Sort-Cut: A Pareto-Optimal and Semi-Truthful Mechanism for Multi-Unit Auctions with Budget-Constrained Bidders*

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ABSTRACT

Motivated by sponsored search auctions with hard budget constraints given by the advertisers, we study multi-unit auctions of a single item. An important example is a sponsored result slot for a keyword, with many units representing its inventory in a month, say. In this single-item multi-unit auction, each bidder has a private value for each unit, and a private budget which is the total amount of money she can spend in the auction. A recent impossibility result [Dobzinski et al., FOCS'08] precludes the existence of a truthful mechanism with Pareto-optimal allocations in this important setting.

We propose Sort-Cut, a mechanism which does the next best thing from the auctioneer's point of view, that we term *semi-truthful*. In our mechanism, it is a weakly dominant strategy for all agents to state their true budgets and to not understate their values. Thus the only way a bidder can possibly benefit from lying in a semi-truthful mechanism is by overstating their value, which leads to good revenue properties for the auctioneer at equilibria.

While we are unable to give a complete characterization of equilibria for our mechanism, we prove that *some* equilibrium of the proposed mechanism optimizes the revenue over all Pareto-optimal mechanisms, and that this equilibrium is the unique one resulting from a natural rational bidding strategy (where every losing bidder bids at least her true value). The latter is similar in spirit to the approach of [Edelman et al. American Economic Review 2007] in their analysis of the equilibria of the Generalized Second Price (GSP) auction implemented for sponsored search ads at Google. Perhaps even more significantly, we show that the revenue of **every** equilibrium of our mechanism differs by at most the budget of one bidder from the optimum revenue (under some mild assumptions).

We extend our results from the setting of hard budget constraints to one where the marginal disutility of spending the next dollar increases with the payment. We model this as a piecewise linear function with the same rate of disutility in each piece and the disutility rates increasing as increasing budget limits are breached. By replicating a bidder into as many copies as the number of pieces in his value/budget function, and allowing them all to participate in our mechanism we are able to preserve all the desirable properties of Sort-Cut.

While earlier work on the problem led to mechanisms that leave some items unallocated [Borgs et al., EC'05], and were variants on earlier ideas of Goldberg et al. [Goldberg et al., SODA'01], Sort-Cut provides a new pricing idea and generalizes second-price auctions in a natural way.

*Part of this work was done while Amin Sayedi was visiting Yahoo! Research.

1. INTRODUCTION

Multi-unit auctions have been studied comprehensively in microeconomics, especially in auction theory [4, 15, 14]. However, the problem of budget-constrained bidders has been paid surprisingly little attention, despite the fact that in practice, bidders face natural budget constraints. In this paper we study multi-unit auctions with budget limits which we motivate in some detail in Appendix A. We suppose there are m identical divisible [1] units of a single item for sale. Each bidder i has a private value v_i for each unit, and a private budget limit b_i on the total amount she may pay. We assume that bidder i's utility from acquiring x_i units and paying price p_i is $u_i = x_i v_i - p_i$ as long as the price is within budget: $p_i \leq b_i$, and is negative infinity if $p_i > b_i$. (i.e. the budget constraint is *hard*. Note that this is the assumption that may be considered unnatural in the more usual quasi-linear utility maximization models. However, we address this in a generalization - see point (5) below.)

Our Contributions:

1. We propose a new mechanism, called Sort-Cut, (Section 2) for selling *all* the units. We prove that Sort-Cut mechanism is *semi-truthful* (Section 3), i.e. no agent can benefit from lying about her budget or understating her value, but may overstate her value at equilibrium (thus address all but one of the four ways in which a bidder may lie about her bidder and value) . We show that the allocation of the Sort-Cut mechanism is Pareto-optimal (defined formally later - Section 4); hence, it is nearly the best possible result that can be obtained for this problem since the recent result of Dobzinski, Lavi and Nisan [9] shows that there is no truthful Pareto-optimal deterministic [2] mechanism for this problem.

2. We obtain an upper-bound R^* (which coincides with the revenue of ascending price auction with truthful bids) for the revenue at equilibrium for *any* Pareto-optimal mechanism

[1]While we describe the mechanism only for the divisible case, many of our results also translate to the indivisible case with minor modifications.

[2]We emphasize that our mechanism is not deterministic.

(Section 5) and then show that if the sum of budgets of all bidders is at least twice of R^*, this upper-bound is achievable at an equilibrium of the Sort-Cut mechanism.

3. Assuming reasonable behavior of the bidders where every losing bidder bids at last her true value (defined as rational bidding - see definition 5.2), we show (also in Section 5) that any equilibrium of Sort-Cut has a revenue of at least $R^* - b_{max}$ where b_{max} is the maximum budget among the winners, and prove this bound is tight for equilibria of all Pareto-optimal mechanisms which are budget-truthful (bidders can not benefit from lying about their budgets).

4. We also study the properties of this auction under greedy bidding behavior (Section 6) and show that under some natural assumptions, if the behavior leads to an equilibrium, the unique one it leads to is the revenue-maximizing one (attains revenue R^*).

5. We generalize the model to one where the disutility due to the payment of the bidder increases in a piecewise linear fashion. In this piecewise representation, the disutility is constant in each piece and is increasing as we move to higher payments. We show how an adaptation of Sort-Cut works in this setting (Section 7, with proofs deferred to the Appendix C).

Previous Work:

The problem of multi-unit auctions with budget-constrained bidders was initiated by Borgs et al. in [6]. Our model is identical to theirs. They introduce a truthful mechanism that is asymptotically revenue-maximizing; however, it may leave some units unsold. The idea is to group the bidders randomly into two groups, and use the market clearing price of each group as an offering price to the other group, following [**?**]. Another paper that uses the same model is by Abrams [1] - it uses techniques similar to [6] but improves upon it; however, it may still leave some units unsold.

A recent paper that analyzes this problem by Dobzinski et al. [9], mainly proves an impossibility result. They assume that budgets of all players are publicly known, and give a truthful mechanism which solves the problem under this assumption. Their mechanism is a direct application of Ausubel's auction [3]. Then they show that this mechanism is the unique mechanism which is both truthful and Pareto-optimal under the assumption of publicly known budgets. Finally by showing that their mechanism is not truthful if the budgets are private knowledge, they conclude that no mechanism for this problem can be both truthful and Pareto-optimal.

Both [6] and [9] argue that lack of quasi-linearity (because of hard budget constraints) is the most important difficulty of the problem. Some papers, nonetheless, have tried to solve the problem by relaxing hard budget constraints [12], or modeling the budget constraint as an upper bound on the value obtained by the bidder rather than her payment [13]. It has also been shown [6] that modeling budget constraints with quasi-linear functions can lead to arbitrarily bad revenue.

Another paper that has studied budget constraints, mainly for advertisement auctions, is the work of Feldman et al.[11]. They give a truthful mechanism for ad auctions with budget-constrained advertisers where there are multiple slots available for each query, and an advertiser cannot appear in more than one slot per query. Their work is related to our work because they also consider the game-theoretic aspects of the problem. However, the utility function that they use is very different from ours. In [11] they define advertisers to be click-maximizers, while in our model, advertisers are profit-maximizers, which we believe to be more realistic in the case of ad auctions.

Other papers that have considered budgets in auctions include [2],[5],[8]. However, [2] only considers the offline optimization problem and does not study the game theoretic aspects of the problem. They also model budget constraints by value functions of the bidders, which means bidders are not willing to get value more than their budget. In [5], they study an auction for selling two single items to budget-constrained bidders. They mainly focus on the effect of bidding aggressively on an unwanted item with the purpose of depleting other bidders budget. A similar effect arises in our model as well, but the focus of our work is generally very different from theirs. Another paper [8] compares first-price and all-pay auctions in a budget-constrained setting and show that the expected payoff of all-pay auctions is better under some assumptions. However, they do not consider multi-unit items.

2. Sort-Cut mechanism DESCRIPTION

In this section we describe how our *Sort-Cut* mechanism allocates the units, and the price each bidder is charged. Throughout the paper, for simplicity of description we always assume there exist a bidder with value ϵ and budget $m\epsilon$ (she has enough money to buy all the items with her value). As ϵ tends to zero, the revenue of this modified instance approaches that of the original, and hence this assumption is without loss of generality.

There are n bidders and m units available of the same item. The bidders have flat demands, i.e. bidder i's value per item is v_i. We consider the game as a game of complete information between the bidders. However, bidders' information (values and budgets) is not available to the designer. A typical bidder i's value per item is denoted by v_i and total budget by b_i. The algorithm will take announced values and budgets and operate on the announced information. As we will show formally, for our mechanism it is in bidder's interest to announce their budgets truthfully, and not understate their values. We describe the mechanism for announced values and budgets.

The Sort-Cut mechanism has two main features: sorting and cutting. First, we sort the bidders in non-increasing order of their values, and assume by relabeling that $v_1 \geq v_2 \geq \dots v_n$. The second part is the cutting by which the algorithm assigns the available m units to the bidders 1 through k, and assigns nothing to the bidders $k+1$ through n. The bidders $1, \dots, k-1$ must have exhausted all their budget, and bidder k may be left with some money from her budget b_k [3]. Bidders $k+1, \dots, n$ do not pay anything. We describe later how we determine the exact cut-point which is the portion of the budget of k that is used up in payments.

Suppose that the money left for bidder k is b'_k, and define $c_k = \frac{b'_k}{v_k}$. For $i > k$ define $c_i = \frac{b_i}{v_i}$. Here, c_i denotes the number of (fractional) units that bidder i can buy according to her value for one unit. We now describe the pricing rule for Sort-Cut: For all bidders whose values are greater than the cut point bidder (bidder i with $i < k$), we charge bidder i for the first c_k items that she wins a price of v_k, for the next c_{k+1} items that she wins a price of v_{k+1}, for the next c_{k+2} items that she wins a price v_{k+2} and so on until her budget is exhausted, i.e., her remaining budget cannot buy any more fractional items[4]. For the cut-point bidder k, the pricing is slightly different: we start charging her a price of v_{k+1} for the first c_{k+1} items that she gets, v_{k+2} for the next c_{k+2} items that she gets and so on until she has exhausted all her allocated spending budget

[3]In the indivisible item case, they may be left with some budget, when this remaining budget is not enough to purchase the next single unit of the item at the current charged price.

[4]In the indivisible case, we do this until her remaining budget cannot buy another whole item.

Algorithm 1 The Sort-Cut Mechanism for divisible units to determine the cut-point x^*

{Initialization} **Let** $B = \sum_i b_i$; **Initialize** $x = B/2$ and allocations $y_j = 0$ for all j.
repeat
  $B \leftarrow B/2$
  {Cut-point determination} **Let** k be the largest index such that $\sum_{i=1}^{k-1} b_i \le x$.
  {Pricing} **Let** $b'_k = \sum_{i=1}^{k} b_i - x$, and define $c_k = b'_k/v_k$. For $i = k+1$ to n define $c_i = b_i/v_i$.
  {Payments and Allocations until cut-point}
  for i = 1 to $k-1$ **do**
    Set the payment $p_i = b_i$; Initialize allocation $y_i = 0$ and $j = k$
    while $b_i > 0$ **do**
      $y_i = y_i + \frac{min(b_i, c_j.v_j)}{v_j}$
      $b_i = b_i - min(b_i, c_j.v_j)$
      $j = j + 1$
    end while
  end for
  {Payments and Allocations for cut-point}
  for bidder k **do**
    Let $b = b_k - b'_k$ and set the payment $p_k = b$; Initialize allocation $y_k = 0$ and $j = k + 1$
    while $b > 0$ **do**
      $y_k = y_k + \frac{min(b, c_j.v_j)}{v_j}$
      $b = b - min(b, c_j.v_j)$
      $j = j + 1$
    end while
  end for
  {Binary Search Update}
  if $\sum_{i=1}^{k} y_i > m$ **then**
    $x \leftarrow x - B/2$
  end if
  if $\sum_{i=1}^{k} y_i < m$ **then**
    $x \leftarrow x + B/2$
  end if
until the sum of allocations y_j is the supply m

of $b_k - b'_k$. We give each bidder i ($i < k$) as many units as she can afford according to our pricing and her budget.

To finish our description, we need to specify how we determine k, and b'_k (the money which is left for the last winner), because they play an important role in our pricing mechanism. If the index k is very large (close to n), then the prices for the units will be very low and the number of units that each winner can afford increases, also the number of winners is large resulting in a shortage of supply. On the other hand, if k is chosen very small (close to 1), we have few winners, and relatively high prices, so a number of units will be left unsold. We seek to find the right point which determines bidder k and the amount of money b'_k left for her, such that the market clears at this point.

To be more precise, define $B = \sum_{i=1}^{n} b_i$. We are looking for a breakpoint (boundary) $x = \sum_{i=1}^{k} b_i - b'_k$ in the interval $[0, B]$ that determines for us both k and b'_k which will make the number of allocated objects exactly equal to m. We want to sell all items. We must also guarantee that the bidders $1, ..., k-1$ (which are determined by x) do not have enough budget to buy any additional item. As we increase x, the prices decrease, the number of winners and items demanded to be allocated increases, and consequently the demand increases. We can find the solution by slowly increasing x from 0 until the demand becomes equal to supply. In other words, we increase x until the total number of units that bidders 1 through k want (assuming that bidder k can use only $b_k - b'_k$ of her budget) becomes equal to the units that we have to sell. (A formal proof of the fact as well as examples of such cut-points for the divisible and indivisible cases appear in Appendix B.

Finally, to keep the bidders from overstating their budget, we add the following to Sort-Cut. Suppose that the Sort-Cut mechanism wants to charge a bidder i an amount equal to p_i, and her announced budget is b_i. Instead of charging her p_i, we charge her b_i with probability p_i/b_i and 0 otherwise. In this way, if somebody overstates her budget, she is accepting the risk of paying more than her budget which makes her expected utility equal to minus infinity (this is because we have hard budget constraints). To make this more practical, even if we perform this alternate pricing process with probability ϵ, and simply charge the bidder i a price of p_i in the rest of the cases, still bidders can not take the risk of overstating their budget[5].

A simple example of Sort-Cut mechanism behavior is when there is only one *indivisible* unit for sale (i.e. $m = 1$). In this case, suppose that j is the smallest index such that there exists some $i < j$ with $b_i \ge v_j$. (Note that j always exists, because we have added a dummy bidder with value ϵ to the set of bidders). Now, take i the smallest index with $b_i \ge v_j$. The mechanism assigns the single unit to bidder i with price v_j. A special case of this example is when bidders do not have budget constraints. In that case, Sort-Cut mechanism is equivalent to the classical second-price auction.

3. SEMI-TRUTHFULNESS

Although Dobzinski et al. [9] shows that no truthful Pareto-optimal mechanism exists for this problem, it is still interesting to know *how truthful* a Pareto-optimal mechanism can be. In other words, we want to know how much the bidders can benefit from lying, and how the different ways of lying can benefit them in a mechanism.

There are four ways that a bidder can lie: overstating budget, understating budget, overstating value and understating value. In different mechanisms, bidders may take different strategies and use either of these ways to increase their utility. We show that in Sort-Cut mechanism, the only way out of these four that the bidder can use to benefit from lying is by overstating value. This result is interesting because first, we know that some kind of lying must be beneficiary for the bidders if the mechanism is Pareto-optimal, and second, among four different ways of lying, this is the most desirable one for the auctioneer – giving good revenue properties (This is formalized in Section 5). It is easy to see that the revenue of Sort-Cut mechanism is monotone with respect to the vector of bids. (i.e. if bidders increase their stated values, the revenue of the mechanism does not decrease.)

DEFINITION 3.1. *We say a mechanism A is semi-truthful if it is a weakly dominant strategy for the bidders to bid their real budgets and not to understate their value.*[6]

THEOREM 3.1. *Sort-Cut mechanism is semi-truthful.*

[5]It is not hard to argue that it suffices to use this kind of randomization for the cut-bidder, bidder k only. We can also construct examples where the cut-bidder can benefit from over stating the budget without this modification.

[6]In other words, every strategy that involves mis-stating the budget or understating the value is dominated by a strategy that does not mis-state the budget and does not understate the value.

PROOF. First it is easy to see that no bidder can benefit by overstating her budget, since by design, if she wins any allocation at all, there is positive probability that she will be charged more than her budget which makes her utility minus infinity.

Next, we argue that no bidder can benefit by understating their budget. A bidder who is a current loser cannot pull the prices down by understating. She will be always a loser, so she cannot benefit. Consider the winners - they (including cut-point bidder) can pull the prices down by understating the budget. Consider $j < k$ (for current prices), if she can pull cut point to $k' > k$ by overstating, she is going to be still a winner and spend all her budget. So, all she cares is the number of goods she will be assigned. But by underbidding she cannot be assigned more number of units as all the other winners are assigned more number of units (same budgets, smaller prices). This also holds for the current cut-point bidder. She cannot benefit by understating the budget when she is still the cut-point bidder: all other winners would be assigned more number of units, she will be assigned less number (with the same pricing schedule). If she understates and the cut-point bidder becomes a smaller value bidder than hers, we can use the above argument for bidders not in the cut-point and argue that she cannot benefit.

For the third kind of deviation (under-stating value), we argue in three cases:

1. Consider a bidder j where $j \leq k-1$. If bidder j understates her value, she may still remain among the first $k-1$ bidders which does not change anything for her, or, she may go to the boundary which makes the situation trickier, or she may go below the boundary which decreases her utility. So the only case that we must handle is when she goes to the boundary.

   Suppose bidder j moves to the boundary by announcing a value $v'_j \leq v_k$ and she spends b of her money while b' of her money will be remaining. (so we have $b + b' = b_j$) We know that before going to the boundary, with b'_k of her budget she bought the units for price v_k per unit, and after that she had unit prices $v_{k+1}, \ldots$. Now when she goes to the boundary, she buys for prices $v_{k+1}, v_{k+2}, \ldots$ which are lower and seem to be better for her, but as we will see, that is not the case because she is not using all her money when she is on the boundary. First note that if $b' \geq b'_k$, she can not benefit from going to the boundary (because previously she was using b'_k of her budget for getting the units for price v_k per unit, but now she has b' of her budget left unused). So we may assume $b' < b'_k$. Now, consider bidder k to see how many units she wins after leaving the boundary. Now the bidder with value v_k gets at least $\frac{b'_k}{v'_j} \geq \frac{b'_k}{v_k} \geq c_k$ units (for a price of v'_j) in addition to all the units that she had before (when she was on the boundary). The prices for all other winners is less than or equal to what it was before. Therefore, the number of units that bidder j wins after going to the boundary must be reduced by at least these c_k units. Bidder j's costliest c_k units were priced v_k units each for a total price of $b'_k = c_k v_k$. After understating her value, she is paying $b_j - b' > b_j - b'_k$ and getting at least c_k units less. Thus her average price per unit has increased so this is not an improvement.

2. Now consider a bidder j where $j > k$. It is obvious that bidder j can not benefit from understating her value, because it keeps her among the losers. However, overstating the value may be beneficial for her in some cases.

3. For the person on the boundary ($j = k$), it is clear that she can not benefit from understating her value, because it can not influence her price and she may even lose the units that she already wins (by reducing the price for earlier winners). Again, overstating the value may increase her utility in some cases.

□

4. PARETO OPTIMALITY

DEFINITION 4.1. *An allocation* $\{(x_i, p_i)\}$ *is* Pareto Optimal *if for no other allocation* $\{(x'_i, p'_i)\}$ *are all players better off:* $u_i(x_i, p_i) > u_i(x'_i, p'_i)$ *(Recall that* $u_i(x_i, p_i) = x_i * v_i - p_i$ *if* $p_i < b_i$ *and* $-\infty$ *otherwise), as well as the auctioneer:* $\sum_i p'_i \geq \sum_i p_i$*, with at least one inequality strict.*

Pareto optimality is simply implied by a proposition from [9].

PROPOSITION 4.1. *An allocation* $\{x_i, p_i\}$ *is Pareto-optimal in the infinitely divisible case if and only if (a) all units are completely sold, and (b) for all* i *such that* $x_i > 0$ *we have that for all* j *with* $v_j > v_i$, $p_j = b_j$*. I.e. a player may get a non-zero allocation only if all higher value players have exhausted their budget.*

Since Sort-Cut mechanism always allocates the units in decreasing order of the values, a bidder with value v_i may be allocated some units only if the bidders with higher values $v_j > v_i$ have exhausted their budget. Therefore, the allocation of the Sort-Cut mechanism is Pareto-optimal by construction.

5. REVENUE ANALYSIS

DEFINITION 5.1. *The ascending price auction mechanism is defined as follows. The price starts at* $p = 0$ *and increases infinitesimally and continuously; At any time, the demand of each bidder* i *is* $d_i = b_i/p$ *if* $p \leq v_i$ *and is* $d_i = 0$ *if* $p > v_i$*. The price continues increasing as long as there is over-demand* $\sum d_i > m$*. The price* $v^* = p$*, the first point when the demand equals the supply* m*, is defined to be the market clearing price. All bidders* i *with value* $v_i \geq v^*$ *are allocated* b_i/v^* *units for price* v^**. (The lowest valued one may be partially allocated, but still for price* v^* *per unit.) We define* R^* *to be the revenue of ascending price auction in which bidders are bidding truthfully.*

Note that ascending price auction is Pareto-optimal, but is not truthful; Specifically, bidders can benefit by understating their budgets.

In the next lemma, we show an upper bound on the revenue of any mechanism which guarantees Pareto-optimality.

LEMMA 5.1. *No Pareto-optimal mechanism, in equilibrium, can guarantee revenue more than* R^*.

PROOF. A mechanism will take announced values and bids and allocate the goods to some bidders at some prices. A mechanism, per-item, should not charge any bidder more than her value.

Suppose that v^* is the market clearing price and let l be the greatest index such that $v_l \geq v^*$. If a mechanism A generates a revenue more that R^*, it must charge some bidder i ($1 \leq i \leq l$) more than v^* per unit. But if bidder i decreases her bid down to $v^* + \epsilon$, the mechanism still has to exhaust all her budget (otherwise, because of Pareto-optimality, it can not charge the bidders who have value v^* or less, and consequently can not even make revenue R^*) but now with price of at most $v^* + \epsilon$. That means that at an equilibrium of the mechanism, no bidder can be charged more than v^* per unit. □

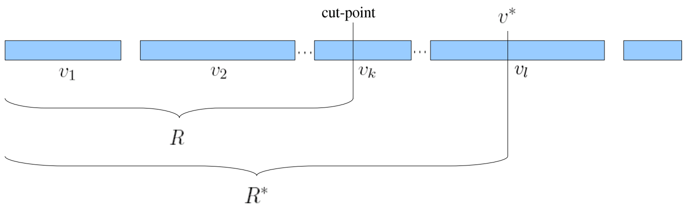

Figure 1: Revenue Comparison

The rest of this section obtains a lower bound for the revenue of Sort-Cut mechanism[7]. Before that, we need to introduce the concept of *Rational Bidding*. Since someone who is not winning anything in the Sort-Cut mechanismcan never benefit from understating her value, we have the following definition.

DEFINITION 5.2. *We say that bidders are bidding rationally, if those who do not win anything bid at least their true value.*

THEOREM 5.1. *Assuming rational bidding, the revenue of the Sort-Cut mechanism at any equilibrium is at least $R^* - b_{max}$ where b_{max} is the maximum budget among the winners.*

PROOF. Suppose that Sort-Cut mechanism has used all the budget of bidders $1, \ldots, k-1$ and a part of the budget of k-th bidder. Also suppose that market clearing price for truthful bids is v^* where $v_l \geq v^* > v_{l+1}$. As we defined, $b_{max} = max_{1 \leq i \leq k} b_i$, where b_i is the budget of i-th bidder. Note that the revenue of Sort-Cut mechanism is $R = \sum_{i=1}^{k} b_i - b'_k$ where b'_k is the amount of money left unused by the k-th bidder. Also note that $R^* \leq \sum_{i=1}^{l} b_i$ (by the definition of ascending price auction).

Now, we are ready to prove the claim. Consider an output of the Sort-Cut mechanism. Since the revenue of Sort-Cut mechanism is less than R^* and both mechanisms sell all m units, there must be some winner i who is getting the item cheaper than v^* per unit. Also we can assume that $k > 1$ since otherwise the claim holds trivially. We can either have $i < k$ or $i = k$. First suppose that $i < k$. According to our pricing scheme, bidder i has to pay at least v^* per unit up to $R^* - R$ of her budget (See Figure 1). That means if i is paying less than v^* on average per unit, her budget must be more than $R^* - R$. Therefore, $b_i > R^* - R$ which implies $R > R^* - b_{max}$. Now consider the case where $i = k$. Here, bidder i is using $b_i - b'_i$ of her budget, and she has to pay at least v^* per unit up to $R^* - R - b'_i$ of her budget. Therefore, if she pays less than v^* on average per unit, the amount of her budget that she is using, $b_i - b'_i$, must be more than $R^* - R - b'_i$. That is $b_i - b'_i > R^* - R - b'_i$, equivalently, $R > R^* - b_i$. □

Our analysis in the above theorem is tight: a simple example with one bidder demonstrates that any Pareto-optimal mechanism which is truthful for the budgets (like Sort-Cut mechanism) can not guarantee a revenue higher than $R^* - b_{max}$. No budget-truthful Pareto-optimal mechanism can charge this bidder more than 0. Therefore, our bound of $R^* - b_{max}$ is tight, and the best achievable in budget-truthful Pareto-optimal mechanisms.

[7]We do the revenue analysis for the announced values and budgets.

6. REVENUE OPTIMAL EQUILIBRIUM BY REPEATED BIDDING

In this section, we are using the same approach that Edelman et al [10] used to model the generalized second price (GSP) auctions. Like in their paper, we assume that the bidders are playing an infinitely repeated game, and use this to obtain some equilibria properties for Sort-Cut mechanism. We then take the approach of a consequent paper [7] which shows that a natural bidding strategy played by all bidders leads to a unique Nash equilibrium of GSP, and that the Nash equilibrium coincides with the outcome of a VCG auction. Here, we show that a natural bidding strategy, called *Greedy Bidding*, when it converges to an equilibrium, leads to one that coincides with the outcome of ascending price auction with optimal revenue.

We focus on simple strategies and impose some assumptions and restrictions. First, we assume that all budgets and values are common knowledge: over time, advertisers are likely to learn all relevant information about each other. Second, since bids can be changed at any time, stable bids must be best responses to one another. We define *Greedy Bidding*, a simple and natural response algorithm for the bidders who are playing the infinitely repeated game without knowing anything about bids and budgets of other bidders. Then we show that if running this algorithm converges, it does so at an unique equilibrium with prices and allocations identical to those of the ascending price auction with truthful bids.

DEFINITION 6.1. *(Greedy Bidding) Assume that each bidder always bids her true budget. Moreover, she revises her bid at each round of the infinitely repeated game by executing the following rules in order.*

1. *If what the bidder is paying per unit on average is higher than her value per unit (her bid is too much above her value), she decreases her bid continuously over time.*
2. *If all or part of her budget is left unspent (trying to deplete the budget of those who are above her, and winning more units), she increases her bid continuously over time.*
3. *If she is using all her budget (she can not influence her own price according to the pricing scheme), she does not change her bid.*

Note that we are not specifying any order for the bidders to change their bids. As it will be clear from the proof below, it does not matter how the bidders take turns to modify their bids as long as they converge to some equilibrium.

LEMMA 6.1. *If all bidders use Greedy Bidding and converge to an equilibrium, this equilibrium has prices and allocations identical to those of the ascending price auction with truthful bids, which provides revenue R^*.*

PROOF. First we claim that all losers have the same bid in an equilibrium. This is because if any of them bids slightly higher, she must be assigned something with a price higher than her value to send her back to her previous bid using rule 1. Because of Pareto-optimality, this can happen only if they all have the same bid. Moreover, this common bid must be the highest possible, otherwise they all can increase their bid. Those who have higher value than the common bid of the losers must bid higher than the losers. Therefore, the unique solution to this greedy bidding system is when all losers are bidding exactly (or slightly lower than) the market clearing price and this completes the proof. □

PROPOSITION 6.1. *If we assume that revenue R^* of ascending price auction for truthful bids is less than half of the total budget of all participants, i.e. $\sum_{i=1}^n b_i \geq 2R^*$, then there exists an equilibrium of the Sort-Cut mechanism in which the payments and utilities are like those of the ascending price auction with truthful bids.*

PROOF. The following vector of bids will be a Nash equilibrium in the game of complete information. All those who have value greater than v^* bid truthfully, those who have value less than or equal to v^* bid v^*. Therefore, all those who are bidding v^* are the losers and will not be assigned anything, and the winners have to pay v^* per unit. (Note that if the last winner is partially using her budget, then v^* is equal to her value, and she has utility 0. Therefore she has no incentive to increase her bid for depleting the budget of other winners.)

The condition on the revenue is required so that there is enough budget of unallocated bidders to set the corresponding market clearing price for the ones that are allocated in the ascending price auction equilibrium. □

7. RELAXING THE HARD BUDGET CONSTRAINT VIA A NEW BIDDING LANGUAGE

We study an extension to the hard budget constraint model by quantifying the "pain" felt in spending the next dollar of the budget. If the pain felt is 1 unit per dollar until a budget and infinite after that, we get back the original model. Alternately, we can allow for different pain values that are felt as the budget allocated increases, which we formalize below as a vector of budgets and values.

Every bidder i submits a vector of values v^i and a vector of budgets b^i (of the same size). The vector v^i must be in descending order. The vector denotes that the value of bidder is v_1^i per unit up to first b_1^i dollars that she spends, v_2^i for the next b_2^i dollars that she spends and so on. A mechanism allocates x units to the bidder for a total price of p. The utility of the bidder is then computed as follows. Suppose that k is the largest integer such that $\sum_1^k b_j^i \leq p$. Let $\rho_j^i = b_j$ for $j \leq k$, $\rho_{k+1}^i = p - \sum_1^k b_j$ and $\rho_j^i = 0$ for $j > k+1$. The utility is defined as $u^i = x.v_1^i - \sum_{j=1}^{k+1} \frac{v_1^i}{v_j^i}\rho_j$. From the earlier discussion, this is as if the pain felt by the bidder until the budget value of b_1^i is 1 per dollar, increases to $\frac{v_1^i}{v_2^i}$ per dollar for the next b_2 dollars, and then to $\frac{v_1^i}{v_3^i}$ per dollar for the next b_3^i dollars and so on.

It is also not hard to see that the formalization above is another interpretation of the model of diminishing marginal utility of wealth. In particular, the disutility due to the payment in the above utility function is increasing as the payment increases in linear pieces. E.g., the second portion of the payment ρ_2^i has disutility factor $\frac{v_1^i}{v_2^i}$, and the third one ρ_3^i has disutility factor $\frac{v_1^i}{v_3^i}$; We require that $v_2^i \geq v_3^i \ldots$ so that these disutility factors are increasing with increasing payment.

We can argue that the above model strictly generalizes the hard budget constraint of b^i with valuation v^i as follows: we set $v_1^i = v$, $b_1^i = b$ and $v_2^i = 0$ for all the remaining budget ($b_2^i = \infty$). If the total payment is more than b, then $\rho_2^i > 0$ implying that the utility is $(x \cdot v - \frac{v}{0}\rho_2)$ which is negative infinity as required. However, such a steep drop-off in valuation to zero is implausible in real-world situations where further units will continue to add at least some nonzero value despite the payment reaching astronomical amounts. Therefore, while getting a utility of negative infinity seems contrived in our setting, it is a natural outgrowth of this more general formulation.

We present a revised version of Sort-Cut for this general setting, whose allocation is Pareto optimal; Also, understating values or budgets in the vectors of bids is weakly dominated by not doing so. However, we show an interesting equivalence in the strategies of a bidder increasing her payoff by overstating her value and overstating her budget; a similar equivalence holds between understating value and budget. This shows that at equilibrium, bidders can either overstate value or budgets in this new model. Nevertheless, we still have an analogue of R^* in this case bounding the revenue of any Pareto optimal mechanism at equilibrium; we show that under a similar form of rational bidding, the revenue of this revised Sort-Cut mechanism is at least $R^* - p_{max}$ where p_{max} is the maximum payment among the bidders and is at most b_{max}. These results are described in detail in the Appendix C.

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APPENDIX

A. MOTIVATION FOR SINGLE-ITEM MULTI-UNIT AUCTIONS WITH BUDGETS

While billions of dollars are spent on keyword-based advertising in the web, a majority of it is cleared through advertisement auctions for sponsored search results in search result pages [6]. When a typical advertiser goes to a typical search engine company to sign up to bid in such auctions, they specify the set of keywords whose search result pages they are interested in bidding for, with a bid value per click; they also specify a total (monthly) budget for the total amount they are willing to spend across all these keywords in this search company's site. We focus our attention on the budget constraint which we model as being a hard constraint (cannot be exceeded) and attempt to design an auction mechanism for clearing this market. However, to better understand the difficulty of the hard budget constraint, we simplify the other aspects by restricting ourself to the problem for a single keyword, and within the keyword search results, restrict our attention to a single sponsored search result slot (in contract with the whole ladder of slots on the right of the results page). Thus, one may view this, e.g. as the auction for the single shaded sponsored search slot right under the search query window and above the organic search results in Google.com.

Our model abstracts the auction for this single item (one slot for a keyword) as a multi-unit auction where the number of units is the inventory of such search results in the period for which the budget is specified. While this is not the way that current sponsored results are allocated (rather an auction is run every time a query is made), it is quite a plausible scenario that the industry may move to, especially as the market for such results mature, and a few major advertisers wish to plan for their internet advertising campaigns in much the same way as for other media advertising (such as the annual Fall market clearing event for prime-time TV advertising in New York). Our study represents the first step towards the design of such a market.

While multi-unit auctions have been studied extensively in auction theory [4, 15, 14], the problem of budget-constrained bidders has been paid surprisingly little attention. One reason the budget constraints have not received much attention is the traditional economist's view that such constraints are unnatural and that if an advertiser can make positive payoff at her current true valuation, she should be able to finance a higher budget by borrowing in the market. However, reality is different and the valuations announced do not scale forever[8]; Furthermore, practical considerations (business planning) also put a cap on how much can be spent by advertisers in each period. Current keyword auctions get around this by treating each search page in the ad inventory as an instance of an online matching of current advertisers to slots, where the bids are assumed to be small compared to the budgets [13], and adjusting participation as the budget gets close to being spent.

Another reason for this lack of attention to the problem may be the technical difficulty that the utility of obtaining the items is compared to the total price at which the items are procured to give a net payoff. The total price is now discontinuously influenced by a hard budget constraint. Perhaps because of this, the theoretical framework of budget-constrained auctions is currently substantially less well-developed than that of unconstrained auctions. This is unsatisfactory both from a theoretical viewpoint, and from the practical viewpoint, where the absence of appropriate framework might potentially result in losses in revenue and efficiency.

[8]While our main results are for the case with the hard constraints, we also present an extension to the one where as the spending from the budget increases, the valuation decreases correspondingly as specified by a piecewise function.

B. SORT-CUT DETAILS

We first sketch here the proof that there exists k, b'_k such that the number of allocated items allocated is exactly m in the description of Short-Cut. First of all, x can span the whole interval $[0, B]$, that is there is continuity in (k, b'_k). The only discontinuity can be when we change k to $k+1$, but b'_k gives us enough continuity to span the whole interval, therefore the number of allocated objects is also going to be continuous in (k, b'_k). Secondly, when x is low (close to zero by setting k equal to 1 and b'_1 close to b_1), the number of allocated objects can be at most $\frac{b_1}{v_2}$ (which we assume to be smaller than m) and is very low. On the other hand, when x is high (close to B, by setting k equal to n) the number of allocated objects is very high (goes to infinity with the assumption that there exists a fictitious last bidder with very low value and high budget). Hence, from intermediate value theorem we can conclude that such k, b'_k exists. Algorithmically, we can use a simple binary search for finding the right value for x. Examples of the mechanism for both the divisible and indivisible cases are presented next.

EXAMPLE B.1. *We give an example to show how our mechanism works for selling* 19 *units of a divisible item to a set of* 4 *bidders with the following (private) values per item and budgets.*

i	v_i	b_i
1	10	55
2	9	60
3	7	40
4	6	30

We start with $x = 128$, *so* $k = 3$ *and* $b'_3 = 27$. *This means that the price of each unit (for the first two bidders) is* 7 *for the first* 27 *that they spend, and after that for the next* 30 *that they spend, the price for each unit is reduced to* 6, *and finally, after that the price is* ϵ *for each unit. Therefore, the first bidder can afford* $x_1 = 27/7 + 28/6$ *units with a total price of* 55. *But the second bidder can afford all the remaining units now (which means nothing will be left for the third bidder, who must be assigned something according to our breakpoint).* $x_2 = 27/7 + 30/6 + 3/\epsilon$. *This means that* x *is too large for the breakpoint.*

Our next guess is $x = 122$, *so* $k = 3$ *and* $b'_3 = 33$. *Here, the price (for the first two bidders) is* 7 *per unit for the first* 33 *that they* *spend, and* 6 *per unit for the next* 30 *that they spend, and* ϵ *per unit after that. Therefore, the first bidder can afford* $x_1 = 33/7 + 22/6$ *and the second bidder can afford* $x_2 = 33/7 + 27/6$ *units. The third bidder can use* 7 *of her money and she has to pay* 6 *per unit for the first* 30 *that she spends. Therefore, she can afford* $x_3 = 7/6$ *units. We can see that* $x_1 + x_2 + x_3 < m$*, therefore,* x *is too small for the breakpoint.*

By continuing the same procedure, we will see that $x \simeq 123.11$ *is the right value for* x*. Therefore,* $x_1 \simeq 8.4$*,* $x_2 \simeq 9.25$ *and* $x_3 \simeq 1.35$*, and the prices they pay are* $p_1 = 55$*,* $p_2 = 60$ *and* $p_3 \simeq 8.11$*.*

In the next example, we use the same numbers as above but demonstrate it for the case of indivisible units.

EXAMPLE B.2. *In this example,* $B = 185$ *and* $m = 19$*. If we assume that Sort-Cut mechanism uses binary search for finding the right breakpoint, the first guess would be* $x = 92$*. Consequently,* $k = 2$ *and* $b'_2 = 115 - 92 = 23$*. Therefore,* $c_2 = \lfloor 23/9 \rfloor = 2$*,* $c_3 = \lfloor 40/7 \rfloor = 5$ *and* $c_4 = \lfloor 30/6 \rfloor = 5$*. Thus, the first bidder wins* 7 *items* ($x_1 = 7$)*; the first* 2 *of them with price* 9 *for each, and the next* 5 *with price* 7 *for each. The total amount of money that the first winner has to pay is* $p_1 = 53$*. Now, we look at the second bidder. Note that she is the one on the boundary, so she has different pricing scheme. She can use up to* $60 - 23 = 37$ *of her budget. The first* 5 *items that she gets cost* 7 *for her each, and other items (if she can afford) cost* 6 *for her each. Therefore, she can get* 5 *items with a total price of* 35*. We see that demand is* $x_1 + x_2 = 7 + 5 = 12$ *while supply is* $m = 19$*. This means that* $x = 92$ *is too small for the breakpoint.*

We continue our binary search with $x = 138$*. Now,* $k = 3$ *and* $b'_3 = 155 - 138 = 17$*.* $c_3 = 2$ *and* $c_4 = 5$*. The price of the first unit, the second unit, and ... that each winner wins (for bidders* 1 *and* 2*) is* $7, 7, 6, 6, 6, 6, 6, \epsilon, \epsilon, \ldots$*. Therefore, the demand of the first bidder according to this pricing scheme is* $x_1 = m$*, which means nothing will be left for the second bidder. This case shows that* x *is too large for the breakpoint, so we have to continue our binary search with* $x = 115$*.*

By continuing the binary search, after trying values $x = 115, x = 128, x = 122, x = 119, x = 121$ *we finally end up with the following pricing and allocation.* $x = 121$*,* $k = 3$*,* $b'_3 = 34$*,* $x_1 = 8$*,* $p_1 = 52$*,* $x_2 = 11$ *and* $p_2 = 58$*. The prices of the first unit, second unit, ... for the first two bidders are:* $7, 7, 7, 7, 6, 6, 6, 6, 6, \epsilon, \epsilon, \ldots$*. If the reader has not followed all the details for this example, it is instructive at least to apply the mechanism for the values of* $x = 120$ *and* $x = 122$ *to see how the value of* $x = 121$ *gives the clearing point.*

C. REVISED SORT-CUT FOR EXPRESSIVE BIDDING

We show that a natural extension of Sort-Cut mechanism works in this new setting. For the sake of simplicity, we drop the superscript on the budgets and values denoting the bidder, and use subscripts to denote the various parts of her vectors.

We now describe the key differences from the original Sort-Cut mechanism. If a bidder has submitted a vector of s entries, we consider her as s different bidders with correspondent budgets and values and we run sort-cut but with a few differences. When we are calculating prices for a bidder, we consider different parts of other bidders as different bidders. However, for pricing, we treat the different parts of this bidder as one, i.e. for each part, we skip over the other smaller value parts of the same bidder in computing the price. Furthermore, for each part, we do not start from the original cut-point for pricing, instead, we continue from where we stopped for the previous part. For example, for the first part (the part with highest value) we do exactly as original sort-cut: we start from the cut-point and we move towards lower values and compute the prices per units. But for the second part of this bidder, we will not start from the cut-point again, instead we will start from where we stopped for the first part. Also for the third part, we start from where we stopped for the second part and so on.

We present the details of the revised Sort-Cut mechanism for expressive bids in Algorithm 2. Without loss of generality, we assume that the size of the vector for every bidder is s.

Our way to model this setting is to assume that every bidder has a true cumulative value curve per dollar which has slope 1 in the first piece $[0, tb_1]$, then has slope $\frac{tv_1}{tv_2}$ in the second piece $[tb_1, tb_2]$, slope $\frac{tv_1}{tv_3}$ in the third piece $[tb_2, tb_3]$, and so on. We have used the t- prefix in the variables to denote that these are from the "true" curve of the bidder.

In the mechanism, the bidders are free to choose how many (budget, value) pairs they would like to submit to the auction. Given this, there is a subtlety in the definition of when the bidder is truthful, which we clarify now. Clearly, stating the original set of budgets $(tb_1, tb_2, \ldots, tb_s)$ along with the corresponding true values $(tv_1, tv_2, \ldots, tv_s)$ is truthful. Here are two other notions of truthfulness.

DEFINITION C.1. *For a given vector of budgets* $(b_1, \ldots, b_s)$*, a vector of values* $(v_1, \ldots, v_s)$ *is called truthful if for every piece* i *with budget* b_i*, the value* v_i *obeys the expression that* $b_i \cdot v_i$ *is exactly the original value* tv_1 *multiplied by the cumulative value per dollar in this segment, namely the cumulative increase in the value per dollar across the segment* $[\sum_{j=0}^{i-1} b_j, \sum_{j=0}^{i} b_j]$*.*

DEFINITION C.2. *For a given vector of values* $(v_1, \ldots, v_s)$*, a vector of budgets* $(b_1, \ldots, b_s)$ *is called truthful if they obey the following conditions: First,* $tv_1 \geq v_1 \geq v_2 \ldots$*. Then,* b_1 *is such that the true cumulative value per dollar at* b_1 *times* tv_1 *is equal to* $b_1 \cdot v_1$*. Also, the cumulative increase in the value per dollar across the segment* $[b_1, b_1 + b_2]$ *times* tv_1 *equals* $b_2 \cdot v_2$ *and so on.*

We are now ready to show that bidders cannot benefit from understating any of the values.

THEOREM C.1. *For any given vector of budgets* b*, an advertiser cannot benefit from understating any of the entries of vector* v*. In other words, understating value is weakly dominated.*

PROOF. Sort-cut defines a pricing scheme for each bidder. The pricing scheme for a particular bidder can be characterized by two vectors $p.value$ and $p.units$. If the bidder is getting x units, the price assigned by sort-cut is

$$price(x) = \sum_{i=1}^{l} p.value_i \times p.units_i + remain \times p.value_{l+1}$$

where l is the largest value such that $\sum_1^l p.units_i \leq x$ and $remain = x - \sum_1^l p.units_i$. More precisely, $p.value$ is the vector $(value(k), value(k+1), \ldots)$ from the formal description of sort-cut, but with entries corresponding to this bidder eliminated, and $p.units$ is the vector $(\gamma_k, \gamma_{k+1}, \ldots)$ again with entries corresponding to this bidder eliminated. The expression given above is computed by the algorithm in step 14.

If a part of this bidder, which is among the winners, understates the value and moves either to the boundary or to the set of losers, then this part will lose a certain number of units say z. The key

Algorithm 2 The Sort-Cut mechanism for new bidding language

{Initialization} **Let** $B = \sum_{i=1}^{n}\sum_{j=1}^{s} b_{ij}$; **Initialize** $x = B/2$ and allocations $y_j = 0$ for all j.
Sort all b_{ij}s in descending order and name them $\beta_1, \ldots, \beta_{ns}$.
{By $value(i)$ and $bidder(i)$we mean the value and the bidder of the bid which corresponds to budget β_i respectively.}
repeat
  $B \leftarrow B/2$
  {Cut-point determination}
  Let k be the largest index such that $\sum_{i=1}^{k-1} \beta_i \leq x$.
  Let $startindex_i \leftarrow k$ and $rem_i \leftarrow 0$ for all $1 \leq i \leq n$.
  Let $\beta'_k = \sum_{i=1}^{k} \beta_i - x$, and define $\gamma_k = \beta'_k/value(k)$, and change $\beta_k \leftarrow \beta_k - \beta'_k$. Define $\gamma_i = \beta_i/value(i)$ for $k+1 \leq i \leq ns$.{β and γ correspond to b and c in original sort-cut}
  {Payments and Allocations}
  for i = 1 to k **do**
    Set the payment $p_i = \beta_i$; Initialize $y_i = 0$, $j = startindex_{bidder(i)}$, and $temp = rem_{bidder(i)}${$startindex$ and rem point to the bidder and the remaining portion of her budget that the pricing must continue from.}
    while $\beta_i > 0$ **do**
      {Computing Pricing}
      while $bidder(i) = bidder(j)$ **do**
        $j \leftarrow j + 1$ {Skipping the parts of the same bidder}
      end while
      $y_i \leftarrow y_i + \frac{min(\beta_i, \gamma_j.value(j) - temp)}{value(j)}$
      $rem_{bidder(i)} \leftarrow min(\beta_i, \gamma_j.value(j) - temp)$
      $\beta_i \leftarrow \beta_i - min(\beta_i, \gamma_j.value(j) - temp)$
      $startindex_{bidder(i)} \leftarrow j$
      $temp \leftarrow 0$
      $j \leftarrow j + 1$
    end while
  end for
  {Binary Search Update}
  if $\sum_{i=1}^{k} y_i > m$ **then**
    $x \leftarrow x - B/2$
  end if
  if $\sum_{i=1}^{k} y_i < m$ **then**
    $x \leftarrow x + B/2$
  end if
until the sum of allocations y_j is the supply m
{Aggregating allocation and price from different parts of a bidder}
for i = 1 to n **do**
  $payment_i = \sum_{j:bidder(j)=i} p_j$
  $allocation_i = \sum_{j:bidder(j)=i} y_j$
end for

claim is that the new pricing scheme is the same as before with *at most* z units shifted to right: z units are removed from the beginning of $p.units$ vector and corresponding $p.value$ vector entries. In other words, the pricing starts from at most z units after where it did before. As a result, also because of descending order of $p.value$ vector, if we name the new price function $price_{new}$ we get the following inequality for any $y \leq x$:

$$price_{new}(x) - price_{new}(y) \geq price(x+z) - price(y+z).$$

Suppose the part that is understating value has real value v', budget b' and allocation x'. We already know that according to definition of sort-cut that $v'x' \geq price(x')$. Also from the corresponding proof in the original sort-cut mechanism, we know $v'(x'-z) - price_{new}(x'-z) \leq v'x' - price(x')$, that is, understating value for the part itself cannot be beneficiary. Therefore, the gain in the utility function of the bidder by understating value will be non-positive if the total payment of other parts increases after understating, or in other words, if the following inequality holds.

$$price_{new}(x) - price_{new}(x'-z) \geq price(x+z) - price(x')$$

But this inequality holds by directly by applying the previous inequality.

□

DEFINITION C.3. *Given the vector of values $(v_1, \ldots, v_s)$ suppose that the truthful vector of budgets for a bidder is $(b'_1, \ldots, b'_s)$ and the stated vector by this bidder is $(b_1, \ldots, b_s)$. If $\sum_1^l b'_i > \sum_1^l b_i$ for some l, we say that the bidder is* understating budget. *If someone is not understating budget and is not truthful either, she is overstating budget.*

We show later that each of understating budget and understating value are each enough to model the other. In other words, every strategy of overstating budget by a bidder can be modeled by one of overstating value and vice-versa. This gives the following implication from the previous theorem.

THEOREM C.2. *In the new bidding language and using new version of sort-cut, understating budget is weakly dominated.*

By construction, since units are allocated to bidders with a value v only if no other bidder of higher value can afford it, the following theorem is immediate.

THEOREM C.3. *The allocation and pricing given by new version of sort-cut for the new bidding language is always Pareto optimal.*

What we showed is that for a fixed vector of values, one cannot benefit from understating her budget, with some more general definition for understating budget according to the new setting. We also showed that given a vector of budgets, one cannot benefit from understating value.

We next show that each of overstating budget and overstating value is enough to model the other. I.e., every way of overstating budget by the bidder can be modeled by an instance of overstating value and vice-versa. This shows that a mechanism cannot prevent the bidders from only one of these two, and therefore, unlike the original sort-cut, here we cannot expect an alteration which prevents bidders from overstating their budget; (Note otherwise that we will get a fully truthful mechanism for this generalization, which according to the result of Dobzinski et al. [9], is impossible.)

DEFINITION C.4. *We say that a mechanism is division-blind if the output of the mechanism does not change if any bidder divides one of her part with value v' and budget b' into two parts both with value v', and one with budget $b1'$, the other with budget $b2'$ such that $b1' + b2' = b'$.*

THEOREM C.4. *If one can benefit from overstating budget in some input of any division-blind mechanism, there is also some other input in which one can benefit from overstating value and vice versa.*

PROOF. Consider a generic budget overstating: $(\ldots, b_i + \epsilon, \ldots, b_j - \epsilon, \ldots)$ instead of $(\ldots, b_i, \ldots, b_j, \ldots)$ (all other forms of overstating budget can be obtained by sequential application of this step). To get the corresponding value overstating with the same effect add one extra part with value v_j and budget ϵ and reduce b_j to $b_j - \epsilon$. Therefore, the vector of values and budgets look like $(\ldots, v_i, \ldots, v_{j-1}, v_j, v_j, \ldots)$ and $(\ldots, b_i, \ldots, b_{j-1}, \epsilon, b_j - \epsilon, \ldots)$ respectively. If the bidder overstates the value of the part with budget ϵ to v_i instead of v_j we get $(\ldots, v_i, v_i, \ldots, v_{j-1}, v_j, \ldots)$ and $(\ldots, b_i, \epsilon, \ldots, b_{j-1}, b_j - \epsilon, \ldots)$ which is equivalent to $(\ldots, b_i + \epsilon, \ldots, b_j - \epsilon, \ldots)$.

For the reverse direction, consider a generic value overstating: If some value v is overstated to value v', we may consider an extra part with value v' and budget 0. The overstating of value can be simply simulated by overstating all the budget corresponding to value v to the budget of value v'.

□

THEOREM C.5. *If one can benefit from understating budget in some input of a division-blind mechanism, there is also some input in which one can benefit from understating value and vice versa.*

The proof to this theorem is very similar to the proof of theorem C.4. Also note that this theorem can be used as a proof to theorem C.2.

C.1 Revenue Analysis

For a price per unit ν and given truthful vectors of values tv_i and budgets tb_i for bidders i, the demand of a bidder is the sum of true-budgets of her parts that have true-value at least ν divided by ν. It is easy to see that demand is a decreasing function in price. Having these, we can define the market clearing price ν^* and its correspondent revenue R^* with respect to this truthful vectors of bids and budgets. We get exactly the same set of results as we had for original sort-cut mechanism. We omit the proof here because they are identical to their corresponding counterparts in the original sort-cut mechanism.

THEOREM C.6. *No Pareto-optimal individual rational mechanism, in equilibrium, can guarantee revenue more than R^*.*

THEOREM C.7. *Assuming rational bidding, the revenue of new sort-cut mechanism at any equilibrium, is at least $R^* - p_{max}$ where p_{max} is the maximum payment among the bidders.*

The last theorem is stated for p_{max} instead of b_{max}; in fact, the original theorem can also be stated for p_{max}. We use the latter here, because in the setting where budgets are stated as vectors, p_{max} is more meaningful and is also stronger.