Notes on higher groups and projective spaces

Ulrik Buchholtz (based on forthcoming joint paper with Egbert Rijke)

CMU HoTT seminar, September 23, 2016

1 Truncatedness and connectedness

Recall that we define $\mathrm{istrunc}_n\, A$ by recursion on $n : \mathbb{N}_{-2}$:

$$\mathrm{istrunc}_{-2}\, A := \mathrm{iscontr}\, A = (a : A) \times \big((x : A) \to (a = x)\big)$$
$$\mathrm{istrunc}_{n+1}\, A := (x\ y : A) \to \mathrm{istrunc}_n(x = y)$$

Then we define $\mathrm{isconn}_n\, A := \mathrm{iscontr}\|A\|_n$.

The type of pointed types is $\mathrm{Type}_{\mathrm{pt}} = (A : \mathrm{Type}) \times (\mathrm{pt} : A)$. The type of n-truncated and n-connected types are $\mathrm{Type}^{\leq n} := (A : \mathrm{Type}) \times \mathrm{istrunc}_n\, A$, $\mathrm{Type}^{>n} := (A : \mathrm{Type}) \times \mathrm{isconn}_n\, A$.

Theorem 1. *For $A : \mathrm{Type}_{\mathrm{pt}}$, we have for $n \geq -1$:*

- *A is n-truncated only if $\pi_k A = 0$ for $k > n$ (and "if" holds when A is hypercomplete).*
- *A is n-connected if and only if $\pi_k A = 0$ for $k \leq n$.*

For $A : \mathrm{Type}_{\mathrm{pt}}$ we define the *n-connected cover* of A to be $A\langle n\rangle := \mathrm{fib}(A \to \|A\|_n)$.

Theorem 2. *For $A : \mathrm{Type}_{\mathrm{pt}}$, $n \geq -2$ and $k \geq 0$ we have*

$$\|\Omega^k A\|_n = \Omega^k \|A\|_{n+k}.$$

2 Higher groups

Recall that types in HoTT may be viewed as ∞-groupoids: elements are objects, paths are morphisms, higher paths are higher morphisms, etc.

It follows that *pointed connected* types A may be viewed as higher groups, with *carrier* $\Omega A = (\mathrm{pt} = \mathrm{pt})$.

Writing G for the carrier, it's common to write BG for the pointed connected type such that $G = \Omega BG$. Let us write

$$\mathrm{Grp} := (G : \mathrm{Type}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G = \Omega BG) = \mathrm{Type}_{\mathrm{pt}}^{>0}$$

for the type of higher groups. *N.B.* For $G : \mathrm{Grp}$ we also have $G : \mathrm{Type}$ using the first projection as a coercion. Using the last definition, this is the loop space map, and not the usual coercion!

$k \setminus n$	0	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	pointed 2-groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	3-group	$\cdots$	∞-group
2	abelian group	braided 2-group	braided 3-group	$\cdots$	braided ∞-group
3	— " —	symmetric 2-group	sylleptic 3-group	$\cdots$	sylleptic ∞-group
4	— " —	— " —	symmetric 3-group	$\cdots$	?? ∞-group
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ω	— " —	— " —	— " —	$\cdots$	connective spectrum

Table 1: Periodic table of k-tuply groupal n-groupoids.

We recover[1] the ordinary set-level groups by requiring that G is a 0-type, or equivalently, that BG is a 1-type. This leads us to introduce

$$n\mathrm{Grp} :\equiv (G : \mathrm{Type}^{\le n}) \times (BG : \mathrm{Type}_{\mathrm{pt}}^{>0}) \times (G = \Omega BG) = \mathrm{Type}_{\mathrm{pt}}^{>0, \le n+1}$$

for the type of *groupal* (group-like) n-groupoids, also known as $(n+1)$-groups. For $G : 0\mathrm{Grp}$ a set-level group, we have $BG = K(G, 1)$.

Of course, double loop spaces are even better behaved than mere loop spaces (e.g., they are commutative up to homotopy). Say a type G is *k-tuply groupal* if we have a k-fold delooping, $B^k G : \mathrm{Type}_{\mathrm{pt}}^{\ge k}$, such that $G = \Omega^k B^k G$.

Mixing the two directions, let us introduce the type

$$(n, k)\mathrm{Grp} :\equiv (G : \mathrm{Type}^{\le n}) \times (B^k G : \mathrm{Type}_{\mathrm{pt}}^{\ge k}) \times (G = \Omega^k B^k G) = \mathrm{Type}_{\mathrm{pt}}^{\ge k, \le n+k}$$

for the type of *k-tuply groupal n-groupoids*.[2]

We can also allow k to be infinite, $k = \omega$, but in this case we can't cancel out the G and we must record all the intermediate delooping steps:

$$(n, \omega)\mathrm{Grp} :\equiv \big(B^{-}G : (k : \mathbb{N}) \to \mathrm{Type}_{\mathrm{pt}}^{\ge k, \le n+k}\big) \times \big((k : \mathbb{N}) \to B^k G = \Omega B^{k+1} G\big)$$

When $n = \infty$, this is the type of stably groupal ∞-groups, also known as *connective spectra*. If we also relax the connectivity requirement, we get the type of all spectra, and we can think of a spectrum as a kind of ∞-groupoid with k-morphisms for all $k \in \mathbb{Z}$.

3 Constructions with higher groups

Here are some constructions with higher groups (giving the action on the carriers as well for clarity, but omitting the third components for readability):

decategorication $\mathrm{Decat} : (n, k)\mathrm{Grp} \to (n-1, k)\mathrm{Grp}$

$\langle G, B^k G \rangle \mapsto \langle \|G\|_{n-1}, \|B^k G\|_{n+k-1} \rangle$

[1]this requires some honest toil!

[2]this is called $n\mathrm{Type}_k$ in [1], but here we give equal billing to n and k.

discrete categorification $\mathrm{Disc} : (n,k)\mathrm{Grp} \to (n+1,k)\mathrm{Grp}$
$\langle G, B^k G\rangle \mapsto \langle G, B^k G\rangle$, and $\mathrm{Disc} \dashv \mathrm{Decat}$ with $\mathrm{Decat} \circ \mathrm{Disc} = \mathrm{id}$

looping $\Omega : (n,k)\mathrm{Grp} \to (n-1,k+1)\mathrm{Grp}$
$\langle G, B^k G\rangle \mapsto \langle \Omega G, B^k G\langle k\rangle\rangle$

delooping $\mathrm{B} : (n,k)\mathrm{Grp} \to (n+1,k-1)\mathrm{Grp}$
$\langle G, B^k G\rangle \mapsto \langle \Omega^{k-1} B^k G, B^k G\rangle$, and $\mathrm{B} \dashv \Omega$ with $\Omega \circ \mathrm{B} = \mathrm{id}$

forgetting $F : (n,k)\mathrm{Grp} \to (n,k-1)\mathrm{Grp}$
$\langle G, B^k G\rangle \mapsto \langle G, \Omega B^k G\rangle$

stabilization $S : (n,k)\mathrm{Grp} \to (n,k+1)\mathrm{Grp}$
$\langle G, B^k G\rangle \mapsto \langle SG, \|\Sigma B^k G\|_{n+k+1}\rangle$,
where $SG = \|\Omega^{k+1}\Sigma B^k G\|_n$ and $S \dashv F$.

Theorem 3 (Freudenthal)**.** *If $A : \mathrm{Type}_{\mathrm{pt}}^{>n}$ with $n \geq 0$, then the map $A \to \Omega\Sigma A$ is $2n$-connected.*

Corollary 1 (Stabilization)**.** *If $k \geq n+2$, then $S : (n,k)\mathrm{Grp} \to (n,k+1)\mathrm{Grp}$ is an equivalence, and any $G : (n,k)\mathrm{Grp}$ is an infinite loop space.*

For example, for $G : (0,2)\mathrm{Grp}$ an abelian group, we have $B^n G = K(G,n)$, an Eilenberg-MacLane space.

The adjunction $S \dashv F$ implies that the free group on a pointed set X is $\Omega\|\Sigma X\|_1 = \pi_1(\Sigma X)$. If X has decidable equality, ΣX is already 1-truncated. It is an open problem whether this is true in general.

Also, the abelianization of a set-level group $G : 0\mathrm{Grp}$ is $\pi_2(\Sigma BG)$. If $G : (n,k)\mathrm{Grp}$ is in the stable range ($k \geq n+2$), then $SFG = G$.

4 Homomorphisms and automorphisms

For $G, H : (n,k)\mathrm{Grp}$, define $\hom_{(n,k)}(G,H) := (B^k G \to_{\mathrm{pt}} B^k H)$. For spectra we need pointed maps between all the deloopings and pointed homotopies showing they cohere.

Note that if $h, k : G \to H$ are homomorphisms between set-level groups, then h and k are *conjugate* if $Bh, Bk : BG \to_{\mathrm{pt}} BH$ are *freely* homotopic (i.e., equal as maps $BG \to BH$).

Also observe that $\pi_j(B^k G \to_{\mathrm{pt}} B^k H) = \|\Sigma^j B^k G \to_{\mathrm{pt}} B^k H\|_0 = 0$ for $j + k - 1 \geq n + k$, that is, for $j > n$, so $\hom_{(n,k)}(G,H)$ is actually n-truncated.

If $k \geq n+2$ (so we're in the stable range), then $\hom_{(n,k)}(G,H)$ becomes a stably groupal n-groupoid. This generalizes the fact that the homomorphisms between abelian groups form an abelian group.

Given *any* type of objects U, any $a : U$ has an *automorphism group* $\mathrm{aut}\, a = (a = a)$ with $B \,\mathrm{aut}\, a = \mathrm{im}(a : 1 \to U) = (x : U) \times \|a = x\|_{-1}$ (the connected component of U at a). Clearly, if U is $n+1$-truncated, then so is $B\,\mathrm{aut}\, a$ and so $\mathrm{aut}\, a$ is n-truncated.

Now, the automorphism group $\mathrm{aut}\, G$ of a $G : (n,k)\mathrm{Grp}$ is in $(n,1)\mathrm{Grp}$. But we can also forget the basepoint and consider the automorphism group $\mathrm{aut}^c\, G$ of $B^k G : \mathrm{Type}^{\geq k, \leq n+k}$. This now allows for (higher) conjugations. We define the *generalized center* of G to be $ZG := \Omega^k \mathrm{aut}^c\, G : (n,k+1)\mathrm{Grp}$ (generalizing the center of a set-level group).

5 Group actions

In this section we consider a fixed group $G : \mathrm{Grp}$ with delooping BG. An *action* of G on some object of type U is simply a function $X : BG \to U$. The object of the action is $X(\mathrm{pt}) : U$, and it can be convenient to consider evaluation at $\mathrm{pt} : BG$ to be a coercion from actions of type U to U. To equip $a : U$ with a G-action is to give an action $X : BG \to U$ with $X(\mathrm{pt}) = a$. The *trivial action* is the constant function at a. Clearly, an action of G on $a : U$ is the same as a homomorphism $G \to \operatorname{aut} a$.

If U is a universe of types, then we have actions on types. If X is an action on types, then we can form the:

invariants $X^{hG} :\equiv (x : BG) \to X(x)$, also known as the *homotopy fixed points*

coinvariants $X_{hG} :\equiv (x : BG) \times X(x)$, also known as the *homotopy quotient* $X /\!\!/ G$.

Every group G carries two canonical actions on itself:

the right action $G : BG \to \mathrm{Type}$, $G(x) = (\mathrm{pt} = x)$

the adjoint action $G^{\mathrm{ad}} : BG \to \mathrm{Type}$, $G^{\mathrm{ad}}(x) = (x = x)$ (by conjugation).

We have $G /\!\!/ G = 1$ and $G^{\mathrm{ad}} /\!\!/ G = LG = (S^1 \to G)$, the free loop space of G.

By definition, BG classifies *principal G-bundles*: pullbacks of the right action of G.

6 Projective spaces

Consider the sequence of actions $GM^n : BG \to \mathrm{Type}$ of G given by

$$GM^{-1}(x) :\equiv 0$$
$$GM^{n+1}(x) :\equiv (\mathrm{pt} = x) * GM^n(x) = G(x) * GM^n(x)$$

i.e., the iterated joins of the right action with itself (M is for Milnor). The types $GM^n(\mathrm{pt})$ are at least $(n+1)(k+2)-2$-connected if G is k-connected, so the colimit $GM^\infty(\mathrm{pt}) = \varinjlim GM^n(\mathrm{pt})$ is contractible, so $GM^\infty /\!\!/ G = 1 /\!\!/ G = BG$. We define the *projective spaces* for G to be $GP^n :\equiv GM^n /\!\!/ G$. Thus, $GP^{-1} = 0$, $GP^0 = 1$, $GP^1 = \Sigma G$, etc. (in general, GP^{n+1} is the mapping cone on the inclusion $GM^n(\mathrm{pt}) \to GP^n$).

The *real* and *complex projective spaces* are $\mathbb{R}\mathrm{P}^n :\equiv O(1)P^n$ and $\mathbb{C}\mathrm{P}^n :\equiv U(1)P^n$, where $O(1)$ is the 2-element group, and $U(1)$ is the circle group. Note that $O(1)M^n = S^n$ and $U(1)M^n = S^{2n+1}$.

Given any group G, the *Hopf construction* gives a principal G-bundle $G \hookrightarrow G * G \to \Sigma G = GP^1$, classified by the projection $H : GP^1 \to BG$, which corresponds under the *clutching construction* to the identity on G. The clutching construction is simply taking the adjoint of a map $A \to \Omega BG$ to give a map $\Sigma A \to_{\mathrm{pt}} BG$.

References

[1] John C. Baez and James Dolan. “Categorification”. In: *Higher category theory (Evanston, IL, 1997)*. Vol. 230. Contemp. Math. Amer. Math. Soc., Providence, RI, 1998, pp. 1–36. DOI: 10.1090/conm/230/03336.