Multi-unit Auctions with Budget Constraints*

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Abstract

Motivated by sponsored search auctions, we study multi-unit auctions with budget constraints. In the mechanism we proposed, Sort-Cut, understating the budgets or values is weakly dominated. Since Sort-Cut's revenue is increasing in budgets and values, all kinds of equilibrium deviations from true valuations turn out to be beneficial to the auctioneer. We obtain a lower bound on the revenue of Sort-Cut and discuss effiency properties of its expost Nash equilibrium.

Keywords: Multi-Unit Auctions, Budget Constraints, Sponsored Search

JEL classification: D44

1 Introduction

Consider the problem of the advertisement departments of Dell, HP or Sony, which is to appear in a particular search engine's query of "laptops." Search engines like Google and Yahoo! use complicated rules to determine the allocation[1] of these advertisements, or "sponsored links" and also their pricing rules. Roughly, the advertisers specify "a value per-click" and a daily maximum budget. Allocation and pricing is then determined by a complex algorithm which makes sure that the advertisers are not, per-click, charged more than their stated values and also are not charged more than their total budget in a day.

Advertisers true (estimated) values per-click and daily budgets are, of course, their private information and given any allocation and pricing rule they will act strategically in bidding their

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[1]Allocations meaning which advertisements will be displayed, also in which order they will be displayed if there are more than one advertisement.

values and budgets. It is then natural to ask whether there is any mechanism in which the participants would prefer to truthfully reveal their types–per-click values and daily budgets in this model. Then there will not be any "gaming" of the mechanism and socially efficient allocations can be implemented. Second-price auctions in single unit auction problems and different versions of Vickrey-Clark-Groves mechanisms in more general setups have been very successful in implementing socially efficient allocations in "dominant strategies." Unfortunately, a recent impossibility result (Dobzinski et al. 2008) precludes the existence of a truthful mechanism with Pareto optimal allocations in this important setting.

In this paper, we propose Sort-Cut, a mechanism which does the next best thing from the auctioneer's point of view. In our mechanism,understating the budgets or values is weakly dominated. Thus the only way a bidder can possibly benefit from lying in our mechanism is by overstating their values or budgets, which leads to good revenue properties for the auctioneer at equilibria.

The idea of Sort-Cut is very similar to the idea of a second-price auction. In second-price auctions without budget constraints, the highest bidder is allocated the object and he pays the highest loser's bid to the auctioneer. Uniform-price auction generalizes this idea to multi-unit auctions. The idea is to charge the winners by the opportunity cost: the losers bids. When the bidders have budget constraints, however, losers might not be able to buy all the items if they were offered, they might simply not afford it. Taken this into account, we modify the algorithm to charge the winners, per item, for the value of the highest value loser, but only up to highest loser's budget. After the highest value loser's budget is exhausted, she would not be able to afford any more items, so we start charging the winners the second highest value, up to her budget and so on. Given this pricing idea[2], the winners and losers are determined via a cut-point to clear the market, i.e. to be able to sell all the available items.

Sort-Cut has a number of desirable properties. First of all, it sells all the items so there is no inefficiency in that sense (whereas Borgs et al (2005) and Goldberg et al (2001) might leave some of the items unallocated). Second, although it is not truthful, bidders can only benefit by overstating their values or budgets, a deviation which is the most desirable one for the auctioneer[3]. Third,

[2] There is a caveat here, which is that the lowest value winner might not be able to exhaust all her budget. Then all higher value bidders are charged first at the lowest value winner's value up to her unused budget. This makes sense as the lowest value winner is still a competitor to other winners to buy further items. The pricing for the lowest value winner, for the same reason, starts from the highest value loser. She cannot be a competitor to herself!

[3] We show that the revenue of Sort-Cut mechanism is monotone with respect to the vector of bids.

allocation in the equilibrium of Sort-cut is almost Pareto optimal in the sense that, all winner's values are greater than cut point bidder's announced value and all losers values are smaller than cut point bidder's announced value. And lastly, Sort-Cut reduces to second-price auction when there are no budget constraints[4].

After introducing Sort-cut, we prove that revenue of every equilibrium of Sort-cut differs at most the budget of one bidder from the revenue of the market clearing price mechanism according to the true valuations. Market clearing mechanism determines a market clearing price and sells all the units for that price. This mechanism however is not truthful and the bidders can benefit from understating their budgets (and thereby decreasing auctioneer's revenue).

After discussing related literature below, we introduce the model and our mechanism in Section 2. Section 3 discusses truthfulness, revenue and Pareto optimality properties of Sort-Cut. Section 4 compares market clearing price mechanism and sort cut. In Section 5 we conclude and discuss possible extensions of our model.

Related Literature The problem of multi-unit auctions with budget-constrained bidders was initiated by Borgs et al. (2005). Our model is similar to theirs except that we do not need to assume that the utility is $-\infty$ when budget constraints are violated. They introduce a truthful mechanism that extracts a constant fraction of the optimal revenue asymptotically; however, their mechanism may leave some units unsold. The idea is to group the people randomly into two groups, and use the market clearing price of each group as a posted price to the other group. Another paper that uses the same model is by Abrams (2006)- it uses techniques similar to Borgs et al. (2005) but improves upon it; however, it may still leave some units unsold.

In an important paper, Dobzinski et al. (2008), proves an impossibility result. They first assume that the budgets of all players are publicly known, and under this assumption, they give a truthful mechanism which is individual rational and Pareto-optimal. Their mechanism basically applies Ausubel's multi-unit auction (Ausubel, 2004) to this budgeted setting. Then they show that their mechanism is the unique mechanism which is both truthful and Pareto-optimal under the assumption of publicly known budgets. Finally by showing that their mechanism is not truthful

[4]Generalized second-price auctions studied by Edelman et al. (2007) also has a similar idea in multi-item auctions. In that mechanism the winner of the best item (first sponsored link) is charged the bid of the second-best item, the winner of the second best item is charged the bid of the third-best item and so on. In this environment there are no budget constraints and second-highest bid is always the competitor of the highest value.

if the budgets are private knowledge, they conclude that no mechanism for this problem can be individual rational, truthful and Pareto-optimal.

Bhattacharya et al. (2010a) show that although the mechanism proposed for public budgets in Dobzinski et al. (2008) is not truthful, the only way that bidders can benefit from lying is to *over-state* their budget; this, together with the fact that the utility of a bidder who is charged more than her budget is $-\infty$, helps them to modify the non-truthful deterministic mechanism into a truthful randomized mechanism. For each bidder, instead of charging her the price specified by Dobzinski et al. (2008), they run a lottery (with appropriate probability) and either charge her 0, or all of her announced budget. Therefore, since a bidder has to pay all of her announced budget with positive probability, the expected utility of over-stating the budget becomes $-\infty$. The assumption of the utility being $-\infty$ when the budget constraints are violated is not very realistic. In our work, we drop that by assuming that the utility of bidder who has to pay more than her budget is an arbitrary negative value. Furthermore, we avoid randomized pricing and allocation to guarantee ex-post individual rationality.

Ashlagi et al. (2010) look at budget constraints in position auctions; in their setting, bidders must be matched to the slots where each slot corresponds to a certain fraction of the total supply. Bidders are profit maximizer and face budget constraints. They assume that violation of budget constraints leads to zero utility for the bidder. They propose a modification of Generalized Second Price mechanism which is Pareto-optimal and envy-free. In their setting, the fraction of supply on each of the slots is fixed; this makes their problem more like a matching problem with discrete structure. However, in our setting, the auctioneer has complete freedom on how much of the supply to give to each of the bidders.

There are other papers that have studied budget constraints in mechanism design but in settings more different from ours. Feldman et al. (2008) give a truthful mechanism for ad auctions with budget-constrained advertisers where there are multiple slots available for each query, and an advertiser cannot appear in more than one slot per query. The utility function that they use is very different from ours. In Feldman et al. (2008), they define advertisers to be click-maximizers, i.e. advertisers do not value their unused budget, they just want to maximize the amount of supply they get; however, in our model, advertisers are profit-maximizers.

Pai and Vohra (2010) look at optimal auctions with budget constraints. In their setting, one indivisible good is to be allocated which makes the setting naturally different from ours; moreover, they assume $-\infty$ utility if budget constraints are violated. In another paper, Malakhov and Vohra (2008) look at the divisible case; however, they assume that there are only two bidder one of which is has no budget constraint while the budget constraint of the other one is publicly known. Kempe et al. (2009) look at budget constraints when the bidders are unit-demand and there are heterogenous items. Bhattacharya et al. (2010b) show that sequential posted price can achieve a constant fraction of the optimal revenue in budgeted setting with heterogenous items; however, in their setting, budgets are publicly known; some of their results carry over to the case where budgets are private knowledge as well, however, for that they need to assume that the utility of being over-budgeted is $-\infty$ and apply the technique used in Bhattacharya et al. (2010a).

Both Borgs et al. (2005) and Dobzinski et al. (2008) argue that lack of quasi-linearity (because of hard budget constraints) is the most important difficulty of the problem. Still some papers have tried to solve the problem by relaxing hard budget constraints (Maskin, 2000), or modeling the budget constraint as an upper bound on the value obtained by the bidder rather than her payment (Mehta, 2007). It has also been shown (Borgs et al., 2005) that modeling budget constraints with quasi-linear functions can lead to arbitrarily bad revenue.

Benoit and Krishna (2001) studies an auction for selling two single items to budget-constrained bidders. They mainly focus on the effect of bidding aggressively on an unwanted item with the purpose of depleting other bidders budget. A similar effect arises in our model as well, but the focus of our work is generally very different from theirs. Another paper is Che and Gale (1996) which compares first-price and all-pay auctions in a budget-constrained setting and show that the expected payoff of all-pay auctions is better under some assumptions. However, they do not consider multi-unit items.

2 Model and Sort-Cut

There are m units of a good for sale. There are n bidders and they have linear demand up to their budget limits. Specifically, each bidder $i \in N = \{1, ..., n\}$ has two dimensional type (v_i, b_i) where v_i denotes private value and b_i denotes budget limit. Bidder i's utility by getting q units of the good and paying p is given by

$$u_i(q,p) = \begin{cases} qv_i - p & \text{if } p \leq b_i \\ -C & \text{if } p > b_i \end{cases}$$

where $\infty \geq C > 0$.

We are interested in mechanisms to sell m items to n bidders which have good truthfulness, efficiency and revenue properties. Equilibrium concept we use is that of "ex-post Nash equilibrium." In an ex-post Nash equilibrium, no bidder would like to deviate after he/she observes all other players' strategies. We believe that this is an appropriate equilibrium concept as we are motivated by sponsored search auctions. Typically, sponsored search auctions are dynamic auctions and bids can be changed anytime. Therefore, it is reasonable that in a stable situation (steady state), no bidder would like to deviate even after bids are revealed. Since the equilibrium concept is ex-post Nash, we do not need to assume strong conditions on private information. Specifically, we can allow for interdependency in two dimensional type within or across bidders.

We focus on direct mechanisms in which bidders announce their types (values and budgets.) A mechanism consists of an allocation rule (how many units to allocate to each bidder) and a pricing rule (how much to charge each bidder). It takes the announcements as inputs and produces allocation and pricing scheme as an output. We consider mechanisms that satisfy the two properties (i) sell all m items (ii) satisfy individual rationality constraints (i.e. all bidders prefer to participate in the mechanism). Note that the latter condition implies that bidders who are not allocated any items (losers) cannot be charged a positive price. Bidders who are allocated with items (winners), however, will be charged a positive price. Let us first introduce a general and an abstract *pricing rule.*

Definition 1 *When we say that the price is set according to a pricing function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, *if the marginal price of the next unit is* $\alpha(y)$ *dollars for a buyer who has already spent* y *dollars in the market. In other words, if pricing of an item is set according to* α, *a buyer with* b *dollars can afford*

$$x(\alpha, b) = \int_0^b \frac{1}{\alpha(y)} dy$$

units of the item. We are interested in pricing rules $\alpha(\cdot)$ *which are nonincreasing and positive.*

Hence, we assume $\alpha(y) \leq \alpha(y')$ *for all* $y \geq y'$ *and also* $\alpha(y) > 0$ *for all* y.

The following definition is also convenient for later discussions.

Definition 2 *(Shifted pricing) For a given pricing function* α *and a positive real number* z, *we define the pricing function* $\alpha^z(y)$ *as:*

$$\alpha^z(y) = \alpha(z + y)$$

In words, $\alpha^z(y)$ is the pricing function obtained by shifting $\alpha(y)$, z units to right. Note that we have, for any $z \in [0, b]$

$$x(\alpha, b) = x(\alpha, z) + x(\alpha^z, b - z)$$

Now, we are ready to introduce a special class of pricing and allocation rules, that we name *Procedure Cut*

Definition 3 *Procedure cut takes values and budgets of the bidders* $(\mathbf{b}, \mathbf{v}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$; *a pricing rule* $\alpha(\cdot)$ *and a real number* $c \in (0, \sum_{i=1}^n b_i]$ *as input. First, it sorts value and bid vectors* $(\mathbf{b}, \mathbf{v})$ *in nonascending*[5] *order of values and reindexes them so that* $v_1 \geq v_2 \geq ... \geq v_n$.[6] *Then, it picks* j *such that* $c \leq \sum_{i=1}^j b_i$ *and* $c > \sum_{i=1}^{j-1} b_i$. *Let* $s = \sum_{i=1}^j b_i - c$. *Procedure cut sets the pricing function of bidders* $1, ..., j-1$ *to* α^c *and the pricing function of bidder* j *to* α^{c+s}. *The allocation of each bidder* $1, .., j-1$ *is such that she spends all her budget, i.e.* $x_i = x(\alpha^c, b_i)$ *for* $i = 1, .., j-1$. *The allocation of bidder* j *is such that she spends* $b_j - s$ *of her budget, i.e.* $x_j = x(\alpha^{c+s}, b_j - s)$. *All bidders* $j+1, ..., n$ *get no allocation and pay nothing.*

Define $X(c, (\mathbf{b}, \mathbf{v}))$ *to be the total number of units allocated to all bidders, i.e.* $X(c, (\mathbf{b}, \mathbf{v})) = \sum_{i=1}^j x_i$. *Bidders* $1, ..., j$ *are called full winners, bidder* j *is called a partial winner*[7] *and bidders* $j+1, ..., n$ *are called losers.*

We consider pricing rules which are not too high, in the sense that it will be able to sell all the

[5] It breaks ties among equal valued bidders arbitrarily.

[6] Note that after reindexing, budgets are not necessarily sorted in a desending way. A bidder with a high valuation could have a small budget.

[7] Note that s denotes the unused budget of the partial winner, where $s \in [0, b_j]$.

items if all budgets are exhausted. Hence we assume that, for $B \equiv \sum_{i=1}^{n} b_i$

$$\alpha(y) \leq \frac{B}{m}$$

With this assumption, we can easily conclude that $X(B, (\mathbf{b}, \mathbf{v})) \geq m$. This is because when $c = B$, all bidders are full winners and their allocations satisfy

$$x\left(\alpha^B, b_i\right) \geq \frac{b_i}{\frac{\sum_{i=1}^{n} b_i}{m}}$$

and hence

$$X(B, (\mathbf{b}, \mathbf{v})) = \sum_{i=1}^{n} x\left(\alpha^B, b_i\right) \geq m$$

We are interested in rules that sell m units. In the following proposition, we show that for any procedure cut rule, $X(c, (\mathbf{b}, \mathbf{v}))$ is strictly increasing and continuous in c. Together with the assumption that $X\left(\sum_{i=1}^{n} b_i, (\mathbf{b}, \mathbf{v})\right) \geq m$, this will imply that there will be a unique c such that $X(c, (\mathbf{b}, \mathbf{v})) = m$.

Proposition 1 $X(c, (\mathbf{b}, \mathbf{v}))$ *is strictly increasing and continuous in* c.

Proof. In the appendix ■

We also want to consider pricing rules which are not too high, in the sense that it will be able to sell all the items if all budgets are exhausted. Hence we assume that, for $B \equiv \sum_{i=1}^{n} b_i$

$$\alpha(y) \leq \frac{B}{m}$$

With this assumption, we can easily conclude that $X(B, (\mathbf{b}, \mathbf{v})) \geq m$. This is because when $c = B$, all bidders are full winners and their allocations satisfy

$$x\left(\alpha^B, b_i\right) \geq \frac{b_i}{\frac{\sum_{i=1}^{n} b_i}{m}}$$

and hence

$$X(B, (\mathbf{b}, \mathbf{v})) = \sum_{i=1}^{n} x\left(\alpha^B, b_i\right) \geq m$$

An important corollary of Proposition 1 is that there will be a unique c^* that will satisfy $X(c^*, (\mathbf{b}, \mathbf{v})) = m$.

Definition 4 *We call the unique c^* with $X(c^*, (\mathbf{b}, \mathbf{v})) = m$ to be the* cut-point*. Given pricing function $\alpha(\cdot)$ and vectors $(\mathbf{b}, \mathbf{v})$, we name Procedure cut that sells m items (with $c = c^*$) to be the* m-Procedure cut*.*

We now can introduce a novel mechanism that we name *Sort-Cut Mechanism.*

Definition 5 *Sort-Cut is a m-Procedure Cut mechanism in which $\alpha(\cdot)$ is a step function defined by (reindexed) $(\mathbf{b}, \mathbf{v})$: $\alpha(y) = v_i$ for $y \in (\sum_{k=1}^{i-1} b_k, \sum_{k=1}^{i} b_k]$.*[8]

In other words, Sort-Cut takes the vectors $(\mathbf{b}, \mathbf{v})$ sorts them in nonascending order of values, calculates the unique cut-point c^* according the pricing function that each full winner (bidders $1, .., j-1$) pays v_j per unit up to a budget of s, then pay v_{j+1} per unit up to a budget of b_{j+1}, then pay v_{j+2} per unit up to a budget of b_{j+2}, and so on, until their budgets are exhausted; and the partial winner (bidder j) pays v_{j+1} per unit up to a budget of b_{j+1}, then pay v_{j+2} per unit up to a budget of b_{j+2}, and so on, until she spends $b_j - s$.

Let us the denote the sort-cut revenue by $R^S(\mathbf{b}, \mathbf{v})$ (note that $R^S(\mathbf{b}, \mathbf{v}) = c^*$ where $X(c^*, (\mathbf{b}, \mathbf{v})) = m$). Now we show that $R^S(\mathbf{b}, \mathbf{v})$ is nondecreasing in $\mathbf{b}$ and $\mathbf{v}$.

Proposition 2 *$R^S(\mathbf{b}, \mathbf{v})$ is continuous and nondecreasing in $\mathbf{b}$ and $\mathbf{v}$*

Proof. In the appendix ■

3 Truthfulness, Revenue and Pareto Optimality

3.1 Truthfulness

In this section, we show that Sort-Cut has good truthfulness properties. More specifically, no bidder benefits from understating his value or budget.

Proposition 3 *For any bidder i with types (b_i, v_i), bidding (b_i, v_i) weakly dominates bidding (b_i, v_i^-) for $v_i^- < v_i$.*

[8] And also $\alpha(y) = \varepsilon > 0$ for $y \in (B, \infty)$

Proof. Consider any $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$. First of all, if i becomes a loser by bidding $\left(b_i, v_i^-\right)$, his utility cannot increase with this deviation. This is because losers' utilities are zero and by construction, a bidder with type (b_i, v_i) achieves a nonnegative utility by bidding (b_i, v_i). We will look at the possible cases one by one:

- if i is a loser by bidding (b_i, v_i), then he will be a loser by bidding $\left(b_i, v_i^-\right)$ (since pricing function gets better for the winners). Hence his utility cannot increase by this deviation.

- if i is a partial winner by bidding (b_i, v_i) and bidding $\left(b_i, v_i^-\right)$ makes him a partial winner, then he will have the same pricing function but he will be able to use less of his budget (since pricing function for winners become better), hence his utility cannot increase. Bidder i cannot become a winner by bidding $\left(b_i, v_i^-\right)$, when he is a partial winner by bidding (b_i, v_i).

- if i is winner by bidding (b_i, v_i) and bidding $\left(b_i, v_i^-\right)$ makes him a winner, his utility does not change. This is because Sort-cut pricing ignores the value of winners in pricing calculation. If i is winner by bidding (b_i, v_i) and bidding $\left(b_i, v_i^-\right)$ makes him a partial winner, then the original partial winner j (with an unused budget s) has to be a winner after the deviation. We argue that i's utility decreases. It is true that i would get the items at a lower per unit price after the deviation, but at the same time he using less of his budget. The argument is that, by this deviation i cannot get to lower priced items and this follows from the fact that revenue of Sort-cut cannot decrease after the deviation. More formally, let us denote the unused budget of i after the deviation by s'. We know that $s' \geq s$ (because revenue cannot increase). Bidder i's utility difference with the deviation can be shown to be nonpositive (where α and c are defined with respect to $(\mathbf{b}, \mathbf{v})$)

$$\begin{aligned}
& \left(x\left(\alpha^{c+s}, b_i - s'\right) v_i - \left(b_i - s'\right)\right) - \left(x\left(\alpha^c, b_i\right) v_i - b_i\right) \\
= & \left(x\left(\alpha^{c+s}, b_i - s'\right) - x\left(\alpha^c, b_i\right)\right) v_i + s' \\
\leq & \left(x\left(\alpha^{c+s'}, b_i - s'\right) - x\left(\alpha^c, b_i\right)\right) v_i + s' \\
= & \left(x\left(\alpha^{c+s'}, b_i - s'\right) - \left(x\left(\alpha^c, s'\right) + x\left(\alpha^{c+s'}, b_i - s'\right)\right)\right) v_i + s' \\
= & s' - x\left(\alpha^c, s'\right) v_i \\
\leq & s' - \frac{s'}{v_i} v_i \\
= & 0
\end{aligned}$$

where the first inequality follows from $s' \geq s$ and second inequality follows from $\alpha^c(y) \leq v_i$.

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Proposition 4 *For any bidder i with types (b_i, v_i) bidding (b_i, v_i) weakly dominates bidding $\left(b_i^-, v_i\right)$ for $b_i^- < b_i$.*

Proof. Consider any $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, first of all, same as above proof if i becomes a loser by bidding $\left(b_i^-, v_i\right)$, his utility cannot increase with this deviation. We look at the possible cases one by one:

- if i is a loser by bidding (b_i, v_i), then he will be a loser by bidding $\left(b_i^-, v_i\right)$ (since pricing function gets better for the winners).

- if i is a partial winner by bidding (b_i, v_i) and bidding $\left(b_i^-, v_i\right)$ makes him a partial winner, then he will have the same pricing function but he will be able to use less of his budget (since pricing function for winners becomes better), hence his utility cannot increase. Bidder i cannot become a winner by bidding (b_i, v_i'), when he is a partial winner by bidding (b_i, v_i).

- if i is winner by bidding (b_i, v_i) and bidding $\left(b_i^-, v_i\right)$ makes him a partial winner, then i would be worse off with this deviation. This is because, (i) he is using less of his budget (ii) his pricing got worse. If i is winner by bidding (b_i, v_i) and bidding (b_i', v_i) makes him a winner, we can argue that his utility decreases. It is true that i may get the items at a lower per unit price after the deviation, but at the same time he is using less of his budget. The argument is that, by this deviation i cannot get to lower priced items and this follows from the fact that revenue of Sort-cut cannot decrease after the deviation. More formally, bidder i's utility difference with the deviation can be shown to be nonpositive (where α and c are defined with respect to $(\mathbf{b},\mathbf{v})$ and $c'(\leq c)$ is the Sort-cut revenue after deviation)

$$\begin{aligned}
& \left(x\left(\alpha^{c'+b_i-b_i^-},b_i^-\right)v_i-b_i^-\right)-\left(x\left(\alpha^c,b_i\right)v_i-b_i\right) \\
= & \left(x\left(\alpha^{c'+b_i-b_i^-},b_i^-\right)-x\left(\alpha^c,b_i\right)\right)v_i+b_i-b_i^- \\
\leq & \left(x\left(\alpha^{c+b_i-b_i^-},b_i^-\right)-x\left(\alpha^c,b_i\right)\right)v_i+b_i-b_i^- \\
= & \left(x\left(\alpha^{c+b_i-b_i^-},b_i^-\right)-\left(x\left(\alpha^c,b_i-b_i^-\right)+x\left(\alpha^{c+b_i-b_i^-},b_i^-\right)\right)\right)v_i+b_i-b_i^- \\
= & b_i-b_i^--x\left(\alpha^c,b_i-b_i^-\right)v_i \\
\leq & b_i-b_i^--\frac{b_i-b_i^-}{v_i}v_i \\
= & 0
\end{aligned}$$

where the first inequality follows from $c\geq c'$ and second inequality follows from $\alpha^c(y)\leq v_i$.

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Similarly, we can argue that bidding $\left(b_i^-,v_i^-\right)$ for $b_i^-<b_i$ and $v_i'<v_i^-$ is weakly dominated by bidding (b_i,v_i). This follows from the proofs above. The same proof methods would work in the sense that we can show both $\left(b_i^-,v_i\right)$ and $\left(b_i,v_i^-\right)$ dominate $\left(b_i^-,v_i^-\right)$ when $b_i^-<b_i$ and $v_i^-<v_i$. Hence, we have the following result.

Proposition 5 *For any bidder i with types (b_i,v_i) bidding (b_i,v_i) weakly dominates bidding $\left(b_i^-,v_i^-\right)$ for $b_i^-<b_i$ and $v_i^-<v_i$.*

Propositions 3, 4 and 5 establish that the revenue decreasing deviations should not occur in equilibrium (they are weakly dominated). There are two deviations, however, which may increase or decrease the revenue. These deviations are "understating budget and overstating value" and "overstating budget and understating value." Now, we show that the former deviation is not reasonable in the sense that it could be a best response only when the utility with that strategy is zero. Then we show that the latter deviation could happen in equilibrium, yet whenever it is a (strict) profitable deviation from truthful revelation, the revenue increases with the deviation.

Proposition 6 *For any bidder i with types (b_i, v_i), for $b_i^- < b_i$ and $v_i^+ > v_i$, bidding $\left(b_i^-, v_i^+\right)$ can never be in the set of best responses unless bidder i's utility in her best response is* 0.

Proof. Given $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, suppose that $\left(b_i^-, v_i^+\right)$ is a best response for i where $b_i^- < b_i$ and $v_i^+ > v_i$. Since bidding (b_i, v_i) would give nonnegative utility to bidder i, utility by bidding $\left(b_i^-, v_i^+\right)$ has to be nonnegative. Now, we claim that bidding $\left(b_i, v_i^+\right)$ is a better response than $\left(b_i^-, v_i^+\right)$, and it is strictly better when the utility with $\left(b_i, v_i^+\right)$ is strictly positive. This implies $\left(b_i^-, v_i^+\right)$ could be a best response only when bidder i's utility in her best response is 0.

Suppose that utility by bidding $\left(b_i^-, v_i^+\right)$ is nonnegative and consider the utility difference between bidding $\left(b_i^-, v_i^+\right)$ versus bidding $\left(b_i, v_i^+\right)$. The utility difference is clearly zero if i is a loser in both cases. For all other cases, i would be either a partial winner or winner by bidding $\left(b_i, v_i^+\right)$. Then, we could see that bidding $\left(b_i, v_i^+\right)$ gives a higher utility then bidding $\left(b_i^-, v_i^+\right)$. The argument is the same as in the proof for Proposition 4: by bidding an extra budget of $b_i - b_i^-$ bidder i can get extra items at a per unit price lower than his value. ■

In other words, we should not expect to see $\left(b_i^-, v_i^+\right)$ to be played, since it is either worse than (b_i, v_i) or $\left(b_i, v_i^+\right)$.

Proposition 7 *For any bidder i with types (b_i, v_i), for $b_i^+ > b_i$ and $v_i^- < v_i$, whenever bidding $\left(b_i^+, v_i^-\right)$ brings a higher utility to i than bidding (b_i, v_i), the auctioneer's revenue with $\left(b_i^+, v_i^-\right)$ is not lower than the revenue with (b_i, v_i).*

Proof. Given $(\mathbf{b}_{-i}, \mathbf{v}_{-i})$, for some $b_i^+ > b_i$ and $v_i^- < v_i$, suppose that $u_i\left(\left(\mathbf{b}_{-i}, b_i^+\right), \left(\mathbf{v}_{-i}, v_i^-\right)\right) > u_i\left((\mathbf{b}_{-i}, b_i), (\mathbf{v}_{-i}, v_i)\right)$. Since bidder i is budget-constrained, he will have to be a partial winner by bidding $\left(b_i^+, v_i^-\right)$ (if he is a winner his utility would be $-C$, and if he is a loser his utility would be 0).

- if he is a loser by bidding (b_i, v_i), auctioneers revenue clearly increases with $\left(b_i^+, v_i^-\right)$. This is because i's ranking with v_i^- is not higher than with v_i and so by deviating from (b_i, v_i) to $\left(b_i^+, v_i^-\right)$, all winners remain winners and i becomes a partial winner.

- if he is a winner by bidding (b_i, v_i), the partial winner with (b_i, v_i) has to become a full winner after i deviates to $\left(b_i^+, v_i^-\right)$ (otherwise, i would be worse off by $\left(b_i^+, v_i^-\right)$ as he will have a worse pricing function) In this case the revenue has to increase. The argument is that, for this deviation to be beneficial, i has to get lower priced items after the deviation. For this to be the case, partial winner's unused budget before the deviation, plus i's used budget after the deviation has to be greater than i's budget, b_i. But then, the revenue increases.

- if he is partial winner by bidding (b_i, v_i), we need to analyze two cases: (i) if i's ranking among the bidders is the same (ii) if i's ranking is different. For (i), the pricing for (b_i, v_i) and $\left(b_i^+, v_i^-\right)$ are the same. Since utility with $\left(b_i^+, v_i^-\right)$ is more than utility with (b_i, v_i), this means i is using more of his budget with $\left(b_i^+, v_i^-\right)$. Therefore the revenue increases. For (ii), i's ranking has to be worse with $\left(b_i^+, v_i^-\right)$. Now, as similar to above, we argue that total budget of "new winners" after deviation plus the used budget of i after deviation has to be greater than b_i. If that is not the case, i cannot get to lower prices.

■

In above propositions we argued that playing $\left(b_i^-, v_i\right)$, $\left(b_i, v_i^-\right)$ and $\left(b_i^-, v_i^-\right)$ are not reasonable (they are dominated by (b_i, v_i)); playing $\left(b_i^-, v_i^+\right)$ is not reasonable in a weaker sense (it is dominated by a combination of (b_i, v_i) and $\left(b_i, v_i^+\right)$); also playing $\left(b_i^+, v_i^-\right)$ is reasonable only when it is done by a winner, which becomes a partial winner after deviation and increases the revenue. We call the equilibria in which the strategies satisfy these conditions a *refined equilibrium.* More formally,

Definition 6 *In a* refined equilibrium, *bidder i does not play $\left(b_i^-, v_i\right)$, $\left(b_i, v_i^-\right)$, $\left(b_i^-, v_i^-\right)$ and $\left(b_i^-, v_i^+\right)$. Moreover, bidder i plays $\left(b_i^+, v_i^-\right)$ only when $u_i\left(\left(\mathbf{b}_{-i}, b_i^+\right), \left(\mathbf{v}_{-i}, v_i^-\right)\right) > u_i\left(\left(\mathbf{b}_{-i}, b_i\right), \left(\mathbf{v}_{-i}, v_i\right)\right)$. In other words, in a* refined equilibrium, *bidders never understate their budgets, and they understate their values only when they overstate their budgets making them better off than truthful announcement.*

Recall that when $u_i\left(\left(\mathbf{b}_{-i}, b_i^+\right), \left(\mathbf{v}_{-i}, v_i^-\right)\right) > u_i\left(\left(\mathbf{b}_{-i}, b_i\right), \left(\mathbf{v}_{-i}, v_i\right)\right)$, $\left(b_i^+, v_i^-\right)$ makes i a partial winner after the deviation and revenue is higher with $\left(b_i^+, v_i^-\right)$ than with (b_i, v_i).

3.2 Revenue

There could be 8 different kinds of deviations from truthful revelation, (b_i, v_i). 5 of them are discussed in above definition, and the remaining 3 of them, namely $\left(b_i, v_i^+\right)$, $\left(b_i^+, v_i\right)$ and $\left(b_i^+, v_i^+\right)$

can only increase the revenue. We hence have the following result almost immediately.

Proposition 8 *In a refined equilibrium of Sort-Cut, revenue is bounded below by the revenue of Sort-Cut with truthful revelations.*

Proof. Consider any refined equilibrium of Sort-Cut. Let b_i^- and v_i^- denote understating the types, and b_i^+ and v_i^+ denote overstating the types (with respect to true types). We know that $\left(b_i^-, v_i\right)$, $\left(b_i, v_i^-\right)$, $\left(b_i^-, v_i^-\right)$ and $\left(b_i^-, v_i^+\right)$ cannot occur. Additionally, $\left(b_i^+, v_i^-\right)$ can only happen when it increases the revenue. Moreover, each of the remaining 3 deviations, $\left(b_i, v_i^+\right)$, $\left(b_i^+, v_i\right)$ and $\left(b_i^+, v_i^+\right)$ increases the revenue. Since the argument about the deviations hold for arbitrary announcement of other players, any refined equilibrium revenue is not smaller than revenue with truthful revelations. Alternatively, consider truthful type announcements, if this is an expost equilibrium, we are done. Otherwise, at least one player has a profitable deviation in refined strategies, with this deviation revenue increases. If after the deviation there is no further deviation, we are done. If not, there is at least one player has a profitable deviation in refined strategies, with this deviation revenue increases (as compared to previous revenue and therefore also to revenue with truthful revelations). And so on. ■

3.3 Pareto Optimality

Among different efficiency concepts that could be considered, we consider that of pareto optimality: we say that an allocation is pareto optimal if there is no other allocation in which all players (including the auctioneer) are better off and at least one player strictly better off[9]. In this setup, Dobzinsky et all (2008) has shown pareto optimality is equivalent to a "no trade" condition: an allocation is pareto efficient if (a) all units are sold and (b) a player get a non-zero allocation only if all higher value players exhausted their budgets. In other words, an allocation is pareto optimal when given *true value* of the partial winner, all winners have higher values and all losers have lower values.

Above subsection illustrates that Sort-Cut has good revenue properties. The following result is regarding the efficiency (pareto optimality) of equilibrium of Sort-Cut. It obtains that in any

[9] Maximizing social welfare dictates all items to be allocated to the bidder with the highest value, even if this bidder very small budget. We follow Dobzinsky at all (2007) and consider pareto optimality as the appropriate efficiency concept.

expost Nash equilibrium of Sort-Cut, the full winners and losers are ordered in the right way given the *announced value* of the partial winner.

Proposition 9 *Consider any expost Nash equilibrium of Sort-Cut where v_j is the announced value of cut point bidder j. Every bidder $i \neq j$ who has true value $v_i^T > v_j$ is a full winner, and every bidder $i \neq j$ who has true value $v_i^T < v_j$ is a loser in this equilibrium of Sort-Cut.*

Proof. First, consider a bidder i whose value is $v_i^T > v_j$. We prove that she must be a full winner in equilibrium. Assume for the sake of contradiction that bidder i is a loser, so her utility is zero. If she deviates and bids $v_j + \varepsilon$ (for $0 < \varepsilon < v_i^T - v_j$) and her true budget, she will become either a full winner or the cut-point bidder (otherwise revenue of Sort-cut will decrease with this deviation, which is not possible because of Proposition 2). Then, obviously her utility becomes strictly positive with this deviation (her price per unit is at most v_j). We thus reached the necessary contradiction.

Now, consider a bidder i whose value is $v_i^T < v_j$. Assume for the sake of contradiction that bidder i is a winner. If b_i is smaller than the unused budget of the cut point bidder (s), then she gets all items at a per unit price v_j and hence she obtains a negative utility. If this is the case, she would be better of by announcing her true valuations and guarantee a nonnegative payoff. If $b_i > s$, then we argue that i would be better of by deviating to $(v_j - \varepsilon, b_i)$ for small enough ε. Let us first look at the limiting case in which i deviates to (v_j, b_i) and becomes the cut-point bidder. After this deviation, unused budget of i would be exactly s. The allocation of original full-winners will not change; bidder j will be getting $\frac{s}{v_j}$ items by paying s more and bidder j will be getting $\frac{s}{v_j}$ less items by paying s less. Therefore, bidder i's utility increases by $\frac{s}{v_j}\left(v_j - v_i^T\right) > 0$ (in a sense by this deviation, bidder i is selling $\frac{s}{v_j}$ units of the items to bidder j at the per unit price of v_j). By deviating to $(v_j - \varepsilon, b_i)$, original winners allocations would slightly increase, therefore bidder i's utility increase will be slightly smaller than $\frac{s}{v_j}\left(v_j - v_i^T\right)$, but for small enough ε, it will be always positive. ■

Above proposition establishes that given equilibrium cut-point value, all winners and losers will be rightly placed. But since the cut-point bidder may be misplaced, this does not imply full-Pareto optimality. Consider the following example.

Example 1 *There are 2 units of the item to be sold, and there are four bidders with budget-value pairs* $(18, 19)$, $(1, 9)$, $\left(\frac{17}{9}, 8\right)$ *and* $(10, 1)$. *For this setup, it can be confirmed that bidders announcing* *their types (budget,value) as* $(18,19),(1,9),(36,18)$ *and* $(10,1)$ *constitute an ex-post equilibrium of Sort-Cut. In this equilibrium, bidder 3 overstates her value and budget and becomes the partial winner. Although winners and losers are rightly ranked according to announced cut-point value, the allocation is not pareto optimal. Bidder 3 gets a positive allocation even though bidder 2 has higher value and zero allocation.*

4 Market Clearing Price Mechanism and Sort-Cut

In this section we compare Sort-Cut to a well known mechanism: *Market Clearing Price Mechanism (MCPM).* MCPM is a mechanism that sells m items to all interested bidders at a fixed price. That is, in MCPM all items are sold p dollars per unit and all bidders whose values are strictly greater than p spend all their budgets to buy these items (the bidders with values equal to p could be partially spending their budgets.) More formally,

Definition 7 *Market Clearing Price Mechanism is a* m*-Procedure cut mechanism with fixed pricing rule,* $\alpha(y)=p^*$ *for all* $y \geq 0$ *where* p^* *satisfies* $v_j \geq p^* > v_{j-1}$.

One can easily argue that there will be a unique p^*. Consider a fixed pricing rule $\alpha(y)=p^*$ that satisfies above definition. Then for any fixed pricing rule $\alpha(y)=p$ with $p>p^*$, we have $p>v_j$ (for v_j defined for $\alpha(y)=p$); and for any $p<p^*$, we have $p \leq v_{j-1}$ (for v_{j-1} defined for $\alpha(y)=p$).

Although it appears as a natural mechanism, as we demonstrate below, MCPM lacks good truthfulness properties.

Proposition 10 *Overstating budget or value is weakly dominated by bidding true types, i.e, for bidder* i *with type* (b_i, v_i)*, announcing* $(b_i^+, v_i),(b_i, v_i^+)$ *and* (b_i^+, v_i^+) *are all weakly dominated by* (b_i, v_i)

Proof. Consider bidder i with type (b_i, v_i) who announces her type truthfully.

- If she is a winner, she is indifferent to announcing v_i^+ and would be strictly worse off by announcing b_i^+ (she would either get negative payoff by staying a winner or will get zero utility by becoming a partial winner or a loser).

- If she is a partial winner, since overstating value or budget can only increase the market clearing price p^*, she never can obtain a strictly positive payoff by deviating to v_i^+ or b_i^+.
- If she is a loser, by overstating value or budget, she may become a winner, but market clearing price after deviation is going to be greater than previous market clearing price and hence greater than her value. v_i^+ or b_i^+ never helps.

■

However, understating the value or budget in general can be beneficial. Consider the following example.

Example 2 *Consider two bidders with budget-value pairs* $(16, 10)$ *and* $(8, 9)$ *and the supply is* $m = 3$. *Under truthful report of types, market clearing price is* $p^* = 8$. *However, if bidder 1 understates her value to* 7, *new market clearing price will be* 7 *with the first bidder spending* 13 *of her budget for an allocation of* $\frac{13}{7}$ *units. Her new payoff is* $(10 - 7)\,\frac{13}{7} \cong 5.57$ *versus* $(10 - 8)\,2 = 4$, *which shows understatement of value is a profitable deviation. Similarly, if bidder* 1 *understates her value to* 10, *new market clearing price will be* 6 *with the first bidder spending* 10 *of her budget for an allocation of* $\frac{10}{6}$ *units. Her new payoff is* $(10 - 6)\,\frac{10}{6} \cong 6.67$ *which shows understatement of budget is a profitable deviation. In fact, for this example an expost equilibrium is when bidders announce their types as* $\left(\frac{2700}{361}, 10\right)$, $\left(\frac{2430}{361}, 9\right)$ *which brings only a revenue of* 14.21.

Above discussion illustrates that the revenue from an (undominated) expost equilibrium of MCPM is bounded above by the revenue of MCPM with truthful revelations. Next, we obtain another lower bound for the revenue of Sort-Cut. For any announcements $(\mathbf{b}, \mathbf{v})$, we can show that the revenue difference between MCPM and Sort-Cut is at most equal to the maximum budget of the players. For the same announcement of the types, since Sort-Cut's pricing function is decreasing with higher budgets of the winners, whereas MCPM's pricing is constant; MCPM's revenue would be higher than the revenue of Sort-Cut, the following proposition shows that the difference in revenues is bounded above by the maximum of the winners' budgets. Let $R^M(\mathbf{b}, \mathbf{v})$ denote MCPM's revenue and $b_{\max}$ denote the maximum budget of the bidders.

Proposition 11 *For any announcements* $(\mathbf{b}, \mathbf{v})$, $R^M(\mathbf{b}, \mathbf{v}) - R^S(\mathbf{b}, \mathbf{v}) \leq b_{\max}$

Proof. Given $(\mathbf{b}, \mathbf{v})$, Sort-Cut's cut point is denoted by c^*, let c^{**} denotes MCPM's cut point. We argue that $c^{**} - c^* \leq b_{\max}$. By the definition of MCPM $c^{**} = m \times p^*$ where p^* satisfies $v_j \geq p^* > v_{j-1}$ and v_j is the partial winner in MCPM. Since $c^* \leq c^{**}$, j cannot be a full winner in Sort-Cut. If he is a partial winner, then $c^{**} - c^* \leq b_{\max}$ obviously holds as the difference between c^{**} and c^* is smaller than b_j. If j is a loser in Sort-Cut, then we can argue as follows. At least one of the winners of Sort-cut has to pay at most p^* per unit (otherwise the revenue of Sort-Cut has to be greater than c^{**}). Now, this bidder's budget has to be greater than $c^{**} - c^*$, because otherwise his price per unit cannot be smaller than p^*. Hence, $c^{**} - c^* \leq b_{\max}$. ■

Let us denote the revenue of MCPM with the truthful revelation of types by R^*. Proposition 11, together with proposition 8, establishes that the revenue of any refined equilibrium of Sort-Cut is not lower than $R^* - b_{\max}$. Unlike Sort-Cut, we next show an example where MCPM obtains a revenue that is order of magnitude (as the number of bidders) lower than R^*

Example 3 *Consider two types of bidders with budget, value pairs* $(b_0, v_0) = (16, 18)$ *and* $(b_1, v_1) = (8, 9)$*; our basic example has one bidders of each type with a supply of* $m = 3$ *units. Under truthful reports of budgets and values, the market-clearing price is* $p = 8$*. Let us look for an expost equilibrium, in which the announcements are* $(a_0, 18)$ *and* $(a_1, 9)$ *. The pair of values* a_0 *and* a_1 *solve the optimization problems of* $\max(v_i - p)\frac{a_i}{p}$ *for* $i = 0, 1$ *where* p *is the market clearing price for the given announcements and supply. In our case* $p = \frac{a_0 + a_1}{3}$*.*

Thus the optimization problem becomes $\max f(a_i) = \frac{3v_i a_i}{a_i + a_{1-i}} - a_i$*. Taking derivatives, we get* $f'(a_i) = \frac{3v_i a_{1-i}}{(a_1 + a_{1-i})^2} - 1$*, with* $f"(a_i) < 0$*. Solving the pair of first order equations by setting* $f'(a_0) = f'(a_1) = 0$*, we get* $a_1 = 6$ *and* $a_2 = 12$ *for a market clearing price of 6. The total revenue of this equilibrium is therefore* 18 *compared to* $R^* = 24$*.*

If we now scale the example to have N *bidders of each type and a supply of* $3N$*, we may assume that all the optimal budget announcements of each type of bidder are the same by symmetry. The clearing price stays unchanged at* $p = \frac{N(a_0 + a_1)}{3N} = \frac{a_0 + a_1}{3}$ *as before. The optimization problem for determining each* a_i *remains identical giving the same solutions as before.*

However, the revenue now is $16N$ *compared to* $R^* = 24N$ *and is thus a whole third less than* R^**, while the maximum bidder's budget is 16.*

5 Conclusion and Discussion

In this paper, we have introduced a mechanism to sell m units to a set of bidders with budget constraints. In this practically important setting where the existence of a truthful and pareto optimal mechanism is precluded, our mechanism, Sort-Cut, achieves good truthfulness, revenue, and efficiency properties. Specifically, in Sort-Cut, (i) there are profitable deviations from truthful revelations of types, but that can only happen in a revenue increasing way, (ii) in a refined expost equilibrium, the revenue of Sort-Cut is bounded below by $R^* - b_{\max}$, and (iii) the equilibrium allocation is *semi pareto efficient* in the sense that full winners and losers are ordered in the right way given the announced value of the partial winner. We then compare Sort-Cut to a well known mechanism, Market Clearing Price Mechanism (MCPM). We show that in MCPM, (i) revenue increasing deviations are dominated, (ii) revenue can be smaller than $R^* - b_{\max}$.

There are many ways our work can be generalized. In the context of online advertisement auctions, our model can be interpreted as "there is a *single* sponsored link that gets m clicks a day (on average) and there are n advertisers." However, in reality, there are many sponsored links. In generalized second-price auctions studied by [Edelman et al. 2007] the winner of the best item (first sponsored link) is charged the bid of the second-best item, the winner of the second best item is charged the bid of the third-best item and so on. In this environment there are no budget constraints and second-highest bid is always the competitor of the highest value. The idea of Sort-cut can be applied in this setup with budget constraints. More specifically, it would be interesting to consider a model in which there are budget-constrained bidders and multiple slots available for a query (in which an advertiser cannot appear in more than one slot per query).

In our model we consider a setting of hard budget constraints in which the bidders definitely cannot spend more than their budgets. Extending our results to a soft-budget problem in which bidders are able to finance further budgets at some cost is a promising direction. One can model this kind of soft-budget constraints as specifying value per-clicks up to some budget, then specifying a smaller value per-click up to some other extra budget and so on. By replicating a bidder into as many copies as the number of pieces in his value/budget function, and allowing them all to participate in our mechanism, it seems reasonable that we may preserve some of the desirable properties of Sort-Cut.

One very important extension is to consider the environment of multi-item auctions with budget constraints. Again consider the problem of the advertisement departments of Dell, HP or Sony, but this time appear in search engine's queries of "laptops" and/or "desktops." These advertisement departments might have a total budget to allocate between all online ads and their per-click values for different items might be different. For instance Dell might have higher per-click values for desktops, but lower per-click values of laptops, as compared to Sony. Designing an allocation and pricing rule which would have good efficiency, truthfulness and revenue properties for his setup is very challenging. Devanur et al. (2002) provided an algorithm for finding the "market clearing prices" (which could be thought of extension of ascending-price auction mechanism of single item case). This mechanism, however, lacks nice truthfulness properties. Bidders would have an incentive to understate their budgets, thereby decreasing the prices. Extension of Sort-cut is not straightforward as how bidders would like to split the budgets between different items would depend on the pricing rule of each of these items. Bidders' effective valuations for different goods are given by the ratios of "per-click values and the average prices" of different items. This multi-item extension seems to be the most important, yet challenging extension of our model.

6 Appendix

6.1 Proof of Proposition 1

First, note that $x(\alpha^c, b)$ is weakly increasing in c : since α is nonincreasing, for $c' \geq c \geq 0$, we have $\alpha^{c'}(y) = \alpha(y + c') \leq \alpha(y + c) = \alpha^c(y)$ and hence

$$x\left(\alpha^{c'}, b\right) = \int_0^b \frac{1}{\alpha^{c'}(y)} dy \geq \int_0^b \frac{1}{\alpha^c(y)} dy = x(\alpha^c, b)$$

Also, obviously $x(\alpha^c, b)$ is strictly increasing in b.

Now, we can show that $X(c, (\mathbf{b}, \mathbf{v}))$ is strictly increasing in c. Consider $c' > c \geq 0$, we have

$$X(c, (\mathbf{b}, \mathbf{v})) = \left(\sum\nolimits_{i=1}^{j-1} x(\alpha^c, b_i)\right) + x\left(\alpha^{c+s}, b_j - s\right)$$

where j satisfies $c \leq \sum_{i=1}^{j} b_i$ and $c > \sum_{i=1}^{j-1} b_i$ (and $s = \sum_{i=1}^{j} b_i - c$). For $c' > c$, we can have one of the two cases, either j is the same or j is bigger.

If j is bigger (this could be the case only when $s > 0$ and $c' > c + s$), then we have

$$\begin{aligned} X\left(c', (\mathbf{b}, \mathbf{v})\right) &> \sum\nolimits_{i=1}^{j} x\left(\alpha^{c'}, b_i\right) \\ &\geq \left(\sum\nolimits_{i=1}^{j-1} x\left(\alpha^{c}, b_i\right)\right) + x\left(\alpha^{c'}, b_j\right) > X\left(c, (\mathbf{b}, \mathbf{v})\right) \end{aligned}$$

This is because $x\left(\alpha^{c'}, b_i\right) \geq x\left(\alpha^{c}, b_i\right)$ for all $i = 1, .., j - 1$ and $x\left(\alpha^{c'}, b_j\right) > x\left(\alpha^{c+s}, b_j - s\right)$ since $c' > c + s$ and $s > 0$.

If j is the same (if $c' < c + s$), then we have

$$\begin{aligned} X\left(c', (\mathbf{b}, \mathbf{v})\right) &= \left(\sum\nolimits_{i=1}^{j-1} x\left(\alpha^{c'}, b_i\right)\right) + x\left(\alpha^{c'+s'}, b_j - s'\right) \\ &> \left(\sum\nolimits_{i=1}^{j-1} x\left(\alpha^{c}, b_i\right)\right) + x\left(\alpha^{c+s}, b_j - s\right) = X\left(c, (\mathbf{b}, \mathbf{v})\right) \end{aligned}$$

where $s' = \sum_{i=1}^{j} b_i - c' < s$. This is because $x\left(\alpha^{c'}, b_i\right) \geq x\left(\alpha^{c}, b_i\right)$ for all $i = 1, .., j - 1$ and $x\left(\alpha^{c'+s'}, b_j - s'\right) > x\left(\alpha^{c+s}, b_j - s\right)$ since $c' + s' = c + s$ and $b_j - s' > b_j - s$.

Next, we show that $X\left(c, (\mathbf{b}, \mathbf{v})\right)$ is continuous in c. By definition, $x\left(\alpha^{c}, b\right)$ is continuous in c and b (this is because $x\left(\alpha^{c}, b\right) = \int_0^b \frac{1}{\alpha(y+c)} dy$ and is continuous in c and b even when α is not a continuous function). Moreover,

$$X\left(c, (\mathbf{b}, \mathbf{v})\right) = \left(\sum\nolimits_{i=1}^{j-1} x\left(\alpha^{c}, b_i\right)\right) + x\left(\alpha^{c+s}, b_j - s\right)$$

if c increases from c to $c + \varepsilon$, j changes only when $s = 0$. If $s \neq 0$, then $X\left(c, (\mathbf{b}, \mathbf{v})\right)$ is obviously continuous in c as all of the terms in the summation are continuous in c. If $s = 0$, then

$$X\left(c + \varepsilon, (\mathbf{b}, \mathbf{v})\right) = \left(\sum\nolimits_{i=1}^{j} x\left(\alpha^{c+\varepsilon}, b_i\right)\right) + x\left(\alpha^{c+\varepsilon+s'}, b_{j+1} - s'\right)$$

and this goes to $X\left(c, (\mathbf{b}, \mathbf{v})\right)$ as ε goes to zero. This is because $\sum_{i=1}^{j} x\left(\alpha^{c+\varepsilon}, b_i\right) \to \sum_{i=1}^{j} x\left(\alpha^{c}, b_i\right) = X\left(c, (\mathbf{b}, \mathbf{v})\right)$ and $x\left(\alpha^{c+\varepsilon+s'}, b_{j+1} - s'\right) \to 0$ since $s' \to b_{j+1}$.

6.2 Proof of Proposition 2

Consider bidder i with announced type $\left(b_i, v_i\right)$.

- First, we show that revenue is increasing in budgets. Consider bidder i who decreases her value to $b_i^- < b_i$. We show that revenue cannot increase with this deviation.
  - If bidder i is originally a loser by announcing (b_i, v_i), then she cannot become a winner or partial winner by deviating to $b_i^- < b_i$. This is because by this deviation pricing function for everybody becomes better and winners pay less per unit. Therefore revenue cannot increase.
  - Next, consider bidder i who is a partial winner by bidding b_i. If bidder i deviates to b_i^- and becomes a loser, then the revenue has to decrease since the set of losers become larger with this deviation. If she deviates to b_i^- and remains a partial winner, since all winners' pricing get better, the revenue has to decrease. If she deviates to b_i^-, she cannot become a full winner. If it were the case, pricing function for every (full or partial) winner gets better, then total number of units to be allocated were to be greater than m.
  - Lastly, consider bidder i who is originally a winner by announcing (b_i, v_i). If she deviates to b_i^- and if she becomes a loser or a partial winner after the deviation, then the revenue clearly decreases. This is because the set of non-full winners after the deviation is a strict superset of the set of non-full winners before the deviation. Now consider the case bidder i deviates to b_i^- and remains a winner. Let us denote $b_i - b_i^-$ by Δ. Suppose that initial cut point is c and new cut point after the deviation is c'. Let α be the n-piece step function defined by (b, v). Note that initial revenue is c and new revenue is c'. We will show that $c \geq c'$.

    Since i has understated her budget, there will be shortage of demand and pricing of all original winners will be better. Therefore, with this deviation all original winners except i will be allocated (weakly) more units of the object. By method of contradiction assume $c' > c$. This means that there will be new winners who use an extra budget strictly greater than Δ, say Δ'. We now argue that extra units allocated to these new winners has to be greater than the number of units i is giving up with the deviation. Extra units allocated to new winners are priced at the values starting from the new cut point $c + \Delta'$ (according to (b, v)) and total budget used is Δ'. The number of units i is giving up are priced at the values in the range of c to $c + \Delta < c + \Delta'$ and the total budget used is Δ. Since extra units are given with higher budget and lesser prices than the units given up, we conclude that with assumption $c' > c$, total units allocated has to be strictly greater than m, which is a contradiction. We can see this argument more formally. For instance consider the case in which Δ is small so that original partial winner j remains a partial winner. All full winners $k \neq i$ with $k < j$ will be allocated more items since j will be using more of his budget after the deviation. Let us consider the difference between the total amounts allocated to bidder i and j before and after the deviation. Bidder i's allocation is decreased by

$$A \equiv x\left(\alpha^{c}, b\right) - x\left(\alpha^{c+\Delta'}, b - \Delta\right)$$

since

$$x\left(\alpha^{c+\Delta'}, b - \Delta\right) > x\left(\alpha^{c+\Delta'}, b - \Delta'\right)$$

we have

$$\begin{aligned} A &< x\left(\alpha^{c}, b\right) - x\left(\alpha^{c+\Delta'}, b - \Delta'\right) \\ &= x\left(\alpha^{c}, \Delta'\right) \end{aligned}$$

On the other hand, bidder j's allocation is increased by

$$\begin{aligned} B &\equiv x\left(\alpha^{c+s}, b_j - s + \Delta'\right) - x\left(\alpha^{c+s}, b_j - s\right) \\ &= x\left(\alpha^{c+b_j}, \Delta'\right) \end{aligned}$$

Since

$$x\left(\alpha^{c+b_j}, \Delta'\right) \geq x\left(\alpha^{c}, \Delta'\right)$$

We conclude $B > A$. Total number of units allocated has to increase after the deviation.

- Now, we show that revenue is increasing in values. Consider bidder i who increases his value to $v_i^+ > v_i$. We show that revenue cannot decrease with this deviation.

  – First, if bidder i is a winner by bidding (b_i, v_i) and she deviates to $v_i^+ > v_i$, then she remains a winner after the deviation, and the revenue does not change. This is because Sort-Cut's allocation and pricing rule is invariant to full winners' values (so long as they remain full winners).

– Second, consider a bidder i who is a loser by bidding (b_i, v_i) and a deviation to $v_i^+ > v_i$. If she remains a loser after deviation, since the pricing function for winners get worse, revenue has to increase. Let us now consider the deviation which makes i a partial winner. If the partial winner becomes a full winner after the deviation ($v_i^+ < v_j$ where j is the original partial winner); the revenue obviously increases with the deviation.

  Let us consider the case in which $v_i^+ > v_j$; i become a partial winner and j becomes a loser after the deviation. By method of contradiction, assume that the revenue decreases with the deviation. If this is the case, it can be seen that the pricing function for all winners become worse after the deviation (total budget of price setters with $v_k \geq v_i$ becomes greater and some of the values increase). Hence all full winners will be allocated less units of items after the deviation. This implies that the number of units allocated i after the deviation has to be greater than number of units allocated to j before the deviation. But again with the same observation, pricing function for i after the deviation is worse than the pricing function for j before the deviation. For i to be allocated more, her budget spent after deviation has to be greater than j's budget spent before the deviation, which is a contradiction.

  If i is currently a loser and deviates to v_i^+ and becomes a winner. We can split this into two deviations. First, i deviates to $v_i^{+\prime} > v_j$ and becomes a partial winner (which increases the revenue), then she deviates to v_i^+ and becomes a full winner which will be shown to increase the revenue in next bullet.

– Lastly, consider bidder i who is a partial winner by bidding (b_i, v_i). It is obvious that she cannot become a loser after deviating to v_i^+. If she deviates to v_i^+ and remains a partial winner, then pricing function for all winners get worse, hence the revenue has to increase. If she deviates to v_i^+ and becomes a full winner. Then we argue that revenue has to increase.

  Consider the case that i is currently the partial winner, and she deviates to $v_i^+ > v_{i-1}$

(where bidder $i-1$ has the next highest value after bidder i) so that $i-1$ is the new partial winner and i is a full winner. Denote original unused budget of bidder i by s'_i and after deviation unused budget of bidder $i-1$ by s'_{i-1}. It suffices to show that $s'_i \geq s'_{i-1}$. By method of contradiction, assume that $s'_{i-1} > s'_i$. Then, it is easy to see that the pricing function for all winners other than i or $i-1$ gets worse, therefore they will be allocated (weakly) less number of items. Similar to above discussion, we can also show that the total number of units allocated to bidder i and $i-1$ has to (strictly) decrease after the deviation, which gives us desired contradiction. Bidder $i-1$'s allocation is decreased by

$$x\left(\alpha^{c+s'_i}, b_{i-1} - s'_{i-1}\right) - x\left(\alpha^c, b_{i-1}\right)$$

which is strictly greater than

$$x\left(\alpha^c, s'_i\right)$$

Bidder i's allocation is increased by

$$x\left(\alpha^{c+s'_i-s'_{i-1}}, b_i\right) - x\left(\alpha^{c+s'_i}, b_i - s'_i\right)$$

which is strictly smaller than

$$x\left(\alpha^{c+s'_i-s'_{i-1}}, s'_i\right)$$

Since $c + s'_i - s'_{i-1} < c$, we conclude that total number of units allocated to players has to be strictly worse, therefore we get a contradiction.

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