1 Truncation and connectedness

Recall that we define istrunc$_n A$ by recursion on $n : \mathbb{N}_2$:

$$
\begin{align*}
\text{istrunc}_{-2} A &= \text{iscontr} A = (a : A) \times (\{x : A \to (a = x)\}) \\
\text{istrunc}_{n+1} A &= (x y : A) \to \text{istrunc}_n(x = y)
\end{align*}
$$

Then we define isconn$_n A := \text{iscontr} \|A\|_n$.

The type of pointed types is $\text{Type}_\text{pt} = (A : \text{Type}) \times (pt : A)$. The type of $n$-truncated and $n$-connected types are $\text{Type}^{\leq n} := (A : \text{Type}) \times \text{istrunc}_n A$, $\text{Type}^{> n} := (A : \text{Type}) \times \text{isconn}_n A$.

**Theorem 1.** For $A : \text{Type}_\text{pt}$, we have for $n \geq -1$:

- $A$ is $n$-truncated only if $\pi_k A = 0$ for $k > n$ (and “if” holds when $A$ is hypercomplete).
- $A$ is $n$-connected if and only if $\pi_k A = 0$ for $k \leq n$.

For $A : \text{Type}_\text{pt}$ we define the $n$-connected cover of $A$ to be $A \langle n \rangle := \text{fib}(A \to \|A\|_n)$.

**Theorem 2.** For $A : \text{Type}_\text{pt}$, $n \geq -2$ and $k \geq 0$ we have

$$
\|\Omega^k A\|_n = \Omega^k \|A\|_{n+k}.
$$

2 Higher groups

Recall that types in HoTT may be viewed as $\infty$-groupoids: elements are objects, paths are morphisms, higher paths are higher morphisms, etc.

It follows that pointed connected types $A$ may be viewed as higher groups, with carrier $\Omega A = (pt = pt)$.

Writing $G$ for the carrier, it’s common to write $BG$ for the pointed connected type such that $G = \Omega BG$. Let us write

$$
\text{Grp} := (G : \text{Type}) \times (BG : \text{Type}^{> 0}_{\text{pt}}) \times (G = \Omega BG) = \text{Type}^{> 0}_{\text{pt}}
$$

for the type of higher groups. *N.B.* For $G : \text{Grp}$ we also have $G : \text{Type}$ using the first projection as a coercion. Using the last definition, this is the loop space map, and not the usual coercion!
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<th>1</th>
<th>2</th>
<th>...</th>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td>connective spectrum</td>
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Table 1: Periodic table of k-tuply groupal n-groupoids.

We recover\(^1\) the ordinary set-level groups by requiring that $G$ is a 0-type, or equivalently, that $BG$ is a 1-type. This leads us to introduce

$$n\text{Grp} := (G : \text{Type}^{\leq n}) \times (BG : \text{Type}_{\text{pt}}^{>0}) \times (G = \Omega BG) = \text{Type}_{\text{pt}}^{0, \leq n+1}$$

for the type of groupal (group-like) $n$-groupoids, also known as $(n+1)$-groups. For $G : 0\text{Grp}$ a set-level group, we have $BG = K(G, 1)$.

Of course, double loop spaces are even better behaved than mere loop spaces (e.g., they are commutative up to homotopy). Say a type $G$ is k-tuply groupal if we have a $k$-fold delooping, $B^k G : \text{Type}_{\text{pt}}^{>k}$, such that $G = \Omega^k B^k G$.

Mixing the two directions, let us introduce the type

$$(n, k)\text{Grp} := (G : \text{Type}^{\leq n}) \times (B^k G : \text{Type}_{\text{pt}}^{>k}) \times (G = \Omega^k B^k G) = \text{Type}_{\text{pt}}^{>k, \leq n+k}$$

for the type of $k$-tuply groupal $n$-groupoids.\(^2\)

We can also allow $k$ to be infinite, $k = \omega$, but in this case we can’t cancel out the $G$ and we must record all the intermediate delooping steps:

$$(n, \omega)\text{Grp} := (B^{-k} G : (k : \mathbb{N}) \to \text{Type}_{\text{pt}}^{>k, \leq n+k}) \times ((k : \mathbb{N}) \to B^k G = \Omega B^{k+1} G)$$

When $n = \infty$, this is the type of stably groupal $\infty$-groups, also known as connective spectra. If we also relax the connectivity requirement, we get the type of all spectra, and we can think of a spectrum as a kind of $\infty$-groupoid with $k$-morphisms for all $k \in \mathbb{Z}$.

## 3 Constructions with higher groups

Here are some constructions with higher groups (giving the action on the carriers as well for clarity, but omitting the third components for readability):

**decategorication** $\text{Decat} : (n, k)\text{Grp} \to (n - 1, k)\text{Grp}$

$$(G, B^k G) \mapsto \langle ||G||_{n-1}, ||B^k G||_{n+k-1} \rangle$$

\(^1\)this requires some honest toil!

\(^2\)this is called $n\text{Type}_k$ in [1], but here we give equal billing to $n$ and $k$. 

2
discrete categorification \( \text{Disc} : (n,k)\text{Grp} \to (n+1,k)\text{Grp} \)\(\langle G, B^k G \rangle \mapsto \langle G, B^k G \rangle, \) and \( \text{Disc} \dashv \text{Decat} \) with \( \text{Decat} \circ \text{Disc} = \text{id} \)

looping \( \Omega : (n,k)\text{Grp} \to (n-1,k+1)\text{Grp} \)
\(\langle G, B^k G \rangle \mapsto \langle \Omega G, B^k G(k) \rangle \)

delooping \( B : (n,k)\text{Grp} \to (n+1,k-1)\text{Grp} \)
\(\langle G, B^k G \rangle \mapsto \langle \Omega^{k-1} B^k G, B^k G \rangle, \) and \( B \dashv \Omega \) with \( \Omega \circ B = \text{id} \)

forgetting \( F : (n,k)\text{Grp} \to (n,k-1)\text{Grp} \)
\(\langle G, B^k G \rangle \mapsto \langle G, \Omega B^k G \rangle \)

stabilization \( S : (n,k)\text{Grp} \to (n,k+1)\text{Grp} \)
\(\langle G, B^k G \rangle \mapsto \langle S G, \| \Sigma B^k G \|_{n+k+1} \rangle, \)
where \( S G = \| \Omega^{k+1} \Sigma B^k G \|_n \) and \( S \dashv F. \)

Theorem 3 (Freudenthal). If \( A : \text{Type}^{>n}_{\text{pt}} \) with \( n \geq 0, \) then the map \( A \to \Omega \Sigma A \) is \( 2n \)-connected.

Corollary 1 (Stabilization). If \( k \geq n+2, \) then \( S : (n,k)\text{Grp} \to (n,k+1)\text{Grp} \) is an equivalence, and any \( G : (n,k)\text{Grp} \) is an infinite loop space.

For example, for \( G : (0,2)\text{Grp} \) an abelian group, we have \( B^n G = K(G,n), \) an Eilenberg-MacLane space.

The adjunction \( S \dashv F \) implies that the free group on a pointed set \( X \) is \( \Omega \| \Sigma X \|_1 = \pi_1(\Sigma X). \) If \( X \) has decidable equality, \( \Sigma X \) is already 1-truncated. It is an open problem whether this is true in general.

Also, the abelianization of a set-level group \( G : 0\text{Grp} \) is \( \pi_2(\Sigma BG). \) If \( G : (n,k)\text{Grp} \) is in the stable range \( (k \geq n+2), \) then \( SFG = G. \)

4 Homomorphisms and automorphisms

For \( G, H : (n,k)\text{Grp}, \) define \( \text{hom}_{(n,k)}(G, H) := (B^k G \to_{\text{pt}} B^k H). \) For spectra we need pointed maps between all the deloopings and pointed homotopies showing they cohere.

Note that if \( h, k : G \to H \) are homomorphisms between set-level groups, then \( h \) and \( k \) are conjugate if \( B h, B k : BG \to_{\text{pt}} BH \) are freely homotopic (i.e., equal as maps \( BG \to BH). \)

Also observe that \( \pi_j(B^k G \to_{\text{pt}} B^k H) = \| \Sigma^j B^k G \to_{\text{pt}} B^k H \|_0 = 0 \) for \( j+k-1 \geq n+k, \) that is, for \( j > n, \) so \( \text{hom}_{(n,k)}(G, H) \) is actually \( n \)-truncated.

If \( k \geq n+2 \) (so we’re in the stable range), then \( \text{hom}_{(n,k)}(G, H) \) becomes a stably groupal \( n \)-groupoid. This generalizes the fact that the homomorphisms between abelian groups form an abelian group.

Given \( \text{any} \) type of objects \( U, \) any \( a : U \) has an automorphism group \( \text{aut} a = (a = a) \) with \( B \text{aut} a = \text{im}(a : 1 \to U) = (x : U) \times \| a = x \|_{-1} \) (the connected component of \( U \) at \( a). \) Clearly, if \( U \) is \( n+1 \)-truncated, then so is \( B \text{aut} a \) and so \( \text{aut} a \) is \( n \)-truncated.

Now, the automorphism group \( \text{aut} G \) of a \( G : (n,k)\text{Grp} \) is in \( (n,1)\text{Grp}. \) But we can also forget the basepoint and consider the automorphism group \( \text{aut}^c G \) of \( B^k G : \text{Type}^{\geq k, \leq n+k}. \) This now allows for (higher) conjugations. We define the generalized center of \( G \) to be \( ZG := \Omega^k \text{aut}^c G : (n, k+1)\text{Grp} \) (generalizing the center of a set-level group).
5 Group actions

In this section we consider a fixed group $G : \text{Grp}$ with delooping $BG$. An action of $G$ on some object of type $U$ is simply a function $X : BG \to U$. The object of the action is $X(\text{pt}) : U$, and it can be convenient to consider evaluation at $\text{pt} : BG$ to be a coercion from actions of type $U$ to $U$. To equip $a : U$ with a $G$-action is to give an action $X : BG \to U$ with $X(\text{pt}) = a$. The trivial action is the constant function at $a$. Clearly, an action of $G$ on $a : U$ is the same as a homomorphism $G \to \text{aut} a$.

If $U$ is a universe of types, then we have actions on types. If $X$ is an action on types, then we can form the:

invariants $X^{hG} := (x : BG) \to X(x)$, also known as the homotopy fixed points

coinvariants $X_{hG} := (x : BG) \times X(x)$, also known as the homotopy quotient $X // G$.

Every group $G$ carries two canonical actions on itself:

the right action $G : BG \to \text{Type}$, $G(x) = (\text{pt} = x)$

the adjoint action $G^{\text{ad}} : BG \to \text{Type}$, $G^{\text{ad}}(x) = (x = x)$ (by conjugation).

We have $G // G = 1$ and $G^{\text{ad}} // G = LG = (S^1 \to G)$, the free loop space of $G$.

By definition, $BG$ classifies principal $G$-bundles: pullbacks of the right action of $G$.

6 Projective spaces

Consider the sequence of actions $GM^n : BG \to \text{Type}$ of $G$ given by

$$GM^{-1}(x) := 0$$

$$GM^n(x) := (\text{pt} = x) * GM^n(x) = G(x) * GM^n(x)$$

i.e., the iterated joins of the right action with itself ($M$ is for Milnor). The types $GM^n(\text{pt})$ are at least $(n + 1)(k + 2) - 2$-connected if $G$ is $k$-connected, so the colimit $GM^\infty(\text{pt}) = \lim\to GM^n(\text{pt})$ is contractible, so $GM^\infty // G = 1 // G = BG$. We define the projective spaces for $G$ to be $GP^n := GM^n // G$. Thus, $GP^{-1} = 0$, $GP^0 = 1$, $GP^1 = \Sigma G$, etc. (in general, $GP^{n+1}$ is the mapping cone on the inclusion $GM^n(\text{pt}) \to GP^n$).

The real and complex projective spaces are $\mathbb{R}P^n := O(1)P^n$ and $\mathbb{C}P^n := U(1)P^n$, where $O(1)$ is the 2-element group, and $U(1)$ is the circle group. Note that $O(1)M^n = S^n$ and $U(1)M^n = S^{2n+1}$.

Given any group $G$, the Hopf construction gives a principal $G$-bundle $G \hookrightarrow G * G \to \Sigma G = GP^1$, classified by the projection $H : GP^1 \to BG$, which corresponds under the clutching construction to the identity on $G$. The clutching construction is simply taking the adjoint of a map $A \to \Omega BG$ to give a map $\Sigma A \to pt BG$.

References