$\forall X. 0 \in X \land (\forall a. a \in X \rightarrow a' \in X) \rightarrow \forall a. a \in X.$$

These define the system $\text{ACA}_0$.

Recall also the comprehension and choice principles:

($\Delta^1_1$-CA)

$$\forall a. (\exists X. A[X, a]) \leftrightarrow \forall X. B[X, a]) \rightarrow \exists Y. \forall a. a \in Y \leftrightarrow \exists X. A[X, a],$$

($\Sigma^1_1$-AC)

$$\forall a. \exists X. C[a, X]) \rightarrow \exists Y. \forall a. C[a, (Y)_a],$$

($\Sigma^1_1$-DC)

$$\forall a. \exists X. \forall Y. D[a, X, Y] \rightarrow \exists Z. \forall a. D[a, (Z)^a, (Z)_a],$$
The *substitution rule* is the rule of inference:

\[
\frac{\forall X. A[X]}{A[\{a : B[a]\}]}
\]

for arithmetic \( A[X] \) and arbitrary \( B[v] \).

Recall a theorem of Avigad: the theory \( \text{ATR}_0 \) is equivalent over \( \mathcal{L}_2 \) to the theory \( \text{FP}_0 \), which is defined as \( \text{ACA}_0 \) plus the fixed point schema:

\[
\frac{}{\exists X. \forall a. a \in X \leftrightarrow A[X, a]},
\]

for \( U \)-positive arithmetic formulas \( A[U, v] \).
Now we extend these theories to language $\mathcal{L}_2^\bullet$ and add the *least fixed point* axioms:

(ID.1) \[ \forall a. \mathcal{A}[P_{\varphi}, a] \rightarrow P_{\varphi}(a), \]
(ID.2) \[ \forall X. (\forall a. \mathcal{A}[X, a] \rightarrow a \in X) \rightarrow \forall a. P_{\varphi}(a) \rightarrow a \in X. \]

We also add the *elementary comprehension axiom*:

(E-CA) \[ \exists X. \forall a. a \in X \leftrightarrow A[a], \]

for elementary formulas $A[u]$. We get the theory $\text{ACA}_0^\bullet$, which conservatively extends the first-order theory $\text{ID}_1$.

We get theories $\Delta^1_1-\text{CA}_0^\bullet$, $\Sigma^1_1-\text{AC}_0^\bullet$ and $\Sigma^1_1-\text{DC}_0^\bullet$ by adding the corresponding schemata with arithmetic replaced with elementary.
Theorem

The following theories all have proof-theoretic ordinal $\psi(\Gamma_{\Omega+1})$:

- $\Delta^1_1$-CA$_0^\ast$ + SUB$^\ast$,
- $\Sigma^1_1$-AC$_0^\ast$ + SUB$^\ast$,
- $\Sigma^1_1$-DC$_0^\ast$ + SUB$^\ast$,
- FP$^\ast_0$,
- ATR$^\ast_0$,
- ATR$_0$ + (LFP), where (LFP) is the schema

\[(LFP) \ (\exists X. \forall a. A[X, a] \to X(a)) \land \forall Y. (\forall a. A[Y, a] \to a \in Y) \to X \subseteq Y,\]


In fact, we have equivalence for elementary $\Pi^1_1$-sentences.
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Type theory, tree ordinals

\[
\Gamma \text{ ctx} \quad \frac{\Gamma \vdash \mathcal{T} : \mathcal{U}_i}{\Gamma \vdash \mathcal{T} : \mathcal{U}_i}
\]

\[
\begin{align*}
\Gamma \text{ ctx} & \quad \frac{\Gamma \vdash a : \mathcal{T}}{\Gamma \vdash \text{suc}(a) : \mathcal{T}} \\
0 : \mathcal{T} & \quad \frac{\Gamma \vdash f : \mathbb{N} \rightarrow \mathcal{T}}{\Gamma \vdash \text{lim}(f) : \mathcal{T}}
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash c : C[0/z] \\
\Gamma, z : \mathcal{T} & \vdash a : \mathcal{T} \\
\Gamma, x : \mathcal{T}, y : C[x/z] & \vdash d : C[\text{suc}(x)/z] \\
\Gamma, z : \mathcal{T} & \vdash C : \mathcal{U}_i \\
\Gamma, x : \mathbb{N} \rightarrow \mathcal{T}, y : \Pi_{n: \mathbb{N}} C[x(n)/z] & \vdash e : C[\text{lim}(x)/z]
\end{align*}
\]

\[
\Gamma \vdash \text{ind}_{\mathcal{T}}(z. C, c, xy. d, xy. e, a) : C[a/z]
\]
Type theory, tree ordinals, computation rules

\[ \Gamma \vdash c : C[0/z] \]
\[ \Gamma, x : T, y : C[x/z] \vdash d : C[\text{suc}(x)/z] \]
\[ \Gamma, z : T \vdash C : U_i \quad \Gamma, x : N \to T, y : \Pi_{n: N} C[x(n)/z] \vdash e : C[\text{lim}(x)/z] \]
\[ \Gamma \vdash \text{ind}_T(z. C, c, xy. d, xy. e, 0) \equiv c : C[0/z] \]

\[ \Gamma \vdash c : C[0/z] \]
\[ \Gamma, x : T, y : C[x/z] \vdash d : C[\text{suc}(x)/z] \]
\[ \Gamma, z : T \vdash C : U_i \quad \Gamma, x : N \to T, y : \Pi_{n: N} C[x(n)/z] \vdash e : C[\text{lim}(x)/z] \]
\[ \Gamma \vdash \text{ind}_T(z. C, c, xy. d, xy. e, \text{suc}(a)) \]
\[ \equiv d[a, \text{ind}_T(z. C, c, xy. d, xy. e, a)/x, y] : C[\text{suc}(a)/z] \]

\[ \Gamma \vdash c : C[0/z] \]
\[ \Gamma \vdash f : N \to T \]
\[ \Gamma, z : T \vdash C : U_i \quad \Gamma, x : N \to T, y : \Pi_{n: N} C[x(n)/z] \vdash e : C[\text{lim}(x)/z] \]
\[ \Gamma \vdash \text{ind}_T(z. C, c, xy. d, xy. e, \text{lim}(f)) \]
\[ \equiv e[f, \lambda n : N. \text{ind}_T(z. C, c, xy. d, xy. e, f(n))/x, y] : C[\text{lim}(f)/z] \]
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Here we introduce a system, UCN\(\mathcal{T}\), in the style of Explicit Mathematics with a type of tree ordinals, \(\mathcal{T}\), and a hierarchy of universes. UCN\(\mathcal{T}\) is, as usually, formulated in classical two-sorted (operations and classes) first-order logic, where for the operation we have partial terms. The operational sort includes

- variables \(a, b, c, f, g, h, u, v, w, x, y, z, \ldots\)
- constants \(k, s\) (combinators), \(p, p_0, p_1\) (pairing and projections), \(0\) (zero), \(s_N\) (numerical successor), \(p_N\) (numerical predecessor), \(d_N\) (definition by numerical cases), \(r_N\) (primitive recursion), \(\ldots\) (we will introduce more later)
The individual terms \((r, s, t, \ldots)\) are inductively generated from variables and constants by application terms \(\cdot (s, t)\), also written \(st\) (left-associative).

We write \(s(t_1, \ldots, t_n)\) for \(st_1 \ldots t_n\), and \(\langle s, t \rangle\) for \(p(s, t)\).

\[
(s \simeq t) := (s \updownarrow \lor t \downarrow) \rightarrow (s = t)
\]
• Partial combinatory algebra.
  1. \( k a b = a. \)
  2. \( s a b \downarrow \& s a b c \simeq a c (b c). \)

• Pairing and projection.
  1. \( p_0 \langle a, b \rangle = a \& p_1 \langle a, b \rangle = b. \)

• Natural numbers.
  1. \( 0 \in \mathbb{N} \& (a \in \mathbb{N} \to a' \in \mathbb{N}). \)
  2. \( a \in \mathbb{N} \to a' \neq 0 \& p_N (a') = a. \)
  3. \( a \in \mathbb{N} \& a \neq 0 \to p_N a \in \mathbb{N} \& (p_N a)' = a. \)

• Definition by cases on \( \mathbb{N}. \)
  1. \( a \in \mathbb{N} \& b \in \mathbb{N} \& a = b \to d_N uvab = u. \)
  2. \( a \in \mathbb{N} \& b \in \mathbb{N} \& a \neq b \to d_N uvab = v. \)

• Primitive recursion on \( \mathbb{N}. \)
  1. \( f \in (\mathbb{N}^2 \to \mathbb{N}) \& a \in \mathbb{N} \to r_N f a \in (\mathbb{N} \to \mathbb{N}). \)
  2. \( f \in (\mathbb{N}^2 \to \mathbb{N}) \& a \in \mathbb{N} \& b \in \mathbb{N} \& h = r_N f a \to h 0 = a \& h (b') = f b (h b). \)
We have additional operational constants: nat (natural numbers), id (identity), co (complement), un (union), dom (domain), inv (inverse image). There are two new relation symbols ∈ (element) and ℜ (naming).
Axioms for classes

- Explicit representation and extensionality.
  1. $\exists x. R(x, U)$.
  2. $R(s, U) \land R(s, v) \rightarrow U = V$.
  3. $(\forall x. x \in U \leftrightarrow x \in V) \rightarrow U = V$.

- Class existence.
  1. $R(\text{nat}) \land \forall x. x \in \text{nat} \leftrightarrow x \in \mathbb{N}$.
  2. $R(\text{id}) \land \forall x. x \in \text{id} \leftrightarrow \exists y. x = \langle y, y \rangle$.
  3. $R(s) \rightarrow R(\text{co}(s)) \land \forall x. x \in \text{co}(s) \leftrightarrow x \notin s$.
  4. $R(s) \land R(t) \rightarrow R(\text{un}(s, t)) \land \forall x. x \in \text{un}(s, t) \leftrightarrow x \in s \lor x \in t$.
  5. $R(s) \rightarrow R(\text{dom}(s)) \land \forall x. x \in \text{dom}(s) \leftrightarrow \exists y. \langle x, y \rangle \in s$.
  6. $R(s) \rightarrow R(\text{inv}(s, f)) \land \forall x. x \in \text{inv}(s, f) \leftrightarrow f x \in s$.

- Class induction.
  $$\forall X. 0 \in X \land (\forall \mathbb{N}. x \in X \rightarrow x' \in X) \rightarrow \forall x \in \mathbb{N}. x \in X.$$
Join adds a constant $j$ such that

$$\mathcal{R}(a) \land (\forall x \in a. \mathcal{R}(f x)) \to \mathcal{R}(j(a,f)) \land$$

$$\forall z. z \in j(a,f) \iff \exists xy. z = \langle x, y \rangle \land x \in a \land y \in f x.$$

Universes add a relation symbol $U$ such that

1. $a \in U \land b \in a \to \mathcal{R}(a)$.
2. $a \in U \to \text{nat} \in a$.
3. Universes are closed under elementary comprehension and join.
4. $\mathcal{R}(a) \to \exists u \in U. a \in u$.
5. Universes are linearly ordered and cumulative.
Some earlier results

- ECJ + \( n \) universes + \( (C-I_N) \equiv \widehat{\text{ID}}_n \) (Feferman ’82).
- \(|\text{UCN}| = \Gamma_0 \) (Marzetta ’93)
Tree ordinals

To obtain $\text{UCN} \mathcal{T}$ we add a name $\text{tree}$ for a class $\mathcal{T}$, and constants $s_T$ (tree successor), $\ell_T$ (tree limit), $r_T$ (tree recursion). We have induction on $\mathcal{T}$ for classes.

All universes are now required to contain tree.
Theorem

The systems ML_{<\omega} T, UCN T^i, UCN T all have proof-theoretic strength ψ(Γ_{Ω+1}).
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Notes on the proofs

Lower bound
The lower bound is established with a well-ordering proof as in my thesis. This needs to be done for $\Delta^1_1$-$CA^*_0 + (SUB^*)$ and for the type theory (using techniques of Setzer and Hancock).

Reductions
$\Sigma^1_1$-$DC^*_0 + (SUB^*)$ reduces to $\text{ATR}_0 + (\text{LFP})$ using existence of enough countably coded $\omega$-models of $\Sigma^1_1$-$DC^*_0$ in $\text{ATR}_0$.
All the mentioned theories then reduce to a theory $\widehat{\text{ID}}^*_<\omega$ of fixed point theories on top of $\text{ID}_1$.

Upper bound
The theory $\widehat{\text{ID}}^*_<\omega$ is handled via ordinal analysis.
Questions or Comments?