Functional interpretation of arithmetical comprehension

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Spector studied the full comprehension schema in type zero:

\[ \text{CA}^0: \quad \exists f^1 \forall x^0 (f(x) =_0 0 \leftrightarrow A(x)), \]

for arbitrary formulas \( A(x) \) of \( \mathcal{L}(\text{WE-PA}^\omega) \).

We’ve discussed how bar recursive functionals solve the ND-translation of this axiom, in fact, they solve \( \text{AC}^{0,\rho} \) via the DNS (Double Negation Shift).
Today we’ll discuss the special case of arithmetical comprehension:

\[ \text{CA}_0^{\text{ar}} : \exists f^1 \forall x^0 (f(x) = 0 \leftrightarrow \Lambda_{\text{ar}}(x)), \]

where \( \Lambda_{\text{ar}}(x) \) is \textit{arithmetical}, that is, contains only quantifiers for type 0.

**Proposition (11.11)**

\( \text{CA}_0^{\text{ar}} \) is equivalent over \( \text{WE-PA}^{\omega} \) to:

\[ \Pi_1^0 - \text{CA: } \forall f^{0(0,0)} \exists g^1 \forall x^0 (g(x) = 0 \leftrightarrow \forall y^0 (f(x, y) = 0)). \]

Let’s prove this. \( \Pi_1^0 - \text{CA} \) is a special case of \( \text{CA}_0^{\text{ar}} \), so let’s go the other way.
Proof of proposition 11.11, I

In WE-PA$\omega + \Pi^0_1$-CA, we are given an arithmetical formula $A_{ar}(x, \bar{a})$, possibly containing $x^0$ free, but not $f^1$, with additional parameters $\bar{a}$. Our goal is:

$$\exists f^1 \forall x^0 \left(f(x) = 0 \leftrightarrow A_{ar}(x, \bar{a})\right)$$

We proceed by induction on $A_{ar}(x, \bar{a})$. The cases are

$$A_{ar} \equiv s =_0 t, \quad A \land B, \quad A \lor B, \quad A \rightarrow B,$$

$$\exists y^0 A, \quad \forall y^0 A.$$
The case $s =_0 t$. The goal is:

$$\exists f^1 \forall x^0 (f(x) =_0 0 \iff s =_0 t)$$

We simply use lambda abstraction and take $f = \lambda x^0. (s - t)$. 
The case $A \land B$. The goal is:

$$\exists g^1 \forall x^0 (g(x) = 0) \leftrightarrow A$$
$$\exists h^1 \forall x^0 (h(x) = 0) \leftrightarrow B$$

$$\exists f^1 \forall x^0 (f(x) = 0) \leftrightarrow A \land B$$

Here we can take $f = \lambda x^0 . g(x) + h(x)$.

The cases $A \lor B$ and $A \to B$ are very similar.
Proof of proposition 11.11, IV

The case $\forall y^0 A$. The goal is:

$$\exists g^1 \forall x^0 \left( g(x) = 0 \iff A \right)$$

$$\exists h^1 \forall x^0 \left( h(x) = 0 \iff \forall y^0 A \right)$$

Let $f^{0(0,0)} = \lambda x, y. g(x)$. Feed $f$ to $\Pi^0_1$-CA to get $h^1$ such that $h(x) = 0 \iff \forall y \left( g(x) = 0 \right)$, as desired.

The case $\exists y^0 A$ is very similar.
Proposition 11.12

Arithmetical comprehension is implied by

\[ \Pi^0_1\text{-AC: } \forall f^{0(0,0,0)}(\forall x \exists y \forall z f(x, y, z) = 0 \rightarrow \exists g^1 \forall x, z f(x, g(x), z) = 0) \].

\[ \Pi^0_1\text{-AC also implies arithmetical number choice: } \]

\[ \text{AC}^{0,0}_{\text{ar}} : \forall x^0 \exists y^0 \text{A}_{\text{ar}}(x, y) \rightarrow \exists f^1 \forall x^0 \text{A}_{\text{ar}}(x, f(x)). \]
Consider the bar recursor of type 0, 1, $B_{0,1}$. It operates on sequence $\chi^1$, decider $y^2$, and is parameterized further by $z$ of type $1(0, 1)$ and $u$ of type $\sigma = 1(1(0), 0, 1)$. So $B_{0,1}$ has type $1(2, 1(0, 1), \sigma, 0, 1)$ with defining axioms $BR_{0,1}$:

\[
\begin{cases}
  y(\chi, n) < 0 \rightarrow B_{0,1}y\lambda \cdot (\chi, n) = 1 \cdot z(n(\chi, n)) \\
  y(\chi, n) \geq 0 \rightarrow B_{0,1}y\lambda \cdot (\chi, n) = 1 \cdot u(\lambda d \cdot B_{0,1}y\lambda \cdot (\chi, n) \cdot d) \cdot n(\chi, n)
\end{cases}
\]
Theorem 11.13

Theorem (11.13)

Let $A(a)$ be an arbitrary formula of $\mathcal{L}(\text{WE-PA}^\omega)$. If

$$\text{WE-PA}^\omega + \text{QF-AC} + \text{AC}_{ar}^{0,0} \vdash A(a),$$

then there is a tuple $t$ of closed terms of $\text{WE-HA}^\omega + \text{BR}_{0,1}$, such that

$$\text{WE-HA}^\omega + \text{BR}_{0,1} \vdash \forall y (A')_D(t, a, y, a).$$

The verification can be carried out in qf-$(\text{WE-HA}^\omega) + \text{BR}_{0,1}$. \quad
Theorem (11.14)

Let $A(a)$ be an arbitrary formula of $L(\textsc{WE-PA}^\omega)$. If

$$\textsc{WE-PA}^\omega \vdash +\text{QF-AC} + \text{AC}_{ar}^{0,0} \vdash A(a),$$

then there is a tuple $t$ of closed terms of $\textsc{WE-HA}^\omega + \text{BR}_{0,1}$ such that

$$\textsc{WE-HA}^\omega \vdash +\text{BR}_{0,1} \vdash \forall y (A')_D(t \overline{a}, y, \overline{a}).$$

The verification can be carried out in qf-($\textsc{WE-HA}^\omega$) + ($\text{BR}_{0,1}$).
Proposition 11.15

Proposition (11.15)

Let $A$ be a prenex sentence of $\text{PA}$. If $\text{PA} \vdash A$, then there are closed terms $\Phi$ of $\text{WE-HA}^\omega \uparrow +\text{BR}_{0,1}$ such that

$$\text{WE-HA}^\omega \uparrow +\text{BR}_{0,1} \vdash \Phi \text{n.c.i. } A.$$ 

Compare with:

Proposition (10.9)

Let $A$ be a prenex sentence of $\text{PA}$. If $\text{PA} \vdash A$, then there are closed terms $\Phi$ of $\text{WE-HA}^\omega$ such that

$$\text{WE-HA}^\omega \vdash \Phi \text{n.c.i. } A.$$