Transfinitely iterated fixpoint theories

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Outline

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We summarize the paper by Jäger, Kahle, Setzer, and Strahm: *The proof-theoretic analysis of transfinitely iterated fixed point theories* [JKSS99].

Our goal is to determine the proof-theoretical ordinals of a family of theories, $\hat{\text{ID}}_\alpha$. Part I will sketch the lower bound. Part II will concern the upper bound.
In proof theory we use ordinals to measure infinitary objects. Ordinals are equivalence classes of well-orderings. If $\alpha$ and $\beta$ are ordinals, then $\alpha < \beta$ if (a representative of) $\alpha$ is isomorphic to a proper initial segment of (a representative of) $\beta$. This relation well-orders the class $\text{On}$ of all ordinals. Therefore, we identify an ordinal with its canonical representative:

$$\alpha = \{ \xi \mid \xi < \alpha \}$$

In this way, ordinals become transitive sets that are well-ordered by $\in$ (von Neumann ordinals).
Every well-founded recursive relation $\prec$ on $\omega$ determines an ordinal less than $\omega_1^{CK}$. Kleene noted that every recursive ordinal can be represented by a primitive recursive (even Kalmár elementary) well-order on $\omega$. [Kle55]

Any well-order $\prec$ is order-isomorphic to its representing von Neumann ordinal, $\text{otyp}(\prec) \in \text{On}$, through an *enumerating* function:

$$\text{en}_\prec : \text{otyp}(\prec) \to \text{field}(\prec)$$

Every subclass $M \subset \text{On}$ is well-ordered with either

- $\text{otyp}(M) = \text{On}$ (if $M$ is a proper class), or
- $\text{otyp}(M) \in \text{On}$ (if $M$ is a set).
Normal functions and club classes

Let $\Omega$ be the least uncountable ordinal.

We say that $M \subset \text{On}$ is *unbounded* in $\Omega$ if for each $\alpha < \Omega$ there is is $\beta \in M \cap \Omega$ with $\alpha < \beta$.

We say that $M \subset \text{On}$ is *closed* in $\Omega$ if for every countable $A \subset M$ we have $\sup(A) \in M$.

An order-preserving function $f : \text{On} \rightarrow \text{On}$ is called *normal* on $\Omega$ if $\text{dom}(f) \supset \Omega$ and $f$ is continuous.

**Lemma**

A class $M \subset \text{On}$ is club in $\Omega$ iff $\text{en}_M$ is normal on $\Omega$.

(Analogues of the above exist for each regular cardinal $\kappa \geq \Omega$.)
Given an ordinal $\alpha < \Omega$, the class $\text{On}_\alpha := \{ \beta \in \text{On} \mid \alpha \leq \beta \}$ is club, so the enumerating function is normal on $\Omega$. Its value on $\xi$ is denoted $\alpha + \xi$ and is called the *ordinal sum* of $\alpha$ and $\xi$.

Note that $\alpha + \xi$ can also be defined by transfinite recursion on $\xi$.

Let $\mathbb{H} := \{ \alpha \in \text{On} \mid \alpha \neq 0 \land \forall \xi, \eta < \alpha. \xi + \eta < \alpha \}$. We call ordinals in $\mathbb{H}$ *additively indecomposable* or *principal*.

**Lemma**

$\mathbb{H}$ *is club in* $\Omega$, so $\text{en}_\mathbb{H}$ *is normal*. We write $\omega^\xi := \text{en}_\mathbb{H}(\xi)$. 
Theorem (Cantor Normal Form)

For all ordinals $\alpha \neq 0$, there are uniquely determined principal ordinals $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n.$$

Thus, noting that powers of $\omega$ enumerate the principal ordinals, and collecting equals terms, we get the form

$$\alpha = \omega^{\beta_1}k_1 + \cdots + \omega^{\beta_n}k_n \quad \text{where} \quad \beta_1 > \beta_2 \cdots > \beta_n.$$

The Cantor Normal Form can be used to form a notation system for ordinals less than $\varepsilon_0$. 
Ordinal class differentiation

For $f: \text{On} \rightarrow \text{On}$ put $\text{Fix}(f) := \{ \eta \in \text{dom}(f) \mid f(\eta) = \eta \}$. The derivative of $f$ is defined by

$$f' := \text{en}_{\text{Fix}(f)}$$

We define the derivative of a class $M \subset \text{On}$ as $M' := \text{Fix}(\text{en}_M)$. Facts:

- If $f$ is normal, then $\text{Fix}(f)$ is club, and so $f'$ is again normal.
- If $M$ is club, then $M'$ is again club.
- A countable intersection of club classes is again club.
The Veblen Hierarchy

The Veblen Hierarchy of critical ordinals is defined by

\[
\begin{align*}
\text{Cr}(0) & := \mathcal{H} \\
\text{Cr}(\alpha + 1) & := \text{Cr}(\alpha)' \\
\text{Cr}(\lambda) & := \bigcap_{\xi<\lambda} \text{Cr}(\xi).
\end{align*}
\]

Then we put \( \varphi_\alpha := \text{en}_{\text{Cr}(\alpha)} \). We also write \( \varphi(\alpha, \beta) = \varphi_\alpha(\beta) \). Then

\[
\varphi(0, \beta) = \omega^\beta \quad \text{and} \quad \varphi(1, \beta) = \varepsilon_\beta
\]

**Theorem**

*For \( \alpha < \Omega \), the class \( \text{Cr}(\alpha) \) is club in \( \Omega \), so \( \varphi_\alpha \) is normal on \( \Omega \).*
Theorem

We have:

(A) $\varphi_{\alpha_1}(\beta_1) = \varphi_{\alpha_2}(\beta_2)$ iff either of the following:
   
   (i) $\alpha_1 < \alpha_2 \land \beta_1 = \varphi_{\alpha_2}(\beta_2)$
   
   (ii) $\alpha_1 = \alpha_2 \land \beta_1 = \beta_2$
   
   (iii) $\alpha_1 > \alpha_2 \land \varphi_{\alpha_1}(\beta_1) = \beta_2$

(B) $\varphi_{\alpha_1}(\beta_1) < \varphi_{\alpha_2}(\beta_2)$ iff either of the following:
   
   (i) $\alpha_1 < \alpha_2 \land \beta_1 < \varphi_{\alpha_2}(\beta_2)$
   
   (ii) $\alpha_1 = \alpha_2 \land \beta_1 < \beta_2$
   
   (iii) $\alpha_1 > \alpha_2 \land \varphi_{\alpha_1}(\beta_1) < \beta_2$

This forms the basis for a notation system for ordinals less than $\Gamma_0$. 
The Small Veblen Ordinal

Define $Cr_\xi(\alpha)$ by

\[
\begin{align*}
Cr_0(0) & := H \\
Cr_\xi(\alpha + 1) & := Cr_\xi(\alpha)' \\
Cr_\xi(\lambda) & := \bigcap_{\eta < \lambda} Cr_\xi(\eta) \\
Cr_{\alpha+1}(0) & := \{ \xi < \Omega \mid \xi \in \bigcap_{\lambda < \xi} Cr_\alpha(\lambda) \} \\
Cr_\lambda(0) & := \bigcap_{\xi < \lambda} Cr_\xi(0)
\end{align*}
\]

and put $\varphi_{\xi,\alpha} := \text{en}_{Cr_\xi(\alpha)}$. Continuing this line of reasoning, we get $n$-ary $\varphi$-functions $\Omega^n \to \Omega$ for each $n < \omega$. Letting $\Phi_0$ be the least ordinal not expressible with these, we also get an elementary notation system $\prec$ on $\omega$ for the ordinals less than $\Phi_0$. 

We define the systems $\hat{\text{ID}}_\alpha$ for $\alpha < \Phi_0$. Let $\mathcal{L}$ be the language of first-order arithmetic. Terms are built from variables by means of primitive recursive functions. The predicates are the primitive recursive relations, as well as a unary predicate $U$. 
If $P$ and $Q$ are fresh unary predicates, then we let $\mathcal{L}(P, Q)$ denote the extension of $\mathcal{L}$ with $P$ and $Q$. We let $\mathcal{A}(P, Q, x, y)$ range over formulas of $\mathcal{L}(P, Q)$ that

1. are $P$-positive (only positive literals $P(t)$ occur), and
2. have at most $x$ and $y$ free.

Such formulas are called *inductive operator forms*. 
Transfinite induction

Let \( \sqsubseteq \) be a primitive recursive relation, and let \( A(x) \) be a formula with a distinguished variable. Let \( s \) be a term. Then we set:

\[
A(\sqsubseteq x) := \forall y \sqsubseteq x. A(y)
\]

\[
\text{Prog}(\sqsubseteq, A) := \forall x. A(\sqsubseteq x) \rightarrow A(x)
\]

\[
\text{TI}(\sqsubseteq, A) := \text{Prog}(\sqsubseteq, A) \rightarrow \forall x. A(x)
\]

\[
\text{TI}(\sqsubseteq, s, A) := \text{Prog}(\sqsubseteq, A) \rightarrow A(\sqsubseteq s)
\]

If we just write \( \text{Prog}(A) \), \( \text{TI}(A) \), or \( \text{TI}(s, A) \), it is understood that we take \( \sqsubseteq \) to be \( \prec \).
The language of $\mathcal{I}D_\alpha$

For each $\alpha < \Phi_0$ we have the theory $\mathcal{I}D_\alpha$ of $\alpha$ times iterated fixpoints. The language is $L_{fix}$, which we obtain from $L$ by adding a new unary predicate symbol $P^A$ for each inductive operator form $A(P, Q, x, y)$.

We define:

$$P^A_s(u) := P^A(\langle u, s \rangle)$$

$$P^A_{\prec a}(t) := (t = \langle u, s \rangle) \land s \prec a \land P^A_s(u)$$

The intended meaning is that $P^A_{\prec a} = \sum_{s \prec a} P^A_s$. 

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Axioms of $\hat{\text{ID}}_\alpha$

The axioms of $\hat{\text{ID}}_\alpha$ comprise

(i) those of Peano arithmetic with the induction scheme for all $\mathcal{L}_{\text{fix}}$-formulas,

(ii) fixpoint axiom: for each inductive operator form $A(P, Q, x, y)$ we have

$$\forall a < \alpha. \forall x. P^A_a(x) \leftrightarrow A(P^A_a, P^A_{\prec a}, x, a),$$

(iii) the axioms $\text{TI}(\alpha, A)$ for each $\mathcal{L}_{\text{fix}}$-formula $A$. 
Previous Results

The theories $\widehat{\text{ID}}_n$ were first introduced by Feferman [Fef82]. It was established that the proof-theoretical ordinal of $\widehat{\text{ID}}_n$ is $\alpha_n$, where $\alpha_0 := \varepsilon_0$, and $\alpha_{n+1} := \varphi \alpha_n 0$. Therefore, the union, $\widehat{\text{ID}}_{<\omega}$, has proof-theoretical ordinal $\Gamma_0$, the first predicatively inaccessible ordinal.

JKSS extends these results to $\widehat{\text{ID}}_\alpha$ for $\alpha \geq \omega$. 
Main Theorem of JKSS

Fix an ordinal $\alpha < \Phi_0$. Let $\varepsilon(\alpha)$ denote the least $\varepsilon$-number greater than $\alpha$. Define inductively

$$(\alpha | 0) := \varepsilon(\alpha), \quad (\alpha | m + 1) := \varphi(\alpha | m).$$

Theorem (Main Theorem)

Let $\alpha$ have Cantor Normal Form

$$\alpha = \omega^{1+\alpha_n} + \cdots + \omega^{1+\alpha_1} + m,$$

where $\alpha_n \geq \cdots \geq \alpha_1$ and $m < \omega$. Then the proof-theoretic ordinal of $\hat{\text{ID}}_\alpha$ is

$$|\hat{\text{ID}}_\alpha| = \varphi_1\alpha_n(\varphi_1\alpha_{n-1}(\cdots \varphi_1\alpha_1(\alpha | m)) \cdots).$$
Lower Bounds: Ordinal Analysis of PA

Let us recall how to show that $|\text{PA}| \geq \varepsilon_0$.
Since $\varepsilon_0 = \sup\{ \exp^n(0) \mid n < \omega \}$, this is done by showing:

$$\text{if } \text{PA} \vdash \text{TI}(\alpha, U) \text{ then } \text{PA} \vdash \text{TI}(\omega^\alpha, U)$$

A basic ingredient is the substitution lemma:

**Lemma**

Let $F(U)$ and $G(x)$ be formulas in $\mathcal{L}(U)$. If $\text{PA} \vdash F(U)$, then $\text{PA} \vdash F(\lambda x. G(x))$. 
Define the jump of $X$ by

$$J(X) := \lambda \alpha. \forall \xi. X(\prec \xi) \rightarrow X(\prec \xi + \omega^\alpha)$$

Then one shows first:

**Lemma**

$\text{PA} \vdash \text{Prog}(X) \rightarrow \text{Prog}(J(X))$.

And then:

**Lemma**

$\text{PA} \vdash \text{TI}(\alpha, J(X)) \rightarrow \text{TI}(\omega^\alpha, X)$.

Combining this with substitution, gives the lower bound for $\text{PA}$. 
To use the well-ordering techniques for predicative systems, we will study theories for self-reflecting truth. We will see that these embed into the $\hat{\text{ID}}_\alpha$, where they will facilitate our well-ordering proofs.

Kripke [Kri75] gave in his “Outline of a theory of truth” a simultaneous inductive definition of predicates $T$ and $F$ over a given language, which give partial truth and falsity predicates grounded in atomic formulas.

Here, we iterate this procedure. We use Quine quotes, $\lceil A \rceil$, to mean terms for Gödel numbers.
Theories for self-reflecting truth, language

The language $\mathcal{L}_{\text{srt}}$ extends $\mathcal{L}$ by two binary relation symbols: $T$ and $F$.

$T_s(t)$ means that $t$ is the code of a true formula at level $s$.

$F_s(t)$ means that $t$ is the code of a false formula at level $s$.

The sublanguage $\mathcal{L}_{\text{srt}}^\alpha$ is obtained by restricting levels $s$ in atoms $T_s(t)$ and $F_s(t)$ to closed terms with $s \leq \alpha$. 
Gödelization of $\mathcal{L}_{srt}^{\alpha}$

We Gödelize the languages $\mathcal{L}_{srt}^{\alpha}$ uniformly in $\alpha < \Phi_0$ to get Gödel numbers $\overline{t}$ and $\overline{A}$ for each $\mathcal{L}$-term $t$ and each $\mathcal{L}_{srt}^{\alpha}$-formula $A$.

We have the following primitive recursive functions and predicates:

- $\text{CTer}(x) - x$ codes a closed term of $\mathcal{L}$.
- $\text{For}_n(f,a) - f$ codes an $\mathcal{L}_{srt}^{\alpha}$-formula with at most $n$ free variables.
- $\text{Atm}(f,a) - f$ codes a positive literal of $\mathcal{L}_{srt}^{\alpha}$.
- $\text{num}(x) -$ the code of the $x$th numeral of $\mathcal{L}$.
- $\text{val}(z) -$ the value of the closed term $z$.
- $\text{neg}(f) -$ the negation of the atom $f$.
- $\text{and}(f,g) -$ the conjunction of $f$ and $g$ (also have $\text{or}(f,g)$).
- $\text{all}(x,f) -$ universal quantification of $f$ with respect to the $x$th variable (also have $\text{ext}(f,g)$).
We write \( \text{Sen}(f, a) \) instead of \( \text{For}_0(f, a) \) and \( \dot{x} \) instead of \( \text{num}(x) \).
If \( \text{For}_n(f, a) \) and \( \text{CTer}(x_1), \ldots, \text{CTer}(x_n) \), then \( f(x_1, \ldots, x_n) \) denotes the code of the formula obtained from \( f \) by simultaneously substituting the \( i \)th variable of \( f \) with \( x_i \) for \( i = 1, \ldots, n \).
Similarly, if \( A \) is an \( \mathcal{L}_{srt}^\alpha \)-formula with at most \( n \) free variables, then
\[
\neg A(\dot{x}_1, \ldots, \dot{x}_n) \equiv \text{an abbreviation for } \neg A(\dot{x}_1, \ldots, \dot{x}_n),
\]
and if we furthermore have \( \text{CTer}(x_1), \ldots, \text{CTer}(x_n) \), then
\[
\neg A(x_1, \ldots, x_n) \equiv \text{an abbreviation for } \neg A(x_1, \ldots, x_n).
\]
If furthermore \( R \) is an \( n \)-ary relation symbol of \( \mathcal{L} \), then
\[
\neg R(x_1, \ldots, x_n) \equiv \text{the code of the corresponding atom.}
\]
Lastly, if \( \text{CTer}(a) \) and \( \text{CTer}(x) \), then
\[
\neg T_a(x) \text{ and } \neg F_a(x) \text{ are codes of the corresponding atoms.}
\]
Let $\alpha$ be an ordinal less than $\Phi_0$.
The language of the system $\text{SRT}_\alpha$ for $\alpha$ times iterated self-reflecting truth is the unrestricted $\mathcal{L}_{\text{srt}}$. The theory comprises:

(i) the axioms of Peano arithmetic with the induction scheme for all $\mathcal{L}_{\text{srt}}$-formulas.

(ii) the axioms $\text{TI}(\alpha, A)$ for all $\mathcal{L}_{\text{srt}}$-formulas $A$.

(iii) the axiom groups I to III on the following slides.
SRT_α axiom group I – atomic truth

(1) For each \(n\)-ary relation symbol \(R\) of \(\mathcal{L}\):

\[
\text{CTer}(x_1) \land \cdots \land \text{CTer}(x_n) \land a \prec \alpha \\
\rightarrow \left( T_a(\lceil R(x_1, \ldots, x_n) \rceil) \leftrightarrow R(\text{val}(x_1), \ldots, \text{val}(x_n)) \right) \\
\land \left( F_a(\lceil R(x_1, \ldots, x_n) \rceil) \leftrightarrow \neg R(\text{val}(x_1), \ldots, \text{val}(x_n)) \right)
\]

(2)

\[
\text{CTer}(x) \land \text{CTer}(b) \land \text{val}(b) \prec a \prec \alpha \\
\rightarrow \left( T_a(\lceil T_b(x) \rceil) \leftrightarrow T_{\text{val} b}(\text{val}(x)) \right) \land \left( T_a(\lceil F_b(x) \rceil) \leftrightarrow F_{\text{val} b}(\text{val}(x)) \right) \\
\land \left( F_a(\lceil T_b(x) \rceil) \leftrightarrow \neg T_{\text{val} b}(\text{val}(x)) \right) \land \left( F_a(\lceil F_b(x) \rceil) \leftrightarrow \neg F_{\text{val} b}(\text{val}(x)) \right)
\]
SRT$_\alpha$ axiom group II – composed truth

(3) $\text{Atm}(f, a) \land a \prec \alpha \rightarrow (T_a(\neg (f)) \leftrightarrow F_a(f)) \land (F_a(\neg (f)) \leftrightarrow T_a(f))$.

(4) $\text{Sen}(f, a) \land \text{Sen}(g, a) \land a \prec \alpha$

$\rightarrow (T_a(\text{and}(f, g)) \leftrightarrow T_a(f) \land T_a(g))$

$\land (F_a(\text{and}(f, g)) \leftrightarrow F_a(f) \lor F_a(g))$.

(5) dual axiom for disjunction

(6) $\text{Sen}(\text{all}(v, f), a) \land a \prec \alpha$

$\rightarrow (T_a(\text{all}(v, f)) \leftrightarrow (\forall x)T_a(f(\dot{x})))$

$\land (F_a(\text{all}(v, f)) \leftrightarrow (\exists x)F_a(f(\dot{x})))$.

(7) dual axiom for existential quantification.
SRT$_\alpha$ axiom group III – self-reflecting truth

(8)

$$CTer(x) \land CTer(a) \land val(a) \prec \alpha$$

$$\rightarrow \left( T_{val(a)}(\neg T_a(x)) \leftrightarrow T_{val(a)}(val(x)) \right)$$

$$\land \left( T_{val(a)}(\neg F_a(x)) \leftrightarrow F_{val(a)}(val(x)) \right)$$

$$\land \left( F_{val(a)}(\neg T_a(x)) \leftrightarrow T_{val(a)}(val(x)) \right)$$

$$\land \left( F_{val(a)}(\neg F_a(x)) \leftrightarrow T_{val(a)}(val(x)) \right)$$
Embedding $SRT_{\alpha}$ into $\widehat{ID}_{\alpha}$

We follow Sol’s paper *Reflecting on incompleteness* [Fef91].

Fix $\alpha < \Phi_0$. We want to model $SRT_{\alpha}$ in $\widehat{ID}_{\alpha}$. The idea is to interpret $T$ and $F$ by a fixpoint family $Q$ so that

$$T_a(t) \leftrightarrow Q_a(\langle t, 0 \rangle) \quad \text{and} \quad F_a(t) \leftrightarrow Q_a(\langle t, 1 \rangle).$$

We need to find a suitable inductive operator form $A(Q_a, Q_{\prec \alpha}, x, a)$. We give an informal definition of this formula on the following slides.
Clauses for truth fixpoint family, 1

\[ A(Q_a, Q_{\prec a}, x, a) := \text{let } x = \langle t, y \rangle \text{ in: case of} \]

\begin{align*}
(1) & \quad t = \llbracket R(x_1, \ldots, x_n) \rrbracket \text{ where} \\
& \quad \text{CTer}(x_1) \land \cdots \land \text{CTer}(x_n) : \\
& \quad (R(val(x_1), \ldots, val(x_n)) \land y = 0) \lor \\
& \quad (\neg R(val(x_n), \ldots, val(x_n)) \land y = 1) \\
(2) & \quad t = \llbracket T_b(f) \rrbracket \text{ where} \\
& \quad \text{CTer}(f) \land \text{CTer}(b) \land \text{val}(b) \prec a : \\
& \quad (Q_{\text{val}(b)}(\langle \text{val} f, 0 \rangle) \land y = 0) \lor \\
& \quad (Q_{\text{val}(b)}(\langle \text{val} f, 1 \rangle) \land y = 1) \\
& \quad \cdots
\end{align*}
Clauses for truth fixpoint family, II

\[ (2') \quad t = \neg F_b(f) \quad \text{where} \]
\[ C Ter(f) \land C Ter(b) \land val(b) < a : \]
\[ (Q_{val(b)}(\langle val \, f, 1 \rangle) \land y = 0) \lor \]
\[ (Q_{val(b)}(\langle val \, f, 0 \rangle) \land y = 1) \]

\[ (3) \quad t = \neg (f) \quad \text{where} \quad Atm(f, a): \]
\[ (Q_a(\langle f, 1 \rangle) \land y = 0) \lor (Q_a(\langle f, 0 \rangle) \land y = 1) \]

\[ \ldots \]

(\text{the other conditions (4)--(7) for composed truth are similar})

\[ \ldots \]
Clauses for truth fixpoint family, III

\[ t = \{T_\alpha(f) \} \quad \text{where } C\text{Ter}(f): \]
\[ (Q_a(\langle \text{val } f, 0 \rangle) \land y = 0) \lor \]
\[ (Q_a(\langle \text{val } f, 1 \rangle) \land y = 1) \]

\[ t = \{F_\alpha(f) \} \quad \text{where } C\text{Ter}(f): \]
\[ (Q_a(\langle \text{val } f, 1 \rangle) \land y = 0) \lor \]
\[ (Q_a(\langle \text{val } f, 0 \rangle) \land y = 1) \]
Ramified sets in $\text{SRT}_\alpha$

The systems $\text{SRT}_\alpha$ has a natural notion of ramified subsets of $\omega$, conceived of as propositional functions. We set:

$$f \in S_a := \text{For}_1(f, a) \land \forall x. T_a(f(\dot{x})) \leftrightarrow \neg F_a(f(\dot{x}))$$

– meaning $f$ is a set of level $a$.

$$x \in_a f := T_a(f(\dot{x}))$$

– meaning $x$ is an element of the set $f$. 
Well-ordering proofs for SRT\(_\alpha\)

We consider the notion of having transfinite induction up to \(a\) for all sets of level less than \(c\), formalized as:

\[
I^c(a) := \forall b < c. \forall f \in S_b. \text{TI}(a, f).
\]

Here \(\text{TI}(a, f)\) means \(\text{TI}(a, \lambda x. x \in_b f)\), which we recall is

\[
(\forall x. (\forall y. y < x \rightarrow y \in_b f) \rightarrow x \in_b f) \rightarrow \forall x < a. x \in_b f.
\]

Lemma

SRT\(_\alpha\) proves:

\[
\forall \ell. \text{Lim}(\ell) \land \ell \leq \alpha \rightarrow \exists f \in S_\ell. \forall a. I^\ell(a) \leftrightarrow a \in_\ell f.
\]
Well-ordering proofs for $\text{SRT}_\alpha$, continued

The first step is from predicative proof theory:

Lemma

$\text{SRT}_\alpha$ proves:

$$\forall \ell, a. \, \text{Lim}(\ell) \land \ell \leq \alpha \land I^\ell(a) \rightarrow I^\ell(\varphi a0).$$

This is done by introducing and using the Veblen-$a$-jumps:

$$V\mathcal{J}_a(\ell) := \lambda b. I^\ell(b) \rightarrow I^\ell(\varphi ab).$$

Note: This step is easier for studying $\text{SRT}_\alpha$ (and so $\hat{\text{ID}}_\alpha$) for $\alpha \geq \omega$. For finite $\alpha$, we need more delicate analysis!
Well-ordering proofs for SRT$_\alpha$, continued

We see that these well-ordering proofs rely on room to maneuver in the levels. So we define:

$$a \uparrow b := \exists c, \ell. \text{Lim}(\ell) \land b = c + a \cdot \ell$$

This is enough to formulate the main lemma:

**Lemma (Main Lemma I)**

*Let Main$_\alpha(a)$ be defined as:*

$$\text{Main}_\alpha(a) := \forall b, c. \; c \preceq \alpha \land \omega^{1+a} \uparrow c \land I^c(b) \rightarrow I^c(\varphi 1ab)$$

*Then SRT$_\alpha$ proves Prog($\lambda a. \text{Main}_\alpha(a)$).*
Well-ordering proofs for $\text{SRT}_\alpha$, completed

**Corollary**

$\text{SRT}_\alpha$ proves:

\[
\forall c, c_0, d. \ c \leq \alpha \land c = c_0 + \omega^{1+d} \rightarrow \text{Prog}(\lambda e. I^c(\varphi 1de)).
\]

The whole well-ordering proof starts from our analysis of number theory:

**Lemma**

Let $\beta < \varepsilon(\alpha)$ and $A$ be an formula of $\mathcal{L}_{\text{srt}}$. Then $\text{SRT}_\alpha$ proves $\text{TI}(\beta, A)$.

Combined, these results give the lower bound of the main theorem.
Upper bounds

We will sketch the cut-elimination arguments used to establish the upper bound in the main theorem. For this, we will employ subsystems $H_\alpha$ of a semi-formal system $H_\infty$. To motivate this, we will briefly recall the goals and the framework of ordinal analysis.
Axioms for number theory

Recall one motivation for proof-theoretical analysis: the desire to quantify how well an axiom system approximates the set of true number theoretical pseudo-$\Pi^1_1$-sentences (those containing free set variables, but no quantification over set variables).

For an axiom system $T$, we define the $\Pi^1_1$-ordinal of $T$ by:

$$|T|_{\Pi^1_1} = \sup\{|\varphi| \mid T \vdash \varphi\}.$$ 

using some measure of complexity, $|\varphi|$, of pseudo-$\Pi^1_1$-sentences. A nice measure is the *truth-complexity*, $tc(\varphi)$, defined using a semi-formal system.

It turns that for most theories we can restrict ourselves to the special sentences, $TI(\square, U)$. 
Proof-theoretical ordinals

Let $T$ be a theory. The proof-theoretical ordinal of $T$, denoted $|T|$ is defined by:

$$|T| := \sup \{ \text{otyp}(\sqsubseteq) \mid \sqsubseteq \text{ primitive recursive and } T \vdash \text{TI}(\sqsubseteq, U) \}$$

For theories with proof-theoretical ordinal less than $\Phi_0$, we have

$$|T| = \sup \{ \alpha \mid T \vdash \text{TI}(\alpha, U) \}$$

The key to determining proof-theoretical ordinals is the following:

**Theorem (Boundedness Theorem)**

Let $\prec$ be a primitive recursive, binary, transitive, well-founded relation. If $\text{Lim}(\text{otyp}(\prec))$, then $\text{otyp}(\prec) = \text{tc}(\text{TI}(\prec))$. 
Aczel’s trick

Now to fixed point theories:
To model one fixpoint, we can use Aczel’s trick:

**Lemma**

*Given an operator form, \( A(P, x) \), there’s a \( \Sigma_1 \)-formula \( \tilde{P}(x) \) such that*

\[
\Sigma_1\text{-AC} \vdash A(\tilde{P}, x) \leftrightarrow \tilde{P}(x)
\]

**Proof.**

Let \( E(z, x, y) \) be a \( \Sigma_1 \)-formula which provably enumerates all \( \Sigma_1 \)-formulas \( Q(x, y) \) as \( z = 0, 1, 2, \ldots \). Consider \( A(\lambda u. E(z, z, u), x) \). This formula is provably equivalent to some \( Q_e(z, x) \) in \( \Sigma_1\text{-AC} \), so take \( \tilde{P} := Q_e(e, x) \). But this is not the route taken in the present paper.
The semi-formal system

We formulate a Tait-style system $H_\infty$. It is formulated in the language $L_\infty$ which extends $L$ by unary relation symbols $P_\beta^A$ and $P_{<\beta}^A$ for each inductive operator form $A$ and each ordinal $\beta < \Phi_0$. The preformulas of $L_\infty$ are generated inductively by:

1. Every literal of $L$ is a $L_\infty$ preformula.
2. If $t$ is a number term, then the literals $P_\beta^A(t)$, $P_{<\beta}^A(t)$, $\neg P_\beta^A(t)$, and $\neg P_{<\beta}^A(t)$ are $L_\infty$ preformulas.
3. $L_\infty$ preformulas are closed under conjunction, disjunction, and universal and existential quantification.

We then define the $L_\infty$ formulas to be the closed preformulas.

We detail the axioms and rules of inferences on the following slides.
The semi-formal system, axioms

I. Axioms, group 1 For all $\Gamma$, numerically equivalent literals $A$ and $B$, and all true literals $C$:

$$\Gamma, \neg A, B \text{ and } \Gamma, C.$$ 

II. Axioms, group 2 For all $\Gamma$, all closed terms $s$ with $\text{pair}(s)$ false, and all closed terms $t$ with $\text{pair}(t)$ and $\beta \leq |(t)_1|$ true:

$$\Gamma, \neg P^A_{<\beta}(s) \text{ and } \Gamma, \neg P^A_{<\beta}(t).$$
The semi-formal system, fixed point rules

III. Fixed point rules, group 1 For all $\Gamma$ and all closed terms $t$ so that $\text{pair}(t)$ and $|(t)_1| = \alpha < \beta$:

$$
\frac{\Gamma, P^A_{\alpha}((t)_0)}{\Gamma, P^A_{<\beta}(t)}, \quad \text{and} \quad \frac{\Gamma, \neg P^A_{\alpha}((t)_0)}{\Gamma, \neg P^A_{<\beta}(t)}
$$

IV. Fixed point rules, group 2 For all $\Gamma$, all closed terms $s$, and all closed terms $t$ with $|t| = \beta$:

$$
\frac{\Gamma, A(P^A_{\beta}, P^A_{<\beta}, s, t)}{\Gamma, P^A_{\beta}(s)}, \quad \text{and} \quad \frac{\Gamma, \neg A(P^A_{\beta}, P^A_{<\beta}, s, t)}{\Gamma, \neg P^A_{\beta}(s)}
$$
V. Propositional rules For all $\Gamma$ and all $\mathcal{L}_\infty$ formulas $A$ and $B$:

$$
\frac{\Gamma, A}{\Gamma, A \lor B}, \quad \text{and} \quad \frac{\Gamma, B}{\Gamma, A \lor B}, \quad \text{and} \quad \frac{\Gamma, A}{\Gamma, A \land B}
$$

VI. Quantifier rules For all $\Gamma$, all $\mathcal{L}_\infty$ preformulas $A(x)$ with only $x$ free, and all closed terms $s$:

$$
\frac{\Gamma, A(x)}{\Gamma, \exists x. A(x)} \quad \text{and} \quad \frac{\Gamma, A(t) \text{ for all closed terms } t}{\Gamma, \forall x. A(x)} \quad (\omega)
$$

VII. Cut rule For all $\Gamma$, and all $\mathcal{L}_\infty$ formulas $A$:

$$
\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}
$$
Partial cut elimination

The language $L_\alpha$ is the sublanguage of $L_\infty$ that allows $P_\beta^A$ only for $\beta < \alpha$, and $P_\beta^A$ only for $\beta \leq \alpha$. Then $H_\alpha$ is just $H_\infty$ restricted to $L_\alpha$.

The notion $H_\alpha \vdash^\beta \Gamma$ is used to express that $\Gamma$ is provable in $H_\alpha$ by a proof of depth less than or equal to $\beta$.

We write $H_\alpha \vdash^\beta_* \Gamma$ if $\Gamma$ can be proved by a derivation of depth less than or equal to $\beta$ only using cuts of the above form.

Lemma

For all $\Gamma$:

if $H_\alpha \vdash^\beta \Gamma$, then $H_\alpha \vdash^{\epsilon(\beta)}_* \Gamma$. 
Elimination of one fixed point

Let us study how to eliminate one fixed point via \textit{asymmetric interpretation}.
[Can85] Fix an ordinal $\beta$.

To see this we introduce a new semiformal system, $H'_\beta$, for studying the \textit{stages} of the construction of the fixpoint $P^A_\beta$. The language $L'_\beta$ is $L_{<\beta}$ extended by unary predicates $I^A_\gamma$, which we interpret as the stages, so $I^A_0 = \emptyset$ and

$$I^A_\gamma = \{ x \mid A(I^A_{<\gamma}, P^A_\beta, x, \beta) \}.$$
Elimination of one fixed point, continued

The system extends $H'_\beta$ extends $H_\beta$ with the axioms:

$$
\Gamma, \neg I_0^A, \quad \text{for all } \Gamma,
$$

and the rules:

**Stage rules, successor** For all $\Gamma$, all closed terms $s$, and all closed terms $t$ with $|t| = \beta$:

$$
\Gamma, A(I^A_\gamma, P^{<_\beta}_A, s, t) \quad \text{and} \quad \Gamma, \neg A(I^A_\gamma, P^{<_\beta}_A, s, t)
$$
Elimination of one fixed point, continued

**Stage rules, limit** For all $\Gamma$, all closed terms $s$, and all limit ordinals $\lambda$, and all $\gamma < \lambda$:

$$\frac{\Gamma, I_\gamma^A(s)}{\Gamma, I_\lambda^A(s)} \text{ and } \frac{\Gamma, \neg I_\delta^A(s) \text{ for all } \delta < \lambda}{\Gamma, \neg I_\lambda^A(s)}$$

Now we introduce the $\beta$-rank of formulas $A$ in $\mathcal{L}'_\beta$:

$$\text{rnk}_\beta(A) = 0, \quad \text{for literals } A \text{ not containing } I_\gamma^A.$$  
$$\text{rnk}_\beta(I_\gamma^A(t)) = \text{rnk}_\beta(\neg I_\gamma^A(t)) = \omega \gamma$$  
$$\text{rnk}_\beta(B \circ C) = \max(\text{rnk}_\beta(B), \text{rnk}_\beta(C)) + 1$$  
$$\text{rnk}_\beta(Qx. B) = \text{rnk}_\beta(B(0)) + 1$$
The asymmetric interpretation

Given any ordinals $\gamma, \delta$ with $0 \leq \gamma \leq \delta < \Phi_0$, we define a translation $A_{\gamma, \delta}$ of $\mathcal{L}_\beta$ formulas to $\mathcal{L}'_\beta$:

\[
\neg P_\beta^A(s) \mapsto \neg I_\gamma^A(s) \quad P_\beta^A(s) \mapsto I_\delta^A(s)
\]

Theorem (Asymmetric Interpretation)

Suppose we have a derivation $H_\beta \vdash^\alpha \Gamma$. Then we can find uniformly in $\delta > 0$ and any sequence $\gamma$ below $\delta$ suitable for $\Gamma$ a derivation:

\[
H'_\beta \vdash^{\lambda(\alpha, \gamma, \delta)} \Gamma[\gamma, \varphi\alpha(|\gamma, \delta| + 1)].
\]

Here $\lambda(\alpha, \gamma, \delta) = \omega \varphi\alpha(|\gamma, \delta| + 1) + \omega + 1$. 
Lemma

Assume $\Gamma$ is a set of formulas of $\mathcal{L}_{\beta+\xi}$. Then we have for all ordinals $\rho$ with $\xi < \omega^{1+\rho}$ and all ordinals $\alpha$:

$$
\text{if } H_{\beta+\omega^{1+\rho}} \models_{\star}^\alpha \Gamma, \quad \text{then } H_{\beta+\xi} \models_{\star}^{\varphi_{1 \rho \alpha}} \Gamma.
$$
Lemma

For all $L_{\text{fix}}$ sentences $A$ we have:

$$\text{if } \widehat{\text{ID}}_\alpha \vdash A, \text{ then } H_\alpha \frac{\varepsilon(\alpha)}{\ast} A^\alpha.$$
Main Theorem of JKSS, again

Fix an ordinal \( \alpha < \Phi_0 \). Let \( \varepsilon(\alpha) \) denote the least \( \varepsilon \)-number greater than \( \alpha \). Define inductively

\[
(\alpha | 0) := \varepsilon(\alpha), \quad (\alpha | m + 1) := \varphi(\alpha | m) 0.
\]

Theorem (Main Theorem)

Let \( \alpha \) have Cantor Normal Form

\[
\alpha = \omega^{1+\alpha_n} + \cdots + \omega^{1+\alpha_1} + m,
\]

where \( \alpha_n \geq \cdots \geq \alpha_1 \) and \( m < \omega \). Then the proof-theoretic ordinal of \( \hat{ID}_\alpha \) is

\[
|\hat{ID}_\alpha| = \varphi 1 \alpha_n (\varphi 1 \alpha_{n-1}(\ldots \varphi 1 \alpha_1(\alpha | m)) \ldots).
\]
Corollaries and connections

Based on the studies of the paper we can add to previous result that $|\hat{\mathcal{D}}_{<\omega}| = \Gamma_0$ the following corollary:

**Corollary**

*We have the following proof-theoretical ordinals:*

$$
|\hat{\mathcal{D}}_{<\omega}| = \varphi 100 = \Gamma_0,
|\hat{\mathcal{D}}_{\omega}| = \varphi 10\varepsilon_0 = \Gamma_{\varepsilon_0},
|\hat{\mathcal{D}}_{<\omega^\omega}| = \varphi 1\omega 0,
|\hat{\mathcal{D}}_{\omega^\omega}| = \varphi 1\omega\varepsilon_0,
|\hat{\mathcal{D}}_{<\varepsilon_0}| = \varphi 1\varepsilon_0 0,
|\hat{\mathcal{D}}_{\varepsilon_0}| = \varphi 1\varepsilon_0\varepsilon_0.
$$
Scaling the metapredicative hierarchy

The theories $\widehat{ID}_\alpha$ can be used to scale (an initial part of) the metapredicative hierarchy. Some results in this direction follow. From Jäger and Strahm [JS00]:

\begin{align*}
\widehat{ID}_{\omega} & \equiv ATR, \\
\widehat{ID}_{<\omega} & \equiv ATR_0 + (\Sigma^1_1\text{-DC}), \\
\widehat{ID}_{<\varepsilon_0} & \equiv ATR + (\Sigma^1_1\text{-DC}).
\end{align*}

The autonomous fixed point theories, and further connections with Explicit Mathematics and Martin-Löf type theories to be discussed in later seminars.
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References IV