Proof-Theoretic Ordinals related to Unfoldings

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1. Background on ordinals

2. Ordinal function hierarchies

3. Relations to Unfoldings
   - Lower bound
   - Upper bound
1. Background on ordinals

2. Ordinal function hierarchies

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   - Lower bound
   - Upper bound
Work with standard Von Neumann set-theoretic definition: an ordinal is a hereditarily transitive set. Then each ordinal is the well-ordered set of all smaller ordinals.

Every ordinal is either:
- the least ordinal, 0;
- a successor ordinal, i.e., an ordinal of the form $\alpha' = \alpha \cup \{\alpha\}$;
- a limit ordinal. The class of limit ordinals is denoted $\text{Lim}$.

Let $\Omega$ be the first uncountable ordinal. Then $\Omega$ is regular: if $M \subseteq \Omega$ is countable, then there exists some $\alpha < \Omega$ with $M \subseteq \alpha$.

Let $\text{ON}$ denote the class of all ordinals. Every subclass $M$ is then well-ordered with order-type $\leq \text{ON}$. If $M$ is bounded in $\text{ON}$, its order-type is an ordinal $\alpha \in \text{ON}$ and we have an enumeration function $\text{en}_M : \alpha \to M$. If $M$ is unbounded in $\text{ON}$, it has the order-type of $\text{ON}$ itself and we have an enumeration function $\text{en}_M : \text{ON} \to M$. 

A class $M \subseteq \mathbb{ON}$ is called *closed*, if $M$ is closed in the order-topology, which is the case precisely when suprema of sets of elements in $M$ themselves are in $M$.

Let $\kappa \in \{\Omega, \mathbb{ON}\}$. An ordinal function $f : \kappa \to \kappa$ is *continuous* when it preserves suprema. And $f$ is called *normal* when it is strictly increasing and continuous. A normal function satisfies $f(\alpha) \geq \alpha$ for all $\alpha$.

A class $M \subseteq \kappa$ is called $\kappa$-*club* if and only if it is closed and unbounded in $\kappa$. A fundamental fact is that $M$ is club if and only its enumeration function $\text{en}_M$ is normal.
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We let AP be the class of *additively principal* ordinals (so $\alpha \in \text{AP}$ if and only if for all $\xi, \eta < \alpha$, $\xi + \eta < \alpha$). Then AP is $\kappa$-club for all $\kappa \in \{\Omega, \text{ON}\}$ and its enumerating function is $\lambda \xi. \omega^\xi$.

By induction on $\alpha > 0$, we prove that there are uniquely determined ordinals $\{\alpha_1, \ldots, \alpha_n\} \subseteq \text{AP}$ such that

$$\alpha = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \alpha_1 \geq \cdots \geq \alpha_n.$$ 

This is called the *additive normal form* of $\alpha$, and $\{\alpha_1, \ldots, \alpha_n\}$ is called the set of *additive components* of $\alpha$.

Using the enumerating function for AP we obtain the *Cantor normal form* for $\alpha > 0$:

$$\alpha =_{\text{NF}} \omega^{\xi_1} + \cdots + \omega^{\xi_n} \quad \text{with} \quad \xi_1 \geq \cdots \geq \xi_n.$$ 

Any ordinal less than the first fixed point $\varepsilon_0$ of $\lambda \xi. \omega^\xi$ has a unique representation in hereditary Cantor normal form starting from 0.
For a class $M \subseteq \kappa$ we define its derivative as the class of the fixed points of its enumerating function:

$$M' := \{ \xi < \kappa \mid \text{en}_M(\xi) = \xi \}$$

The derivative $f'$ of a function $f$ is defined by $f' := \text{en}_{\text{Fix}(f)}$ where

$$\text{Fix}(f) := \{ \xi < \kappa \mid f(\xi) = \xi \}.$$ 

It is easily shown that if $M$ is $\kappa$-club, then so is $M'$. Hence, if $f$ is $\kappa$-normal, then so is $f'$.

If $\{ M_\xi \}$ is a collection of less than $\kappa$ many $\kappa$-club classes, then $\bigcap_\xi M_\xi$ is again $\kappa$-club.

Thus we can iterate the derivation process transfinitely to get a hierarchy of club classes. Starting with $\text{AP}$ we obtain

$$\text{Cr}(0) := \text{AP}$$
$$\text{Cr}(\alpha + 1) := \text{Cr}(\alpha)'$$
$$\text{Cr}(\lambda) := \bigcap_{\xi < \lambda} \text{Cr}(\xi) \quad \text{for } \lambda \in \text{Lim}.$$ 

Then we put $\varphi_\alpha := \text{en}_{\text{Cr}(\alpha)}$ to obtain the usual Veblen hierarchy.
Ordinals closed under the binary Veblen function, $\lambda \alpha, \beta. \varphi_{\alpha}(\beta)$, are called **strongly critical ordinals**. Their class, SC, is $\kappa$-club, and we let $\lambda \xi. \Gamma_{\xi}$ be their enumerating function.

The first strongly critical ordinal, $\Gamma_0$, is also known as the Feferman-Schütte ordinal, and is well-known from the analysis of predicativity given the natural numbers.

If $\alpha \in \text{AP} \setminus \text{SC}$, then $\alpha$ can be written in normal form

$$\alpha =_{\text{NF}} \varphi_{\alpha_1}(\alpha_2), \quad \text{with } \alpha_1, \alpha_2 < \alpha.$$
In Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordnungszahlen, (published 1950 in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich), Heinz Bachmann in effect used fundamental sequences for ordinals larger than $\Omega$ to define a longer hierarchy of $\Omega$-normal functions, and thus gave notations for larger ordinals less than $\Omega$.

He wanted to then assign fundamental sequences $(\nu[\xi])_{\xi<\tau_\nu}$ to these ordinals such that

1. $\tau_\nu \leq \Omega \land ((\tau_\nu \in \operatorname{Lim} \land \nu \in \operatorname{Lim}) \lor (\tau_\nu = 1 \land \nu = \nu[0] + 1))$,
2. for $\nu \neq 0$ we have $\nu = \sup \{ \nu[\xi] + 1 \mid \xi < \tau_\nu \}$,
3. if $0 < \xi_1 = \xi_2 + 1 < \tau_\nu$, then either $\nu[\xi_1] \in \operatorname{Lim}$ or $\nu[\xi_1] = \nu[\xi_2] + 1$,
4. if $\xi \in \operatorname{Lim} \cap \tau_\nu$, then $\nu[\xi] \in \operatorname{Lim}$, $\tau_\nu[\xi] = \xi$, and $\nu[\xi][\eta] = \nu[\eta]$ for $0 \leq \eta < \xi$,
5. if $\nu[\xi] < \zeta \leq \nu[\xi + 1]$, then $\nu[\xi] \leq \zeta[0]$. 
Bachmann fundamental sequences

For each limit $\alpha \leq \Gamma_{\Omega+1}$ define a fundamental sequence $(\alpha[\xi])_{\xi<\tau_\alpha}$:

1. If $\alpha \leq \Omega$, then $\tau_\alpha := \alpha$ and $\alpha[\xi] := \xi$.
2. If $\Omega < \alpha =_{\text{NF}} \beta + \gamma$, then $\tau_\alpha := \tau_\gamma$ and $\alpha[\xi] := \beta + \gamma[\xi]$.
3. If $\Omega < \alpha =_{\text{NF}} \varphi_\beta(\gamma)$ ($\gamma \in \text{Lim}$), then $\tau_\alpha := \tau_\gamma$ and $\alpha[\xi] := \varphi_\beta(\gamma[\xi])$.
4. If $\Omega < \alpha =_{\text{NF}} \varphi_0(\gamma + 1) = \omega^{\gamma+1}$, then $\tau_\alpha := \omega$ and $\alpha[\xi] := \omega^{\gamma} \cdot \xi$.
5. If $\Omega < \alpha =_{\text{NF}} \varphi_{\beta+1}(0)$, then $\tau_\alpha := \omega$ and $\alpha[\xi] := \varphi^{(\xi)}(\beta)(0)$.
6. If $\Omega < \alpha =_{\text{NF}} \varphi_{\beta+1}(\gamma + 1)$, then $\tau_\alpha := \omega$ and $\alpha[\xi] := \varphi^{(\xi)}(\varphi_{\beta+1}(\gamma) + 1)$.
7. If $\Omega < \alpha =_{\text{NF}} \varphi_\beta(0)$ ($\beta \in \text{Lim}$), then $\tau_\alpha := \tau_\beta$ and $\alpha[\xi] := \varphi_\beta[\xi](0)$.
8. If $\Omega < \alpha =_{\text{NF}} \varphi_\beta(\gamma + 1)$ ($\beta \in \text{Lim}$), then $\tau_\alpha := \tau_\beta$ and $\alpha[\xi] := \varphi_\beta[\xi](\varphi_\beta(\gamma) + 1)$.
9. If $\alpha = \Gamma_{\Omega+1}$, then $\tau_\alpha := \omega$, $\alpha[0] := \Omega + 1$, $\alpha[\xi + 1] := \varphi_\alpha[\xi](0)$, $\alpha[\omega] := \alpha$. 
Bachmann’s hierarchy

Bachmann used these fundamental sequences to define a more powerful variant of the Veblen hierarchy:

1. \( \tilde{\varphi}_0(\xi) := \omega^\xi \).
2. \( \tilde{\varphi}_{\alpha+1}(\xi) := \tilde{\varphi}'_\alpha(\xi) \).
3. If \( \alpha \in \text{Lim} \) with \( \tau_\alpha < \Omega \), then \( \tilde{\varphi}_\alpha(\xi) := \text{en}_R(\xi) \) with \( R = \bigcap_{\xi < \tau_\alpha} \text{Ran}(\tilde{\varphi}_\alpha[\xi]) \).
4. If \( \alpha \in \text{Lim} \) with \( \tau_\alpha = \Omega \), then \( \tilde{\varphi}_\alpha(\xi) := \tilde{\varphi}_\alpha[\xi](0) \).

(It is more efficient to directly to the successor in the last case.)

Bachmann then gave refined fundamental sequences for countable ordinals less than \( \tilde{\varphi}_{\varphi_\Omega(1)+1}(0) \) \((H(1) \text{ in Bachmann’s article})\).
By recursion on $\alpha$ we define simultaneously

$$B(\alpha) := \text{the least set containing } \{0, \Omega\}$$

and closed under $+$, the Veblen function $\lambda \xi \eta. \varphi_\xi(\eta)$,

and the restricted function $\psi \upharpoonright \alpha := \lambda \xi < \alpha. \psi(\xi)$,

$$\psi(\alpha) := \min \Omega \setminus B(\alpha).$$

Then we have:

1. Each $B(\alpha)$ is countable.
2. $\psi(\alpha) < \Omega$ and $\psi(\alpha) \not\in B(\alpha)$.
3. $\psi(\alpha) \in SC$ (strongly critical ordinals: closed under $\varphi$).
4. If $\alpha \leq \beta$, then $\psi(\alpha) \leq \psi(\beta)$ and $B(\alpha) \subseteq B(\beta)$.
5. If $\alpha < \beta$ and $\alpha \in B(\beta)$, then $\psi(\alpha) < \psi(\beta)$.
6. $\psi$ is continuous.
Some ordinals

- $\varepsilon_0$.
- $\Gamma_0$ (Feferman-Schütte).
- $\psi(\Omega^\omega)$ (small Veblen ordinal).
- $\psi(\Omega^\Omega)$ (large Veblen ordinal).
- $\psi(\varepsilon_{\Omega+1})$ (Howard ordinal).
- $\psi(\varphi_\Omega(1))$ (Bachmann’s $H(1)$).
- $\psi(\Gamma_{\Omega+1})$ (ordinal of unfolding of $ID_1$).
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- $\psi(\varphi_\Omega(1))$ (Bachmann’s $H(1)$).
- $\psi(\Gamma_{\Omega+1})$ (ordinal of unfolding of $\text{ID}_1$).

(Fundamental sequences on whiteboard.)
Outline

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What is unfolding?

The unfoldings are defined for *schematic systems* in first-order logic.

Schematic systems, $S$, are formulated with a free predicate variable $U$ accompanied by a rule of formula substitution

$$
(\text{Subst}) \quad \frac{A(U)}{A(\{ x \mid B(x) \})}
$$

**Examples of schematic systems**

- **Non-finitist arithmetic, NFA** with
  
  \[ U(0) \land (\forall x. U(x) \rightarrow U(x')) \rightarrow \forall x. U(x) \]

- **Zermelo’s set theory** with
  
  \[ \forall a. \exists b. \forall x. x \in b \iff x \in a \land U(x). \]
Operational unfolding

- The universe of the operational unfolding consists of the original sorts of $S$, embedding into a single sort of operations by means of predicates $V_s$.
- Each $n$-ary function symbol $f$ of $S$ determines an total $n$-ary operation $f^*$ on the corresponding sorts.
- Machinery to define new operations by recursion and explicit definition (the universe forms a PCA, in fact a model of BON).
- Each $n$-ary predicate symbol $R$ of $S$ determines a predicate $R^*$.
- The axioms of $S$ are included, relativized to the $V_s$.
- The logic of $U_0(S)$ is Beeson’s *logic of partial terms*. 
The substitution rule

We include in \( \mathcal{U}_0(S) \) the substitution rule:

\[
\frac{A(U)}{A(\{ x \mid B(x) \})} \quad (\text{SUBST})
\]
The language of $\mathcal{U}(S)$ extends the language of $\mathcal{U}_0(S)$ by additional constants for the predicate symbols of $S$ plus eq, pr$_U$, inv, neg, conj, un, join.

Remark

Our formulation here allows us to focus on the rôle of the join operation. We get the intermediate unfolding by leaving out the join axioms.
\[ \Pi(\text{nat}) \land \forall x (x \in \text{nat} \leftrightarrow N(x)) . \]
\[ \Pi(\text{eq}) \land \forall x (x \in \text{eq} \leftrightarrow \exists y (x = \langle y, y \rangle)) . \]
\[ \Pi(\text{pr}_R) \land \forall x (x \in \text{pr}_R \leftrightarrow R^*(x)) . \]
\[ \Pi(a) \rightarrow \Pi(\text{inv}(a, f)) \land \forall x (x \in \text{inv}(a, f) \leftrightarrow f x \in a) . \]
\[ \Pi(a) \land \Pi(b) \rightarrow \Pi(\text{conj}(a, b)) \land \forall x (x \in \text{conj}(a, b) \leftrightarrow x \in a \land x \in b) . \]
\[ \Pi(a) \rightarrow \Pi(\text{neg} a) \land \forall x (x \in \text{neg}(a) \leftrightarrow \neg (x \in a)) . \]
\[ \Pi(a) \rightarrow \Pi(\text{un} a) \land \forall x (x \in \text{un}(a) \leftrightarrow \forall y (N(y) \rightarrow \langle x, y \rangle \in a)) . \]

The dependent join axiom:

\[ \Pi(a) \land (\forall y \in a) \Pi(f y) \rightarrow \Pi(\text{join}(f, a)) \]
\[ \land \forall x (x \in \text{join}(f, a) \leftrightarrow \exists y, z (x = \langle y, z \rangle \land y \in a \land z \in f(y))) . \]
For $\mathcal{U}(S)$ we restrict the substitution rule,

$$\frac{A(U)}{A(\{ x \mid B(x) \})} \text{ (\text{SUBST}) ,}$$

by requiring $A$ to be in the language of $\mathcal{U}_0(S)$. This is needed, because the full unfolding language reflects the free predicate $U$. 
Theorem (Feferman and Strahm, 2000)

We have the following proof-theoretic equivalences

- $U_0(\text{NFA}) \equiv \text{PA}$
- $U_1(\text{NFA}) \equiv \text{RA}_{<\omega}$
- $U(\text{NFA}) \equiv \text{RA}_{<\Gamma_0}$

In each case the systems prove the same arithmetical sentences.
Unfolding of $\text{ID}_1$

Feferman proposed to look at $\mathcal{U}(\text{ID}_1)$, as another example of predicative closure.

Formal system for one arithmetical inductive definition, $\text{ID}_1$

The language of $\text{ID}_1$, $\mathcal{L}_1$, is that of first-order arithmetic, PA, (with a free predicative variable $U$) augmented by predicate symbols $I_{\mathcal{A}}$ for each arithmetical positive operator form $\mathcal{A}(X,x)$ that does not contain $U$.

The universal case is that of Kleene’s $\mathcal{O}$ consisting of ordinal notations for recursive ordinals.

Theorem

$\mathcal{U}(\text{ID}_1)$ has proof-theoretic strength $\psi(\Gamma_{\Omega+1})$. 
Strategy

The strategy for the lower bound proof consists of combining two elements:

- A lower bound proof for $\text{ID}_1$ (recall that $|\text{ID}_1| = \psi(\varepsilon_{\Omega + 1})$)
- The techniques for reaching a strongly critical ordinal using the predicate unfolding machinery from Feferman and Strahm ’00.
There is a simultaneous finitary inductive definitions of terms, $\alpha$, finite sets $K(\alpha)$ and an ordering $\alpha < \beta$.

Definition of $\text{SC} \subseteq \text{H} \subseteq \text{ON}$:
- $\langle 0 \rangle \in \text{ON}$ (denoting 0),
- $\langle 1 \rangle \in \text{SC}$ (denoting $\Omega$),
- if $n > 1$, $\alpha_1, \ldots, \alpha_n \in \text{H}$ and $\alpha_1 \geq \cdots \geq \alpha_n$, then $\langle 2, \alpha_1, \ldots, \alpha_n \rangle \in \text{ON}$ (denoting $\alpha_1 + \cdots + \alpha_n$),
- $\alpha, \beta \in \text{ON}$, then $\langle 3, \alpha, \beta \rangle \in \text{H}$ (denoting $\bar{\varphi}_\alpha \beta$),
- if $\alpha \in \text{ON}$ and $K(\alpha) \subseteq \alpha$, then $\langle 4, \alpha \rangle \in \text{SC}$ (denoting $\psi(\alpha)$).
Definition of $K(\alpha)$:

\[
K(0) := \emptyset, \\
K(\Omega) := \emptyset, \\
K(\alpha_1 + \cdots + \alpha_n) := K(\alpha_1) \cup \cdots \cup K(\alpha_n), \\
K(\bar{\varphi}_{\alpha\beta}) := K(\alpha) \cup K(\beta), \\
K(\psi(\alpha)) := \{\alpha\} \cup K(\alpha).
\]
For $\alpha, \beta \in \text{ON}$, put $\alpha < \beta$ if one of the following conditions obtains:

- $\alpha = 0$ and $\beta \neq 0$,
- $\alpha = \alpha_1 + \cdots + \alpha_m$, $\beta = \beta_1 + \cdots + \beta_n$, and either
  - $m \geq n$ and $\exists i \leq n. \alpha_i < \beta_i \land \forall j < i. \alpha_j = \beta_j$, or
  - $m < n$ and $\forall i \leq m. \alpha_i = \beta_i$.
- $\alpha = \alpha_1 + \cdots + \alpha_n$, $\beta \in \text{H}$, and $\alpha_1 < \beta$.
- $\alpha \in \text{H}$, $\beta = \beta_1 + \cdots + \beta_n$, and $\alpha \leq \beta_1$.
- $\alpha = \varphi_{\alpha_1} \alpha_2$, $\beta = \varphi_{\beta_1} \beta_2$ and one of the following obtains
  - $\alpha_1 < \beta_1$ and $\alpha_2 < \beta$.
  - $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.
  - $\beta_1 > \alpha_1$ and $\beta_2 \leq \alpha$.
- $\alpha = \varphi_{\alpha_1} \alpha_2$, $\beta \in \text{SC}$, and $\alpha_1, \alpha_2 < \beta$.
- $\alpha \in \text{SC}$, $\beta = \varphi_{\beta_1} \beta_2$, and $\alpha \leq \beta_1$ or $\alpha \leq \beta_2$.
- $\alpha = \psi(\alpha_1)$, $\beta = \psi(\beta_1)$ and $\alpha_1 < \beta_1$.
- $\alpha = \psi(\alpha_1)$ and $\beta = \Omega$. 
Lemma

The class Acc is closed under all the parts of the notation system that are “from below”, i.e., 0, +, $\bar{\varphi}$. 
Lemma

- $\text{ID}_1 \vdash \text{TI}_1(\Omega + 1, U) \land K(\Omega + 1) \subseteq \Omega + 1 \land \Omega + 1 \in M$.
- If $\text{ID}_1 \vdash \text{TI}_1(\alpha, U) \land K(\alpha) \subseteq \alpha \land \alpha \in M$, then $\text{ID}_1 \vdash \text{TI}_1(\omega^\alpha, U) \land K(\omega^\alpha) \subseteq \omega^\alpha \land \omega^\alpha \in M$.

Lemma

If $\text{ID}_1 \vdash \text{TI}_1(\alpha, U) \land K(\alpha) \subseteq \alpha \land \alpha \in M$, then $\text{ID}_1 \vdash \psi(\alpha) \in \text{Acc}$.

Corollary

For any $\alpha < \psi(\varepsilon_{\Omega+1})$, $\text{ID}_1 \vdash \text{TI}(\alpha, U)$. 
Let $A(X, \alpha, x)$ be a formula of $\text{ID}_1$ with at most $X, \alpha, x$ free. We wish to define segments (in terms of the $<_1$-relation) of the $A$ jump hierarchy starting with $U$, given set-theoretically by the transfinite recursion

$$
Y_0 := \{ x \mid U(x) \}, \\
Y_\alpha := \{ x \mid A(Y_\alpha, \alpha, x) \}
$$

where $Y_\alpha := \{ (\beta, m) \mid \beta < _1 \alpha \land m \in Y_\beta \}$.

Define a term $\text{hier}_A : (\iota \rightarrow \pi_1) \in U(\text{ID}_1)$ by

$$
\text{hier}_A := \text{LFP} \left( \lambda f, \alpha. \{ \text{if } \alpha = 0 \text{ then } \text{Pr}_U \text{ else } r_A(\text{join}(f, (<_1 \alpha)), \alpha) \} \right).
$$

Note that we really need the dependent version of the join operation.
Lemma

*If* $U(ID_1) \vdash TI_1(\alpha, U)$, *then* $U(ID_1) \vdash \forall \beta <_1 \alpha. \text{hier}_A(\beta)\downarrow$.

By a clever choice of $A$ (following Feferman and Schütte), we obtain:

**Lemma**

*If* $U(ID_1) \vdash TI_1(\alpha, U) \land K(\alpha) \subseteq \alpha \land \alpha \in M$, *then*

$U(ID_1) \vdash TI_1(\varphi_\alpha(0), U) \land K(\varphi_\alpha(0)) \subseteq \varphi_\alpha(0) \land \varphi_\alpha(0) \in M$.

**Corollary**

*For any* $\alpha < \psi(\Gamma_{\Omega+1})$, $U(ID_1) \vdash TI(\alpha, U)$.
Upper bound strategy

The strategy for the upper bound is:

- Embed $\mathcal{U}(\text{ID}_1)$ in an intermediate system $(\text{ID}_1)^+_{\text{ON}} + (\text{SUBST})$ (in analogy with Feferman and Strahm ’00).
- Interpret $(\text{ID}_1)^+_{\text{ON}} + (\text{SUBST})$ in infinitary systems for numbers and ordinals based on ID$_1$.
- Extract the upper bound using cut-elimination and asymmetric interpretation for the infinitary systems.
We introduce a theory \((\text{ID}_1)^+_\text{ON} + (\text{SUBST})\), analogous to the system \(\text{PA}^+_\Omega + (\text{SUBST})\) from Strahm ’00.

\((\text{ID}_1)^+_\text{ON} + (\text{SUBST})\) is formulated in the language \(\mathcal{L}^1_{\text{ON}}\), which is obtained from the language of \(\text{ID}_1\), \(\mathcal{L}^1\), by

- adding a new sort for ordinal variables (with \(<\) and \(=\) relations), and
- an \((n+1)\)-ary predicate symbol \(P_{\mathfrak{A}}\) for each inductive operator form \(\mathfrak{A}(X, \vec{x})\) over \(\text{ID}_1\)

(that is, \(\mathfrak{A}\) is an \(\mathcal{L}^1\)-formula so it can contain \(U\) and both positive and negative occurrences of \(I_\mathcal{O}\), but of course only positive occurrences of the fresh \(n\)-ary predicate variable \(X\)).

As a matter of notation we write \(P_{\mathfrak{A}}^\alpha(\vec{x})\) instead of \(P_{\mathfrak{A}}(\alpha, \vec{x})\), and we put \(P_{\mathfrak{A}}^{<\alpha}(\vec{x}) := \exists \beta < \alpha. P_{\mathfrak{A}}^\beta(\vec{x})\).
Intermediate system, axioms

We axiomatize \((\text{ID}_1)^+_{\text{ON}} + (\text{SUBST})\) by:

- Number-theoretic axioms
- Schematic induction on the natural numbers
- Schematic induction and closure of the arithmetical inductive definition.
- Inductive operator axioms:

\[
P^\sigma_\mathcal{A}(\vec{x}) \leftrightarrow \mathcal{A}(P^\leq\sigma_\mathcal{A}, \vec{x}).
\]

- Linearity axioms for the ordinals.
- \(\Sigma\)-reflection scheme on the ordinal sort.
- \(\Sigma\)-induction scheme on the ordinal sort.
- Substitution rule: For \(A\) an \(\mathcal{L}^1\)-formula, and \(B(x)\) an \(\mathcal{L}_{\text{ON}}^1\)-formula:

\[
\frac{A(U)}{A(\{ x \mid B(x) \})} \quad (\text{SUBST})
\]
We embed $\mathcal{U}(\text{ID}_1)$ into $(\text{ID}_1)^+_{ON} + (\text{SUBST})$ by

- Interpreting each partial operation by a code for a partial recursive function.
- Writing an inductive operator form that simultaneously defines:
  - A collection $\Pi$ of (non-unique) codes of predicates of the unfolding.
  - A complimentary pair of relations $\in$ and $\bar{\in}$ that determine the extension for each such code.

(The dependent join operator causes $\Pi$ to depend on $\bar{\in}$, for example.)

- Note that the substitution rule of $(\text{ID}_1)^+_{ON} + (\text{SUBST})$ interprets the substitution rule of $\mathcal{U}(\text{ID}_1)$. 

Embedding $\mathcal{U}(\text{ID}_1)$
Asymmetric Interpretation and Substitution

\( \mathcal{T}^r \)
(restricted system with no free variables)

\( \mathcal{T} \)
(derived system with free ordinal variables and \( \Sigma \cup \Pi \) axioms and rules)

substitution

asymmetric interpretation
Substitution and Asymmetric Interpretation

Lemma (Substitution lemma for $T^r$ into $T$)

Let $\Gamma(U)$ be a finite set of $\mathcal{L}^{1,rc\text{-}}$formulas with no occurrences of $P_{2\text{-}}$literals, and let $B(x)$ be any formula of $\mathcal{L}^{1,c\text{-}}$. Assume $T^r, \mathcal{H}\models_{\Omega+1}^{\alpha} \Gamma(U)$ for some infinite ordinal $\alpha$. Then we have $T, \mathcal{H}[\text{par } B]\models_{<\omega}^{\alpha} \Gamma(\{x \mid B(x)\})$.

Theorem (Asymmetric interpretation of $T$ into $T^r$)

Assume $\Gamma$ is a finite set of $\Sigma \cup \Pi$-formulas of $T$ so that $T, \mathcal{H}\models_{1}^{\alpha} \Gamma$. Let $\beta \geq \Omega$ be a limit ordinal and put $\gamma := \varphi\alpha(\beta + \beta)$. Then for every $(\beta, \gamma)$-instance $\Lambda$ of $\Gamma$ with $\text{par}(\Lambda) \cup \{\beta\} \subseteq \mathcal{H}$ we have $T^r, \mathcal{H}\models_{\gamma}^{\gamma} \Lambda$. 
Theorem (Reduction of \((\text{ID}_1)^+_\text{ON} + (\text{Subst}))\)

Let \(C\) be a formula of \(\mathcal{L}^1_{\text{ON}}\), and let \(A\) be a closed formula of \(\mathcal{L}^1\). Then we have for all natural numbers \(n\), and all acceptable operators \(\mathcal{H}\):

1. \((\text{ID}_1)^+_\text{ON} + (\text{Subst})^{\leq n} \vdash C \quad \rightarrow \quad T, \mathcal{H} \vdash^{\xi_{2n}} \frac{\xi}{1} C^*.
2. \((\text{ID}_1)^+_\text{ON} + (\text{Subst})^{\leq n} \vdash A \quad \rightarrow \quad T^r, \mathcal{H} \vdash^{\xi_{2n+2}} \frac{\xi}{\Omega + 1} A^*.

Zigzagging cut elimination
Theorem (Collapsing Theorem for $T^r$)

Let $\Delta$ be a set of $\Sigma^\Omega$-sentences of $\mathcal{L}^1_{\infty}^r$, and assume $T^r, \mathcal{H}_0 \models \Delta$. Then

$$T^r, \mathcal{H}_{\omega^{\Omega+1+\alpha}} \models \frac{\psi(\omega^{\Omega+1+\alpha})}{\psi(\omega^{\Omega+1+\alpha})} \Delta.$$ 

Corollary

$$| (\text{ID}_1)_{ON}^+ + (\text{SUBST}) | \leq \psi(\Gamma_{\Omega+1}).$$
Open problems

- Determine the strength of $\mathcal{U}_1(ID_1)$ (no join).
- Determine the strength of $\mathcal{U}_2(ID_1)$ (restricted join).
- Give a system of strength $\psi(\varphi_\Omega(1))$. 
Questions or Comments?