Formalizing forcing arguments in subsystems of second-order arithmetic

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We will review the paper *Formalizing forcing arguments in subsystems of second-order arithmetic* by Jeremy Avigad.[2]

We will also draw on Avigad’s survey article about forcing in proof theory.[1]

The main result of [2] is an effective version of the following theorem:

**Theorem (Harrington)**

\[ \text{WKL}_0 \text{ is conservative over } \text{RCA}_0 \text{ for } \Pi^1_1\text{-sentences.} \]

(Friedman had earlier shown conservativity for \( \Pi^0_2 \)-sentences.)
The Brown-Simpson Extension

In fact, Avigad is able to treat the following extension:

**Theorem (Brown-Simpson)**

\( \text{WKL}^+_0 \text{ is conservative over } \text{RCA}_0 \text{ for } \Pi^1_1\text{-sentences, where } \text{WKL}^+_0 \text{ is } \text{WKL}_0 \text{ plus} \)

\[\forall n \forall \sigma \exists \tau \subset \sigma \varphi(n, \tau) \rightarrow \exists f \forall n \exists m \varphi(n, f[m]), \quad \text{(BCT)}\]

where \(\sigma\) and \(\tau\) range over binary sequences, \(f\) is a function with range \(\{0, 1\}\) and \(f[m]\) denotes the sequence \(\langle f(0), \ldots, f(m - 1)\rangle\).

(BCT) implies a version of the Baire Category Theorem, and the Open Mapping Theorem for separable Banach spaces.
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Kripke structure

Recall that a Kripke structure for a first-order relational language consists of a tuple $\langle P, D, \models \rangle$ where

- $P$ is an inhabited poset, elements of which are called “conditions,”
- $D$ assigns to each $p \in P$ a set, $D(p)$, to be the “domain at $p$”,
- for each relation symbol $R$ and each $p \in P$, $p \models R(\bar{a})$ denotes a relation on $D(p)$.

These data are required to satisfy the monotonicity requirements: for $q \leq p$ (“$q$ is stronger than $p$”):

- $D(q) \supset D(p)$,
- if $p \models R(\bar{a})$, then $q \models R(\bar{a})$. 
The classical forcing relation

1. \( p \models \varphi \land \psi \text{ if and only if } p \models \varphi \text{ and } p \models \psi, \)
2. \( p \models \varphi \lor \psi \text{ if and only if } p \models \varphi \text{ or } p \models \psi, \)
3. \( p \models \varphi \rightarrow \psi \text{ if and only if } \forall q \leq p \,(q \models \varphi \rightarrow q \models \psi), \)
4. \( p \models \forall x \varphi(x) \text{ if and only if } \forall q \leq p \forall a \in D(q) \,(q \models \varphi(a)), \)
5. \( p \models \exists x \varphi(x) \text{ if and only if } \exists a \in D(p) \,(p \models \varphi(a)). \)
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The above considerations take models at face value. By formalizing a forcing analysis of one theory $T_1$ inside another theory, $T_2$, we may be able to obtain stronger results, for instance concerning lengths of proofs.

To do this, define in $T_2$ predicates $\text{Cond}(p)$, $q \leq p$, and $p \models \text{Name}(x)$ (of the displayed variables). Then we define, also in $T_2$, for each relation symbol $R$ in the language of $T_1$, a relation $p \models R(\vec{a})$. Finally, we prove in $T_2$ that this determines a Kripke structure where the axioms of $T_1$ are forced, and that forcing respects the logic of $T_1$. 
Assume the setup of the previous slide. Then the upshot is that whenever $T_1$ proves $\varphi$, $T_2$ proves that $\varphi$ is forced.

Now, if $T_2$ proves that $\bot$ is not forced, then this shows that $T_1$ is consistent relative to $T_2$.

Further, if for a class $\mathcal{F}$ of formulae $\varphi$, $T_2$ proves that $\models \varphi$ is equivalent to $\varphi$, then the interpretation shows that $T_1$ is conservative over $T_2$ relative to $\mathcal{F}$.
Examples of results

Let’s consider a few examples of this approach, the first of which will be the focus of the presentation:

Recall that the theory WKL₀ extends RCA₀ with the following axiom

\[ \forall T \ (T \text{ is an infinite binary tree} \rightarrow \exists P \ (P \text{ is a path through } T)) \].

It is an old result of Friedman that WKL₀ is conservative over RCA₀ for \( \Pi^0_2 \)-sentences. Harrington strengthened this to \( \Pi^1_1 \)-conservativity. These relied on model-theoretic arguments that gave no effective means of translating proofs using the above axiom to proofs without it.

Hájek [3] provided an effective version using recursion-theoretic coding techniques, whereas Avigad obtained an effective version by formalizing Harrington’s forcing argument.
Goodman’s theorem

Another example is provided by Beeson’s version of Goodman’s theorem:

**Theorem**

\[ \text{HA}^\omega + (\text{AC}) + (\text{Ext}) \text{ is a conservative extension of HA}^\omega. \]

Beeson formulated this as composition of:

- a realizability argument exploiting the fact that HA\(^\omega\) proves that the axiom of choice is realizable, with
- the observation that in the negative fragment,”\( \varphi \) is realizable” is equivalent to \( \varphi \), with
- a forcing argument adding “generic” functions to verify the axiom of choice, coding up witnesses to \( \lor \)- and \( \exists \)-subformulas.
As a final example, let us mention Thierry Coquand’s observation that forcing can be used to show that $I\Sigma_1$ is $\Pi^0_2$-conservative over its intuitionistic counterpart, $I\Sigma^i_1$.

Here, the double-negation translation isn’t adequate by itself, since the translation of an instance of $\Sigma_1$-induction isn’t again an instance of $\Sigma_1$-induction. However, adding Markov’s principle

$$\neg\forall x \ A \rightarrow \exists x \ \neg A,$$

for quantifier-free $A$; translated $\Sigma_1$-sentences become equivalent to $\Sigma_1$-sentences. Thus, we need to interpret $I\Sigma^i_1 + (\text{MP})$ in $I\Sigma^i_1$. 

Conservativity of $I\Sigma_1$ over $I\Sigma^i_1$, continued

To do this, take conditions to be codes of finite sets of $\Pi^0_1$-sentences,

$$p = \neg \{ \forall x A_1(x), \forall x A_2(x), \ldots, \forall x A_n(x) \} \neg,$$

with $p \leq q$ if and only if $p \supseteq q$. For atomic $\theta$, define $p \models \theta$ to be

$$\exists y (A_1(y) \land \cdots \land A_n(y) \rightarrow \theta).$$

Then Markov’s Principle is forced.

For further applications: see [1].
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In general outline, Harrington’s argument starts with a model of RCA$_0$ and constructs a sequence of models

\[ M = M_0 \subset_\omega M_1 \subset_\omega M_2 \subset_\omega \cdots \subset_\omega M_i \subset_\omega \cdots \]

where each $M_i$ is a model of RCA$_0$, and if $T$ is an infinite binary tree in $M_i$, there is a $j > i$ such that $M_j$ contains an infinite path through $T$. Then $\bigcup M_i$ models WKL$_0$.

Avigad replicates this argument syntactically as follows.
Good strong forcing notions

Definition
Assume $\text{Cond}(P)$ and $P \leq Q$ have been defined so that the base theory proves that the class of conditions forms a partial order. Definitions “$P \models^s \text{Name}(x)$” and “$P \models^s \varphi$” (for atomic $\varphi$) form a good strong forcing notion if the following holds:

1. The free variables of $P \models^s \varphi$ are $P$ together with those of $\varphi$; the free variables of $P \models^s \text{Name}(X)$ are $P$ and $X$,
2. Monotonicity; the base theory proves that $P \models^s \varphi$ and $P \models^s \text{Name}(X)$ are monotone in $P$,
3. Substitution; for each term $t$,

$$P \models^s \varphi(x/t) \iff (P \models^s \varphi)(x/t).$$
Good strong forcing notions extended

Definition

We extend a given good strong forcing notion to arbitrary formulas of $\mathcal{L}^2$ as follows:

1. $P \vdash^s \neg \varphi :\iff \forall Q \leq P \neg (Q \vdash^s \varphi)$,
2. $P \vdash^s \varphi \land \psi :\iff (P \vdash^s \varphi) \land (P \vdash^s \psi)$,
3. $P \vdash^s \varphi \lor \psi :\iff (P \vdash^s \varphi) \lor (P \vdash^s \psi)$,
4. $P \vdash^s \varphi \rightarrow \psi :\iff \forall Q \leq P (Q \vdash^s \varphi \rightarrow \exists R \leq Q (R \vdash^s \psi))$,
5. $P \vdash^s \exists x \varphi :\iff \exists x (P \vdash^s \varphi)$,
6. $P \vdash^s \forall x \varphi :\iff \forall x \forall Q \leq P \exists R \leq Q (R \vdash^s \varphi)$,
7. $P \vdash^s \exists X \varphi :\iff \exists X (P \vdash^s \text{Name}(X) \land P \vdash^s \varphi)$,
8. $P \vdash^s \forall X \varphi :\iff \forall X \forall Q \leq P \exists R \leq Q (R \vdash^s \text{Name}(X) \rightarrow R \vdash^s \varphi)$.

The extended forcing relation satisfies monotonicity and substitution.
Good weak forcing notions

A problem with good strong forcing notions is that they don’t necessarily preserve logic! Basically this occurs when they’re not “not-not stable,” so we define:

Definition

A good strong forcing notion \( P \models \phi \) is a good weak forcing notion if the base theory additionally proves \( P \models \phi \iff P \models \neg \neg \phi \). The extension to arbitrary formulae differs from strong forcing in the following clauses:

3. \( P \models \phi \lor \psi :\iff \forall Q \leq P \exists R \leq Q ((R \models \phi) \lor (R \models \psi)) \),

5. \( P \models \exists x \phi :\iff \forall Q \leq P \exists R \leq Q \exists x (R \models \phi) \),

6. \( P \models \forall x \phi :\iff \forall x (P \models \phi) \),

7. \( P \models \exists X \phi :\iff \forall Q \leq P \exists R \leq Q \exists X (R \models \text{Name}(X) \land R \models \phi) \),

8. \( P \models \forall X \phi :\iff \forall X (P \models \text{Name}(X) \rightarrow P \models \phi) \).
An extended good weak forcing notion satisfies monotonicity, substitution and stability:

\[ P \models \varphi \iff P \models \neg \neg \varphi. \]

**Lemma**

*Suppose \( \models^s \) is a good strong forcing notion. Then the relation \( \models \) given by

\[ P \models \varphi \iff P \models^s \neg \neg \varphi \]

\((*)\)

is a good weak forcing theory, and \((*)\) will hold as well for the extended relations.*
To force WKL, the first step is to add a generic path through some infinite tree. We do this in two steps: first we a single generic path ($\frac{1}{2}$-forcing); then we add all sets recursively definable from this new path and old set, so as to model RCA$_0$ (1-forcing). Both will be weak forcing relations.

**Definition**

1/2-conditions are infinite binary trees:

$$\text{Cond}_{\frac{1}{2}}(P) :\Leftrightarrow P \text{ is an binary tree } \land \forall n \exists \sigma \in P \left( \text{len}(\sigma) = n \right),$$

(this is equivalent to a $\Pi^0_1$-formula), and we let $P \leq_{\frac{1}{2}} Q :\Leftrightarrow P \subset Q$.

The $\frac{1}{2}$-names are $\hat{X} = \{ \langle 0, x \rangle \mid x \in X \}$ for old sets and $\hat{G} = \{ \langle 1, 0 \rangle \}$ for the new generic.
The $\frac{1}{2}$-forcing relation

We define:

\[
P \models_{\frac{1}{2}} t_1 = t_2 :\iff t_1 = t_2
\]

\[
P \models_{\frac{1}{2}} t \in \hat{X} :\iff t \in X
\]

\[
P \models_{\frac{1}{2}} t \in \hat{G} :\iff \exists n \forall \sigma (\sigma \in P \land \text{len} \sigma = n \rightarrow t \subset \sigma).
\]

The intuition of the last clause is that we want $G$ to be an infinite path through $P$, so at some height $n$, all nodes have $t$ as a prefix (thus, since $P$ is prefix closed, all nodes of height greater than $n$ will also have $t$ as a prefix). Thus, all but finitely many nodes of $P$ have $t$ as a prefix.

This condition is $\Sigma^0_1$. 
Now, the key facts about $\frac{1}{2}$-forcing are:

1. $\frac{1}{2}$-forcing is a good weak forcing notion,
2. $\text{RCA}_0$ proves that for $\varphi$ not mentioning $G$, $\models \frac{1}{2} \varphi(\hat{X})$ is equivalent to $\varphi(X)$,
3. if $\varphi$ is $\Sigma^0_1$ (resp. $\Pi^0_2$), then $\text{RCA}_0$ proves that $\models \frac{1}{2} \varphi$ is equivalent to another $\Sigma^0_1$ (resp. $\Pi^0_2$) formula.
4. $\text{RCA}_0$ proves that $\Sigma^0_1$-induction is $\frac{1}{2}$-generically valid.

The next step is to add names for all sets recursively definable from the $\hat{X}$ and $\hat{G}$. 
We take 1-names to be triples $\langle X, \psi, \chi \rangle$, where $\psi(x, X, G)$ and $\chi(x, X, G)$ are codes of $\Sigma^0_1$ and $\Pi^0_1$-formulas determining a set which is recursive in $X$ and $G$.

Let $\text{Tr}_{\Sigma^0_1}$ and $\text{Tr}_{\Pi^0_1}$ be suitable truth predicates. Then we define

$$P \models_{1} \text{Name}(\langle X, \psi, \chi \rangle) :\iff P \models_{\frac{1}{2}} \forall x \ (\text{Tr}_{\Sigma^0_1}(\psi, x, \hat{X}, \hat{G}) \iff \text{Tr}_{\Pi^0_1}(\chi, x, \hat{X}, \hat{G})).$$

Then we can set

$$P \models_{1} t \in \langle X, \psi, \chi \rangle :\iff P \models_{\frac{1}{2}} \text{Tr}_{\Sigma^0_1}(\psi, t, \hat{X}, \hat{G}).$$
Now, the key facts about $1$-forcing are:

1. $1$-forcing is a good weak forcing notion,
2. $\text{RCA}_0$ proves that for $\varphi$ not mentioning $G$, $\models_1 \varphi(\hat{X})$ is equivalent to $\varphi(X)$,
3. if $\varphi$ is $\Sigma^0_1$ (resp. $\Pi^0_2$), then $\text{RCA}_0$ proves that $\models_1 \varphi$ is equivalent to another $\Sigma^0_1$ (resp. $\Pi^0_2$) formula.
4. $\text{RCA}_0$ proves that $\Sigma^0_1$-induction is $1$-generically valid.
5. $\text{RCA}_0$ proves that each axiom of $\text{RCA}_0$ is $1$-generically valid.
6. $\text{RCA}_0$ proves that if $P$ is a $1$-condition, then $P \models_1 \exists X \ (X$ is an infinite path through $\hat{P})$. 
From $n$- to $n + 1$-forcing

Once $n$-forcing has been defined, we define:

1. An $n + 1$-condition is a pair $\langle P, P' \rangle$ such that
   \[ \text{Cond}_n(P) \land P \models_n \text{Name}(P') \land P \models_n \text{Cond}_1(P'). \]

2. If $\langle P, P' \rangle$ and $\langle Q, Q' \rangle$ are $n + 1$-conditions, then $\langle P, P' \rangle \leq_{n+1} \langle Q, Q' \rangle$ if and only if
   \[ P \leq_n P' \land P \models_n (P' \leq_1 Q'). \]

3. $\langle P, P' \rangle \models_{n+1} \text{Name} X$ if and only if
   \[ P \models_n (P' \models_1 \text{Name}(X)). \]

4. $\langle P, P' \rangle \models_{n+1} \varphi$ if and only if
   \[ P \models_n (P' \models_1 \varphi). \]
Uniform \( n \)-forcing

By carefully pushing the formula-complexity of \( n \)-forcing to \( \Pi_2^0 \), Avigad is able to find primitive recursive functions of \( n \), giving the notions of condition, order, name, and “element of” as codes of \( \Pi_2^0 \)-formulas (with \( n \) as a parameter).

Then, using the fact that RCA\(_0\) is finitely axiomatizable, we can prove that “if \( \bigwedge \) RCA\(_0\) is \( n \)-forced, then \( \bigwedge \) RCA\(_0\) is \( n + 1 \)-forced.”

Then we take an \( \omega \)-condition to be an \( n \)-condition where \( \bigwedge \) RCA\(_0\) is \( n \)-forced.
ω-forcing: summing up

Now, the key facts about ω-forcing are:

1. ω-forcing is a good weak forcing notion,
2. RCA₀ proves that for φ not mentioning G, \( \models_\omega \varphi(\hat{X}) \) is equivalent to \( \varphi(X) \),
3. if \( \varphi \) is \( \Sigma^0_1 \) (resp. \( \Pi^0_2 \)), then RCA₀ proves that \( \models_1 \varphi \) is equivalent to another \( \Sigma^0_1 \) (resp. \( \Pi^0_2 \)) formula.
4. RCA₀ proves that \( \Sigma^0_1 \)-induction is ω-generically valid.
5. RCA₀ proves that each axiom of RCA₀ is ω-generically valid.
6. RCA₀ proves that (WKL) is ω-generically valid.
The effective version of Brown-Simpson

More work forces also BCT, and by analyzing the transformation, we get

Theorem (Avigad)

There is a recursive function $f$ and a polynomial $p$ such that: if $d$ codes a proof in $\text{WKL}^+_0$ of a $\Pi^1_1$-formula $\varphi$, then $f(d)$ codes a proof of $\varphi$ in $\text{RCA}_0$, and the length of $f(d)$ is less than $p(\text{length of } d)$. 
