Expressive Auctions for Externalities in Online Advertising*

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ABSTRACT

When online ads are shown together, they compete for user attention and conversions, imposing negative externalities on each other. While the competition for user attention in sponsored search can be captured via models of clickthrough rates, the post-click *competition for conversions* cannot: since the value-per-click of an advertiser is proportional to the conversion probability conditional on a click, which depends on the other ads displayed, the private value of an advertiser is no longer one-dimensional, and the GSP mechanism is not adequately expressive. We study the design of expressive GSP-like mechanisms for the simplest form that an advertiser's private value can have in the presence of such externalities— an advertiser's value depends on *exclusivity*, *i.e.*, whether her ad is shown exclusively, or along with other ads.

Our auctions take as input two-dimensional (per-click) bids for exclusive and nonexclusive display, and have two types of outcomes: either a single ad is displayed exclusively, or multiple ads are simultaneously shown. We design two expressive auctions that are both extensions of GSP— the first auction, GSP_{2D}, is designed with the property that the allocation and pricing are identical to GSP when multiple ads are shown; the second auction, NP_{2D}, is designed to be a next price auction. We show that both auctions have high efficiency and revenue in all reasonable equilibria; further, the NP_{2D} auction is guaranteed to always have an equilibrium with revenue at least as much as the current GSP mechanism. However, we find that unlike with one-dimensional valuations, the GSP-like auctions for these richer valuations do not always preserve efficiency and revenue with respect to the VCG mechanism.

1. INTRODUCTION

Online advertisements shown alongside each other compete—first, for the user's attention, and then for a conversion. The effectiveness of an ad, therefore, depends not only on targeting it accurately to a relevant user, but also on the set of other advertisements that are displayed along with it: when an ad for a product is shown along with other high-quality competing ads, the chance that a user will purchase from the first advertiser is diminished. Online ads that are shown together thus impose negative externalities on each other.

This externality effect comes from two factors. First, the presence of other advertisements decreases the amount of *attention* an ad gets from a user: the user may not notice or click on an ad because of other competing ads. Second, even if a user notices or clicks on an ad, he may not *convert* on it, but instead convert on a competing advertisement. Indeed, a user looking to purchase a product would arguably click on multiple ads before deciding which one to convert on. Such ads, which have already successfully competed for attention, now compete with each other for a conversion from the user. This externality effect from *post-click competition for conversions* cannot be captured by models of clickthrough rates, which only model the effect of other ads on user attention; rather, as we argue below, they affect advertisers' private values per-click. In this paper, we will focus on the effect of externalities on conversions, and the design of adequately expressive auctions for such externalities.

In sponsored search auctions, where advertisers bid per click but ultimately derive value from conversions, the presence of externalities affects the private value. An advertiser's value-per-click is the product of her value-per-conversion times the probability of a conversion conditional on a click, *i.e.*, $v_{click} = v_{conv} \cdot Pr(conv|click)$. If $Pr(conv|click)$ depends on whether or not other ads are simultaneously displayed, the advertiser's (private) value-per-click will be different as well — that is, the private per-click-value is no longer one-dimensional. In such a situation, a mechanism such as the existing GSP mechanism[1], which solicits only a one-dimensional bid and always displays a full slate of ads, can be arbitrarily inefficient[2]. To achieve efficiency, the outcomes and bidding languages offered by the auction mechanism must be adequately expressive.

The most general form of an advertiser's valuation in an auction with n bidders and externalities is a function $\mathbf{v} : 2^n \rightarrow \mathrm{R}$, which is exponential in the number of bidders. Allowing very general valuations, even with restrictions on the model to ensure reasonably-sized reports, has two problems. First, reporting high-dimensional valuations imposes a heavy cognitive burden on advertisers (particularly the less sophisticated ones), who may not be able to determine their

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[1](with either separable or cascade model based CTRs)

[2]Suppose each advertiser has value 1 if only her ad is shown, and ϵ if any other ads are shown along with. Since GSP does not have an outcome which displays a single ad, advertisers simply bid according to their value of ϵ, resulting in an efficiency of most $n\epsilon$ compared to the optimal efficiency of 1 (display a single ad).

values for a wide range of outcomes [17, 19], making such a bidding language highly impractical. Second, in general, set-based valuations can lead to computational hardness and inapproximability in the winner determination problem, by reduction from independent set[3].

We will adopt a very simple valuation model for externalities, based on *exclusivity*: an advertiser's value depends on whether or not other ads are shown along with her ad, i.e., whether she is shown exclusively or not. Such a valuation model is very reasonable in the context of online advertising: first, only *one* additional bid is solicited from advertisers, in addition to the bid they already place in the existing GSP auction. Second, given that bidding languages that involve competitor identities can lead to computational hardness, a natural, expressive bidding language is one that specifies a value for each possible number of other ads displayed alongwith —*i.e.*, if k is the number of slots, this valuation would be represented by a (decreasing) k-dimensional vector. Our two-dimensional valuation is a simple approximation, especially from an advertiser's point of view, for such a vector. Also, the two-dimensional language based on exclusivity can arguably better represent[4] valuations where the identity of competitors actually influences value, such as for keywords where some competing ads cause much greater decrease in value than others.

We will be interested in designing expressive *GSP-like mechanisms* for this setting with two-dimensional private values (*i.e.*, with and without exclusivity). By GSP-like auctions, we mean auctions that are extensions, in ways we will make precise, of the generalized second price auction (GSP) currently used to sell sponsored search ads. Such auctions have two major advantages over auctions that deviate significantly from GSP (such as the VCG auction): first, advertisers face a smooth transition between the existing and new system, and do not find themselves faced with an unfamiliar and complex auction; second, they are also are easier to build and integrate with the existing system. Auctions which are extensions of GSP are therefore far more likely to actually be deployed in practice. Other research on designing auctions for sponsored search with more complex valuations has focused on extending the GSP auction as well ([2, 16]), reflecting the practicality of designing such GSP-like auctions.

Results and Organization. We design two expressive auctions that are both extensions of the GSP auction, and analyze their equilibria for revenue and efficiency. Both auctions take as input two-dimensional (per-click) bids (b, b') for exclusive and nonexclusive display, and have two types of outcomes: S, where a single ad is displayed, and M, where a full slate of multiple ads is shown. Since both auctions are not truthful and can have multiple equilibria, we compare the equilibria of each auction to VCG_{2D}, the VCG auction for two-dimensional valuations (we also provide additive bounds or pointwise comparisons of equilibria of the two auctions where possible).

There are two natural ways to extend the GSP auction to this two-dimensional setting. The first is to require that when multiple ads are displayed, the outcome should exactly match GSP. Our first auction, GSP_{2D}, has the property that when the outcome is M, the allocation and pricing is exactly as if the bids b' had been submitted to the original GSP auction (it then remains to design the rule deciding whether the outcome will be S or M, and the pricing for S, to ensure good equilibrium properties). The second is to extend the 'next-price' principle of GSP to the two-dimensional setting: our second auction, NP_{2D} has the property that every winner pays the minimum amount necessary to keep his position (note that GSP_{2D} is not a next price auction: in the two-dimensional setting, maintaining one's position in outcome M involves both maintaining the outcome type (M versus S), and maintaining one's ranking amongst the winners in M).

The comparison between the two auctions is rather subtle. NP_{2D} has better efficiency guarantees than GSP_{2D} when the efficient outcome is M, while GSP_{2D} is, roughly speaking, slightly better when the efficient outcome is S. In all cases, the welfare in all 'good' equilibria where losers bid at least their true value is guaranteed to be at least half the optimal efficiency (we show that such good equilibria always exist for both auctions). In terms of revenue, NP_{2D} has better revenue guarantees when the equilibrium outcome is S, dominating the VCG_{2D} revenue, while GSP_{2D} is guaranteed to have at least half the VCG_{2D} revenue. When the equilibrium outcome is M, all envy-free equilibria of GSP_{2D} dominate the VCG_{2D} revenue, whereas there is no corresponding multiplicative bound for NP_{2D}. However, all envy-free equilibria of NP_{2D} revenue-dominate VCG_M, the revenue from the one-dimensional VCG mechanism; also, all the high-revenue M-equilibria of GSP_{2D} are M-equilibria of NP_{2D} as well. Further, unlike GSP_{2D}, the NP_{2D} auction is guaranteed to always have an equilibrium with revenue at least as much as the current GSP mechanism.

Related Work. There is a rapidly growing literature on externalities in online advertising, starting with the work in [9] which addresses the conversion aspect, but does not directly apply to sponsored search. Externalities in sponsored search are studied theoretically by [3], [15], [11] and [21], and empirically by [12, 14]. However, all of these focus on the effect of externalities on the clickthrough rate, that is, the attention aspect. The work in [7] does address the conversion aspect; however, it assumes a specific form for the conversion rates and more importantly, focuses on analyzing equilibria for this model of conversion rates under the *existing* GSP mechanism. In contrast, we focus on designing mechanisms with a more expressive bidding language and outcome space.

In simultaneous and independent work, [16] proposes the agenda of designing auctions where advertisers bid for configurations; our work provides a thorough design and analysis for one type of configuration, namely exclusivity (see §5 for a more detailed discussion on the relation between the

[3]Each node corresponds to an advertiser who derives value 0 , if any of her neighbors are included in the set of winners, and 1 otherwise. Choosing the optimal set of advertisers to display corresponds exactly to finding the largest independent set in the graph.

[4]It is likely that when advertiser is displayed along with a full slate of other ads, the competitors that cause the greatest decrease in value to her are included as well, causing the maximum decrease in value; in contrast, when only one other ad is displayed along with, the value obtained would depend on whether that ad is a strong competitor (minimum value) or not (maximum value). So an advertiser can simply use his highest and lowest valuations in the exclusivity-based language, but deriving his value vector when some competitors affect value more than others is much harder for the k-dimensional bidding language.

two problems).

[18] studies auctions with share-averse bidders, i.e., bidders suffering from negative externalities when an item is shared amongst multiple competitors, exactly as in online advertising. However, [18] focuses on characterizing the revenue-maximizing single item auction for this setting, whereas we want to design GSP-like auctions for sponsored search. Finally, a primary motivation for our work is the loss in efficiency due to limited expressiveness. The work in [5] provides a general theory for expressiveness in mechanisms, and relates the efficiency of mechanisms to their expressiveness in a domain independent manner.

2. MODEL

There are n advertisers bidding for a page with k slots. Advertiser i's private value is the two-tuple (v_i, v_i'), where v_i is her value-per-click for being displayed *exclusively*, *i.e.*, with no other ads on the page, and v_i' is her value-per-click if other ads are shown as well. We make the natural assumption that each advertiser (weakly) prefers exclusivity, i.e., $v_i \geq v_i'$.

There are two *types of outcomes*: S, where only a single ad is displayed on the page, and M, where multiple ads are displayed. (Note that the maximum possible number of ads are always displayed in M, since bidders do not have, or express, higher values for displaying i ads, $1 < i < k$.) We denote the clickthrough rate (CTR) of the i-th slot in outcome M by θ_i, and assume, without loss of generality, that the CTR of the only slot in outcome S is 1. It is natural to expect that the CTR with outcome S is at least as large as than that of any slot in M, *i.e.*, $1 \geq \theta_1 \geq \ldots \geq \theta_k$. We also define $\theta_i = 0$ for $i > k$ for convenience. (We point out that our results also extend to the case of separable clickthrough rates, where the CTR of an ad in a slot is a product of an ad-dependent clickability and a slot-dependent clickability; we use the simpler model for clarity of exposition.)

Each advertiser's two-dimensional bid is denoted by (b_i, b_i'), where b_i and b_i' represent her bids for outcomes S and M respectively. We refer to the v_i and b_i as S-values and S-bids, and v_i' and b_i' as M-values and M-bids. We order advertisers in decreasing order of their M-bids so that $b_1' \geq b_2' \geq \ldots \geq b_n'$ [5], and use $[i]$ to refer to the advertiser with i-th highest M-value, *i.e.*, $v_{[1]}' \geq v_{[2]}' \geq \ldots \geq v_{[n]}'$. We will use the indices max and $\max^2$ to denote the bidders with the highest and second highest S-values, so $v_{\max} \geq v_i$ for every i, and $v_{\max^2} \geq v_i$ for every $i \neq \max$. We will also abuse notation to use $b_{\max}$ and $b_{\max^2}$ to denote the highest and second highest S-bids, respectively (in all equilibria of interest, these will actually correspond to the same bidders as with the true S-values). Furthermore, we define $v_{\max_{-i}}$ and $b_{\max_{-i}}$ to be the advertisers who have highest S-value and highest S-bid excluding advertiser i. In other words, $v_{\max_{-i}} = v_{\max}$ and $b_{\max_{-i}} = b_{\max}$ if $i \neq \max$, and $v_{\max_{-i}} = v_{\max^2}$ and $b_{\max_{-i}} = b_{\max^2}$ if $i = \max$. Finally, to simplify notation, we sometimes skip the lower bound of summation when it is 1; e.g., the summation $\sum_{i=1, i\neq j}^{k} v_i'$ is abbreviated to $\sum_{i\neq j}^{k} v_i'$.

A *mechanism* for this setting decides on the winning configuration, *i.e.*, whether the outcome is S or M, and the winning advertisers (and their ranking if the outcome is M), and the prices for the winners. The VCG mechanism, of course, applies to this setting, and is a truthful mechanism which always produces an efficient (i.e., welfare maximizing) outcome.

DEFINITION 2.1 (VCG_{2D}). *The VCG mechanism compares* $v_{\max}$ *and* $\sum_{i=1}^{k} \theta_i v_i'$.

- *If* $v_{\max} \geq \sum_{i=1}^{k} \theta_i v_i'$, *VCG allocates the page to only one advertiser, namely* max, *and charges him either the sum of the k highest* $\theta_i v_i'$*s (excluding himself) or the second highest* S *value, whichever is larger, i.e., the winner's payment is*

$$\max(v_{\max^2}, \sum_{i=1}^{\max -1} \theta_i v_i' + \sum_{i=\max}^{k} \theta_i v_{i+1}').$$

- *If* $v_{\max} < \sum_{i=1}^{k} \theta_i v_i'$, *then VCG allocation is* M, *but the expression for the payments is more complicated. When advertiser i is removed, the efficient reallocation can be either* S *or* M. *If it is* S, *the winner is* $v_{\max_{-i}}$, *and hence, the increase in the sum of the values of all advertisers other than i is* $v_{\max_{-i}} - \sum_{j\neq i}^{k} \theta_j v_j'$. *If the efficient reallocation is* M, *all advertisers below i will move one slot up, therefore, the sum of their values increases by* $\sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}'$. *Therefore, i-th advertiser's payment,* $\theta_i p_i$, *is (for* $i \leq k$*):*

$$\max(\sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}', v_{\max_{-i}} - \sum_{j=1}^{k} \theta_j v_j' + \theta_i v_i).$$

We use VCG_{2D} to denote the VCG mechanism applied to our setting where bidder values are two-dimensional, and VCG_M to denote VCG for the one-dimensional setting studied in [20, 8], where the only possible outcome is M and advertisers have one-dimensional valuations. We make this distinction to easily distinguish between the VCG revenues in the various settings.

Restricting Equilibria. We will be interested only in equilibria where losers bid at least their true value, which we will refer to as "good" equilibria. This is particularly relevant when the outcome is S: while there might be Nash equilibria where the losing bidders must bid $b_i' < v_i'$ to ensure that the winner has no incentive to deviate to outcome M, it is unreasonable to expect that the losing bidders will not bid higher in an effort to change the outcome to M, which would give them positive utility. Thus, Nash equilibria where the outcome is S but losers bid less than their true values simply to maintain equilibrium are unlikely to exist in practice.

3. GSP_{2D}

In this section, we design an auction with the following property: given a set of bids (b_i, b_i'), suppose the auction decides to display multiple ads, *i.e.*, the outcome is M. Then, the allocation and pricing for the winning ads is exactly the same as when GSP_M is applied to the bids b_i'. This requirement ensures the practical benefit that when multiple ads are displayed, advertisers see no difference at all between the new auction and the existing system.

Given that the allocation pricing for outcome M is completely specified, it remains to design the rule that decides

[5]In the case of ties between advertisers, we will assume oracle access to the true values for tiebreaking; this assumption is made only for clarity of presentation and is not at all essential to the proofs.

whether the outcome will be S or M, as well as the pricing for S. The GSP_{2D} auction is defined below.

DEFINITION 3.1 (THE GSP_{2D} AUCTION). *The mechanism GSP_{2D} takes as input bids (b_i, b'_i) and compares $b_{\max}$ to $\sum_{i=2}^{k+1} \theta_{i-1} b'_i$ to decide whether the outcome should be* S *or* M.

- *If $b_{\max} \geq \sum_{i=2}^{k+1} \theta_{i-1} b'_i$, the outcome is* S *with winning bidder* max*, whose payment is $\sum_{i=2}^{k+1} \theta_{i-1} b'_i$ per click.*
- *If $b_{\max} \leq \sum_{i=2}^{k+1} \theta_{i-1} b'_i$, assign the page to bidders $1, \ldots, k$ and charge them according to GSP_M pricing, i.e. bidder i (for $i \leq k$) has to pay b'_{i+1} per click.*

Note that the allocation rule compares against $\sum_{i=2}^{k+1} \theta_{i-1} b'_i$, rather than against $\sum_{i=1}^{k} \theta_i b'_i$, *i.e.*, the highest M-bid is completely ignored when deciding the outcome. This is because the natural allocation rule, which would be to compare $b_{\max}$ with $\theta_1 b'_1 + \theta_2 b'_2 + \ldots + \theta_k b'_k$, does not quite work: if the bidder with the highest M-value is different from the bidder max with the highest S-value, that bidder will always set $b'_1 = b_{\max} - \epsilon$ which changes the outcome to M *at no cost to her* (as long as there is some other non-zero bid b'_i), since the pricing when the outcome is M according to GSP remains b'_2. That is, using the natural allocation rule would imply that the only possible equilibria are those with outcome M, defeating the purpose of designing a more expressive auction.

In the remainder of this section, we will investigate the efficiency and revenue of the equilibria of this mechanism. The restriction to using GSP_M when the outcome is M does cause a potential loss in efficiency and revenue with respect to VCG_{2D}, unlike the case with one-dimensional valuations where all envy free equilibria of GSP_M are efficient and dominate VCG_M in terms of revenue. However, as we show below, GSP_{2D} has fairly nice properties nonetheless: both the efficiency and revenue of all reasonable equilibria of GSP_{2D} are guaranteed to be at least within a factor $1/3$ and $1/2$ respectively of the optimal efficiency and revenue. (By reasonable equilibria, we mean equilibria of the mechanism where losers bid at least their true value; we show such equilibria always exist. Further, when the outcome is M, we will restrict ourselves, as in [20] and [8], to *envy free* equilibria, since the efficiency and revenue guarantees for GSP_M relative to VCG_{1D} themselves hold only for envy-free equilibria of GSP_M.)

The easy lemma below, which follows immediately from individual rationality, will be used repeatedly in the following two subsections.

LEMMA 3.1. *In any equilibrium of GSP_{2D} with outcome* M, *$b'_{i+1} \leq v'_i$ for every $i \leq k$.*

3.1 Efficiency

We consider two cases, one where the efficient outcome is S, and the other where the efficient outcome is M, and analyze the efficiency of the equilibria of GSP_{2D}. Note that we prove our efficiency results for *all* reasonable equilibria, rather than only showing that there exists one equilibrium with these properties.

Due to want of space, the proofs of the following results are omitted, and can be found in the full version of the paper [10].

THEOREM 3.1. *If the efficient outcome is* S *($v_{\max} > \sum_{i=1}^{k} \theta_i v'_{[i]}$), there is no equilibrium of GSP_{2D} with outcome* M.

PROPOSITION 3.1. *Suppose the efficient outcome is* M. *Every equilibrium of GSP_{2D} with outcome* S *where losers bid at least their true values has efficiency at least $1/3$ of the optimal. Any envy-free equilibrium with outcome* M *is efficient.*

3.2 Revenue

In this section, we compare the revenues of equilibria in GSP_{2D} with the revenue of VCG_{2D}.

THEOREM 3.2. *Suppose the efficient outcome is* S. *The revenue in any equilibrium of GSP_{2D} where losers bid at least their true values is at least half of the revenue of VCG_{2D}.*

PROOF. First, recall that the only possible equilibrium outcome is S, so that the revenue of GSP_{2D} is $\max(b_{\max^2}, \sum_{i=1}^{k} \theta_i b'_{i+1})$. We give lower-bounds for both terms and then show that the revenue of VCG_{2D} cannot be larger than the sum of the lower bounds; therefore, the revenue of VCG_{2D} cannot be more than twice of the revenue of GSP_{2D}.

First we assume max $\neq [1]$. We have $b_{\max^2} \geq v_{\max^2} \geq v'_{[1]} \geq \theta_1 v'_{[1]}$, since $v'_i \leq v_i$, and losers bid at least their true values. For the other term, we know that all bidders except max are losers in outcome S. Therefore, $b'_i \geq v'_i$ for every $i \neq$ max. So we get

$$\sum_{i=1}^{k} \theta_i b'_{i+1} \geq \sum_{i=2}^{j-1} \theta_{i-1} v'_{[i]} + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_{[i]}$$

where $j = \min(\max, k+2)$. On the other hand, the revenue of VCG_{2D} is

$$\max(v_{\max^2}, \sum_{i=1}^{l-1} \theta_i v'_{[i]} + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_{[i]})$$

where $l = \min(\max, k+1)$. To finish the proof, we need to show that the sum of the lower bounds we have for GSP_{2D} is greater than or equal to revenue of VCG_{2D}:

$$v_{\max^2} + \sum_{i=2}^{j-1} \theta_{i-1} v'_{[i]} + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_{[i]} \geq$$
$$\max(v_{\max^2}, \sum_{i=1}^{l-1} \theta_i v'_{[i]} + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_{[i]}).$$

If the first term in the VCG_{2D} revenue is the dominant term, the inequality obviously holds. Otherwise, we need to show that

$$v_{\max^2} + \sum_{i=2}^{j-1} \theta_{i-1} v'_{[i]} + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_{[i]} \geq \sum_{i=1}^{l-1} \theta_i v'_{[i]} + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_{[i]},$$

i.e., it is enough to show that

$$\theta_1 v'_{[1]} + \sum_{i=2}^{j-1} \theta_{i-1} v'_{[i]} + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_{[i]} \geq$$
$$\theta_1 v'_{[1]} + \sum_{i=2}^{l-1} \theta_i v'_{[i]} + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_{[i]},$$

but this inequality clearly holds using term-by-term comparison.

It remains to prove the the theorem for the case where $\max = [1]$. In this case, $b_{\max^2} \geq v_{\max^2} \geq v'_{[2]}$, and the revenue of GSP_{2D} is

$$\max(b_{\max^2}, \sum_{i=1}^{k} \theta_i b'_{i+1}) \geq \max(b_{\max^2}, \sum_{i=1}^{k} \theta_i v'_{[i+2]})$$

while the revenue of VCG_{2D} is $\max(v_{\max^2}, \sum_{i=1}^{k} \theta_i v'_{[i+1]})$. As before, if the dominant term in revenue of VCG_{2D} is $v_{\max^2}$ we are done. Otherwise, the sum of the two terms of the GSP_{2D} revenue is at least $\theta_1 v'_{[2]} + \sum_{i=1}^{k} \theta_i v'_{[i+2]}$ which dominates the second term in the VCG_{2D} revenue term by term. $\square$

A simple modification to Example 3.1 at the end this section shows that this factor of 2 is tight (set $v_1 = 3$ so that the efficient outcome is S).

The following additive bound on revenue follows immediately from the previous proof:

$$R_{GSP_{2D}} \geq R_{VCG_{2D}} - \theta_1 v'_{[1]} + \theta_k v'_{[k+2]}.$$

THEOREM 3.3. *Suppose the efficient outcome is* M. *Any envy-free equilibrium of* GSP_{2D} *with outcome* M *has revenue greater than or equal to that of* VCG_{2D}.

PROOF. First, note that since the equilibrium is envy-free, the ordering of M-bids is the same as ordering of M-values ([20]), *i.e.*, $v'_i = v'_{[i]}$ for any $i \leq k+1$. We show that the payment of advertiser i (for $i \leq k$) in GSP_{2D} is at least as much as his payment in VCG_{2D}.

Recall from 2.1 that the payment for advertiser i in VCG_{2D} is

$$p_i = \max(\sum_{j=i}^{k} (\theta_j - \theta_{j+1}) v'_{j+1}, v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v'_j).$$

First we prove that GSP_{2D} payment of bidder i, $\theta_i b'_{i+1}$, is at least $v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v'_j$. We will prove this by contradiction: if not, we show that there is a bidder with a profitable deviation to S. Let l be the bidder with the highest S-value excluding i, i.e. $v_l = v_{\max_{-i}}$. By the contradiction hypothesis, $\theta_i b'_{i+1} < v_l - \sum_{j \neq i}^{k} \theta_j v'_j$. If l is not a winner, he has a profitable deviation by bidding (v_l, b'_l) which changes the outcome to S because $v_l > \sum_{j \neq i}^{k} \theta_j v'_j + \theta_i b'_{i+1} \geq \sum_{j=1}^{k} \theta_j b'_{j+1}$. (Of course, bidding $(v_l, 0)$ is a "more profitable" deviation, but is unnecessary for the argument.)

So suppose that l is a winner. Adding and subtracting $\theta_l b'_{l+1}$ and rearranging we get

$$\theta_l (v'_l - b'_{l+1}) < v_l - (\theta_i b'_{i+1} + \theta_l b'_{l+1} + \sum_{j \neq i,l}^{k} \theta_j v'_j).$$

Note that the term in parentheses on the right hand side is an upper-bound on the price that l has to pay for S if he deviates and bids (v_l, b'_l): the price for S is at most $\sum_{j=1}^{k} \theta_j b'_{j+1}$ (since the outcome with the original vector of bids was M, $b_{\max} \leq \sum_{j=1}^{k} \theta_j b'_{j+1}$, so the price for S is always dominated by this term). Since $b'_{j+1} \leq v'_j$ (the original vector of bids was in equilibrium), the price for S is upper-bounded by $(\theta_i b'_{i+1} + \theta_l b'_{l+1} + \sum_{j \neq i,l}^{k} \theta_j v'_j)$ as claimed, showing that l can deviate profitably. (Note that as before, this bid does change the outcome to S.)

The fact that $\theta_i b'_{i+1} \geq \sum_{j=i}^{k} (\theta_j - \theta_{j+1}) v'_{j+1}$ follows from the lower bound on bids in envy-free equilibria in [20], which also holds for envy-free equilibria in outcome M of GSP_{2D}. $\square$

THEOREM 3.4. *Suppose the efficient outcome is* M. *The revenue in any equilibrium of* GSP_{2D} *with outcome* S *where losers bid at least their true values is at least half of the revenue of* VCG_{2D}.

PROOF. The proof, unfortunately, proceeds by considering cases. The revenue of GSP_{2D} is $\max(b_{\max^2}, \sum_{i=1}^{k} \theta_i b'_{i+1})$. For ease of notation let $p_i^1 = \sum_{j=i}^{k} (\theta_j - \theta_{j+1}) v'_{[j+1]}$, and $p_i^2 = v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v'_{[j]}$. The revenue of VCG_{2D} is $\sum_{i=1}^{k} p_i$, where $p_i = \max(p_i^1, p_i^2)$. First note that $p_i^1 \leq \theta_i v'_{[i+1]}$. Also, from individual rationality we have $p_i \leq \theta_i v'_{[i]}$.

We consider the following three cases, and will prove for each case that $b_{\max^2} + \sum_{i=1}^{k} \theta_i b'_{i+1} \geq \sum_{i=1}^{k} p_i$. Therefore, the revenue of GSP_{2D} is at least half the revenue of VCG_{2D}.

1. If $\max \notin \{[1], \ldots, [k]\}$: Each bid b'_i, $i \leq k+1$, is at least v'_i in this case, so the revenue of GSP_{2D} is at least $\max(v_{\max^2}, \sum_{i=1}^{k} \theta_i v'_{[i+1]})$. The revenue of VCG_{2D} is at most $p_1 + \sum_{i=2}^{k} \theta_i v'_{[i]}$. Since $v_{\max^2} \geq v_{[1]} \geq v'_{[1]} \geq \theta_1 v'_{[1]}$, we have $v_{\max^2} \geq p_1$, and hence

$$v_{\max^2} + \sum_{i=1}^{k} \theta_i v'_{[i+1]} \geq p_1 + \sum_{i=2}^{k} \theta_i v'_{[i]},$$

which implies that the revenue of GSP_{2D} is at least half the revenue of VCG_{2D}.

2. If $\max \in \{[2], \ldots, [k]\}$: The revenue of GSP_{2D} in this case is at least $\max(v_{\max^2}, \sum_{j=1}^{\max-2} \theta_j v'_{[j+1]} + \sum_{j=\max-1}^{k} \theta_j v'_{[j+2]})$ because all losers bid at least their true values. We first consider the case where $p_i^1 \geq p_i^2$ for every i. The revenue of VCG_{2D} cannot be more than $\sum_{j=1}^{k} p_j^1 \leq \sum_{j=1}^{k} \theta_j v'_{[j+1]}$. Since $v_{\max^2} \geq v'_{[1]} \geq v'_{[\max]} \geq \theta_{\max} v'_{[\max]}$,

$$v_{\max^2} + \sum_{j=1}^{\max-2} \theta_j v'_{[j+1]} + \sum_{j=\max-1}^{k} \theta_j v'_{[j+2]} \geq \sum_{j=1}^{k} \theta_j v'_{[j+1]},$$

which shows the revenue of VCG_{2D} cannot be more than twice the revenue of GSP_{2D} in this case.

For the other case, let l be some index for which $p_l^1 < p_l^2$. We consider two cases depending on whether $p_{\max}^1 > p_{\max}^2$ or $p_{\max}^2 \geq p_{\max}^1$. For both cases, we upper-bound the VCG_{2D} payment of bidder i (for $i \neq \max$ and $i \neq l$) by $\theta_i v'_{[i]}$. First, if $p_{\max}^2 \geq p_{\max}^1$, the revenue of VCG_{2D} is at most

$$\begin{aligned} p_l^2 + p_{\max}^2 + \sum_{j \neq \max, j \neq l}^{k} \theta_j v'_{[j]} &= v_{\max} - \sum_{j \neq l}^{k} \theta_j v'_{[j]} \\ &+ v_{\max^2} - \sum_{j \neq \max}^{k} \theta_j v'_{[j]} + \sum_{j \neq \max, j \neq l}^{k} \theta_j v'_{[j]} \\ &= \left(v_{\max} - \sum_{j=1}^{k} \theta_j v'_{[j]} \right) + v_{\max^2}. \end{aligned}$$

Since the efficient outcome is M, the term in parentheses is non-positive; therefore, the revenue of VCG_{2D} is bounded above by $v_{\max^2}$, which is clearly less than or equal to the revenue of GSP_{2D}.

Now, if $p^1_{\max} \geq p^2_{\max}$, the revenue of VCG_{2D} is

$$p^2_l + p^1_{\max} + \sum_{j \neq \max, j \neq l}^{k} \theta_j v'_{[j]} = v_{\max} - \sum_{j \neq l}^{k} \theta_j v'_{[j]}$$

$$+ \sum_{j=\max}^{k} (\theta_j - \theta_{j+1}) v'_{[j+1]} + \sum_{j \neq \max, j \neq l}^{k} \theta_j v'_{[j]}.$$

Since $v_{\max} - \sum_{j \neq l}^{k} \theta_j v'_{[j]} \leq \theta_l v'_{[l]}$ (the efficient outcome is M), the revenue of VCG_{2D} is at most

$$\sum_{j=1}^{\max -1} \theta_j v'_{[j]} + \sum_{j=\max}^{k} \theta_j v'_{[j+1]}$$

$$= \theta_1 v'_{[1]} + \sum_{j=2}^{\max -1} \theta_j v'_{[j]} + \sum_{j=\max}^{k} \theta_j v'_{[j+1]}.$$

Since $v_{\max^2} \geq \theta_1 v'_{[1]}$, by term-by-term comparison we get

$$v_{\max^2} + \sum_{j=1}^{\max -2} \theta_j v'_{[j+1]} + \sum_{j=\max -1}^{k} \theta_j v'_{[j+2]} \geq$$

$$\theta_1 v'_{[1]} + \sum_{j=2}^{\max -1} \theta_j v'_{[j]} + \sum_{j=\max}^{k} \theta_j v'_{[j+1]},$$

which implies the revenue of GSP_{2D} is at least half of the revenue of VCG_{2D}.

3. If $\max = [1]$: The revenue of GSP_{2D} in this case is at least $\max(v_{\max^2}, \sum_{j=1}^{k} \theta_j v'_{[j+2]})$ because all losers bid at least their true values. As before, we first consider the case where $p^1_i \geq p^2_i$ for every i; the revenue of VCG_{2D} cannot be more than $\sum_{j=1}^{k} p^1_j \leq \sum_{j=1}^{k} \theta_j v'_{[j+1]}$. Since $v_{\max^2} \geq v'_{[2]} \geq \theta_1 v'_2$,

$$v_{\max^2} + \sum_{j=1}^{k} \theta_j v'_{[j+2]} \geq \theta_1 v'_{[2]} + \sum_{j=2}^{k} \theta_j v'_{[j+1]}$$

which shows that the revenue of VCG_{2D} cannot be more than twice of revenue of GSP_{2D} in this case.

The analysis of the other case is almost identical to when $\max \in \{[2], \ldots, [k]\}$, so we omit repeating it here.

□

Example 3.1 shows that this factor 2 is tight as well.

How does GSP_{2D} compare to GSP_M in terms of revenue? Suppose bidders have two-dimensional valuations (v_i, v'_i), but are only offered the GSP_M mechanism with its one-dimensional bidding language. Since the outcome of GSP_M is never S, bidders will bid according to valuations v'_i in GSP_M. The example below shows that the revenue of GSP_{2D} (in every equilibrium) can actually be smaller than the revenue in GSP_M, i.e., if the search engine had persisted with the old mechanism. However, the mechanism we design in the next section does not suffer from this potential loss in revenue with respect to GSP_M.

EXAMPLE 3.1. *Suppose there are two slots with $\theta_1 = \theta_2 = 1 - \epsilon$, and three bidders with values $v_1 = 1 + \epsilon, v_2 = v_3 = 1$, and $v'_i = 1$ for $i \leq 3$. The revenue of GSP_M for this example is $2 - 2\epsilon$ for all equilibria, and the utility is 0 for all bidders. However, if advertiser 1 bids $(\infty, 0)$, and advertisers 2 and 3 bid truthfully, this is an equilibrium with revenue $1 - \epsilon$ and payment $1 - \epsilon$ with utility $2\epsilon > 0$ for the winner. In fact, this is the highest possible revenue in any equilibrium outcome of GSP_{2D}.*

Finally, we conclude with showing that good equilibria (where losers bid their true values) always exist, so that the theorems we proved so far are not vacuous.

THEOREM 3.5. *For GSP_{2D}, a good equilibrium always exists.*

PROOF. Suppose (v_i, v'_i) are the S-value and M-value of the i-th bidder, and suppose that v'_i's are sorted in descending order. We construct a good equilibrium of GSP_{2D}. Let $\hat{v}'_i$ be the i-th highest M-value excluding $v'_{\max}$, where max is the bidder who has the highest S-value. (In the efficient ordering of advertisers excluding max in outcome M, the advertisers occupying the i-th slot has M-value $\hat{v}'_i$.) Note that $\hat{v}'_i = v'_i$ if $i < \max$, and $\hat{v}'_i = v'_{i+1}$ otherwise. Let $S_0 = \infty$ and $S_l = \sum_{i=1}^{l-2} \theta_i \hat{v}'_{i+1} + \theta_{l-1} \hat{v}'_{l-1} + \sum_{i=l}^{k} \theta_i \hat{v}'_i$ for $1 \leq l \leq k+1$ (define $\theta_0 = 0$ and $\hat{v}'_0 = 0$). Intuitively, S_l (for $l \geq 1$) is an upper-bound on $\sum_{i=1}^{k} \theta_i b'_{i+1}$ in which everyone except max is bidding truthfully, and max is bidding the maximum possible bid, $\hat{v}'_{l-1}$, to get the l-th slot. Clearly, $S_1 \geq S_2 \geq \ldots \geq S_k$. Let $0 \leq j \leq k$ be the largest index such that $S_j > v_{\max^2}$. Let $u_M = \max_{1 \leq i \leq j} \theta_i (v'_{\max} - \hat{v}'_i)$, and let $t = arg \max_{1 \leq i \leq j} \theta_i (v'_{\max} - \hat{v}'_i)$; in other words, u_M is the maximum utility that bidder max can get if the outcome is M and all other bidders are bidding truthfully. Also, let $u_S = v_{\max} - \max(\sum_{i=1}^{k} \theta_i \hat{v}'_{i+1}, v_{\max^2})$ which means u_S is the maximum utility that bidder max can get if the outcome is S and all other bidders are bidding truthfully. If $j = 0$, $v_{\max^2} > S_1$ so the outcome will always be S irrespective of max's bid; so there is no deviation for max that changes the outcome to M. Therefore, every bidder except max bidding truthfully and max bidding $(\infty, 0)$ is a good equilibrium of GSP_{2D} with outcome S. So, for the rest of the proof, we assume $j \geq 1$, and hence t exists.

If $u_M \geq u_S$, everyone except max bidding truthfully and max bidding $(S_t - 2\epsilon, \hat{v}'_{t-1} - \epsilon)$ is an equilibrium of GSP_{2D} with outcome M. The outcome is M by definition of S_t, and also because $S_t \geq v_{\max^2}$. Consider a bidder $a \neq \max$. If bidder a decreases her M-bid, the outcome switches to S leading to utility 0 for her. Furthermore, since a is bidding her true M-value, any overstating value which results in change of allocation leads to negative utility for a; therefore, a has no profitable deviation. We know that bidder max is already getting the slot which has maximum utility for her among slots $1, \ldots, j$, and there is no deviation for her leading to outcome M with slot lower than j. Therefore, any deviation which leads to outcome M is not profitable for max. Also, since $u_S < u_M$, we know that max prefers outcome M to S, and hence, any deviation which switches the outcome to S can not be profitable. Next, consider the case where $u_M < u_S$. In this case, all bidders except max bidding truthfully and max bidding $(\infty, 0)$ is a good equilibrium of GSP_{2D} with outcome S. No loser can change the outcome to M profitably, and max prefers the current outcome to any M-outcome. □

4. NP_{2D}: A NEXT PRICE AUCTION

The current GSP auction, GSP_M, is a next price auction—every winner pays the "next price", *i.e.*, the minimum bid necessary in order to maintain her position, which in GSP_M is the bid of the next highest bidder. In our two-dimensional setting, where there are two types of outcomes in addition to multiple slots, maintaining one's position consists of two things for a winner in outcome M: first, the outcome must remain M and not switch to S; second, the bid must enable the bidder to maintain her position amongst the k slots. In a next price auction for our more expressive setting, therefore, the payment of a winner in slot i of outcome M is the larger of two terms— the first being the minimum value at which the outcome still remains M, and the second being the bid of the next bidder, b'_{i+1}, as in GSP_M. The auction is formally defined below.

DEFINITION 4.1 (THE NP_{2D} AUCTION). *Bidders submit bids* (b_i, b'_i). *Assume* max $= j$, *i.e., the bidder corresponding to* $b_{\max}$ *has the* j*th largest* M*-bid, and let* $\Gamma = \sum_{i=1}^{k} \theta_i b'_i$.

- *If* $b_{\max} \geq \Gamma$*, the outcome is* S *with payment*

$$\max(b_{\max^2}, \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}).$$

- *If* $b_{\max} \leq \Gamma$*, the outcome is* M *and the bidder winning slot* $i \neq$ max *pays*

$$\theta_i p_i = \max(\theta_i b'_{i+1}, b_{\max} - \Gamma + \theta_i b'_i)$$

while the bidder max *winning slot* j *pays*

$$\theta_j p_j = \max(\theta_j b'_{j+1}, b_{\max^2} - \Gamma + \theta_j b'_j)$$

Note that in computing the price for outcome S, the second term is smaller than Γ.

In the next two subsections, we will analyze the efficiency and revenue respectively in the equilibria of NP_{2D}. As before, we will prove guarantees for the revenue and efficiency of good equilibria, where losers bid at least their true value (such equilibria always exist, as we show in Proposition 4.1). Some proofs have been removed for want of space, and can be found in the full version of the paper [10].

4.1 Efficiency

As before, we consider two cases corresponding to the efficient outcome being S or M. We first start with the following lemma, which allows us to prove the efficiency result for S.

LEMMA 4.1. *Assume that bidder* max *is bidding truthfully. If the outcome of* NP_{2D} *for a given vector of bids is* S*, then the winner* max *cannot benefit from any deviation that changes the outcome to* M.

PROOF. Assume max $= j$, *i.e.*, the bidder max has the jth largest M-bid for the given M-bids b'_i from the remaining bidders. By assumption that max bids truthfully, $b'_j = v'_j$ and $b_{\max} = v_{\max}$. We need to show that bidder max $= j$ prefers outcome S to any position in outcome M. Consider an M-bid $\overline{b}'$ of bidder j with $b'_l \leq \overline{b}' < b'_{l-1}$, *i.e.*, targeting slot l, and assume the deviation changes the outcome to M.

First notice that if $\overline{b}' > b'_{j-1}$, outcome M gives bidder j negative utility because her payment would be at least $b'_{j-1} > b'_j$ in this case. So without loss of generality we may assume $\overline{b}' \leq b'_{j-1}$, *i.e.*, $l \geq j$. We have to show

$$v_{\max} - \max(b_{\max^2}, \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}) \geq$$
$$\theta_l v'_j - \max(\theta_l b'_{l+1}, b_{\max^2} - \Gamma + \theta_l \overline{b}').$$

We know $b'_j \geq b'_i$ for $(i \geq j)$, therefore,

$$(\theta_j - \theta_l) b'_j = \sum_{i=j}^{l-1} (\theta_i - \theta_{i+1}) b'_j \geq \sum_{i=j}^{l-1} (\theta_i - \theta_{i+1}) b'_{i+1}$$

and since $b'_j = v'_j$ we can write

$$\theta_j b'_j - \sum_{i=j}^{l-1} (\theta_i - \theta_{i+1}) b'_{i+1} - \theta_l v'_j \geq 0.$$

By adding

$$\sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{l-1} \theta_i b'_{i+1} + \theta_l v'_j + \sum_{i=l+1}^{k} \theta_i b'_i$$

to both sides of the inequality we get

$$\sum_{i=1}^{k} \theta_i b'_i \geq \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{l-1} \theta_i b'_{i+1} + \theta_l b'_j + \sum_{i=l+1}^{k} \theta_i b'_i.$$

Since the outcome is S, we have $v_{\max} \geq \sum_{i=1}^{k} \theta_i b'_i$, and therefore, using the inequality above,

$$v_{\max} \geq \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{l-1} \theta_i b'_{i+1} + \theta_l v'_j + \sum_{i=l+1}^{k} \theta_i b'_i. \quad (1)$$

We consider two cases, based on whether the dominant term for the price of max in outcome S is $b_{\max^2}$ or not.

1. Assume the dominant term is not $b_{\max^2}$, *i.e.*, $\max(b_{\max^2}, \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}) = \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}$. Then, by inequality (1), and since $b'_{i+1} \leq b'_i$ for any i, we have

$$v_{\max} \geq \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{l-1} \theta_i b'_{i+1} + \theta_l v'_j + \sum_{i=l+1}^{k} \theta_i b'_{i+1}$$

and, by adding and subtracting $\theta_l b'_{l+1}$ to the right hand side of the inequality we get

$$v_{\max} \geq \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{l} \theta_i b'_{i+1} + \theta_l (v'_j - b'_{l+1}) + \sum_{i=l+1}^{k} \theta_i b'_{i+1}.$$

The final inequality can be written as

$$v_{\max} - (\sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}) \geq \theta_l (v'_j - b'_{l+1})$$

which implies

$$v_{\max} - \max(b_{\max^2}, \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}) \geq$$
$$\theta_l v'_j - \max(\theta_l b'_{l+1}, b_{\max^2} - \Gamma + \theta_l \overline{b}').$$

2. Assume $\max(b_{\max^2}, \sum_{i=1}^{j-1}\theta_i b'_i + \sum_{i=j}^{k}\theta_i b'_{i+1}) = b_{\max^2}$. Since after deviation,

$$\Gamma = \sum_{i=1}^{j-1}\theta_i b'_i + \sum_{i=j}^{l-1}\theta_i b'_{i+1} + \theta_l \bar{b}' + \sum_{i=l+1}^{k}\theta_i b'_i,$$

we can rewrite inequality (1) as $v_{\max} \geq \theta_l v'_j + \Gamma - \theta_l \bar{b}'$. Now, by just subtracting $b_{\max^2}$ from both sides we get

$$v_{\max} - b_{\max^2} \geq \theta_l v'_j - b_{\max^2} + \Gamma - \theta_l \bar{b}'$$

which implies

$$v_{\max} - \max(b_{\max^2}, \sum_{i=1}^{j-1}\theta_i b'_i + \sum_{i=j}^{k}\theta_i b'_{i+1}) \geq$$

$$\theta_l v'_j - \max(\theta_l b'_{l+1}, b_{\max^2} - \Gamma + \theta_l \bar{b}').$$

□

This allows for an easy proof of the following result:

THEOREM 4.1. *Suppose the underlying valuations are such that the efficient outcome is* S, *i.e.*, $v_{\max} > \sum_{i=1}^{k}\theta_i v'_{[i]}$. *There exists an equilibrium with outcome* S *where losers bid at least their true values. Further, there is no inefficient equilibrium where all bidders play undominated strategies.*

Unlike in GSP_{2D}, inefficient equilibria with outcome M (with arbitrarily large inefficiency) can occur in NP_{2D} when the efficient outcome is S. However, all such equilibria are 'bullying' equilibria (such equilibria occur in GSP_M as well) where some bidder bids above her true value, which (Lemma 7.1 in the full version of the paper) is a weakly dominated strategy in NP_{2D}.

Next, suppose the efficient outcome is M. Here, similar to GSP_{2D}, inefficiency can occur in NP_{2D} as well; however, the extent of inefficiency is less than that in GSP_{2D}, as the following multiplicative and additive bounds show.

THEOREM 4.2. *Suppose the efficient outcome is* M. *Then the efficiency in any good equilibrium of* NP_{2D} *with outcome* S *is at least* $1/2$ *of the optimal efficiency.*

COROLLARY 4.1. *Suppose the efficient outcome is* M. *Then the welfare in any good equilibrium of* NP_{2D} *with outcome* S *is at least* $OPT - \sum_{i=j}^{k}\theta_i(v'_{[i]} - v'_{[i+1]})$, *where* OPT *is the optimal welfare and* j *is the rank of* M*-value of bidder* max, *i.e.* $\max = [j]$.

Note that if the marketplace is competitive, *i.e.*, the v'_i's are not very different, the additive bound shows that the loss in welfare will be small even when the inefficient outcome occurs.

Finally, suppose the efficient outcome is M and the equilibrium outcome is M as well. A result similar to that [8, 20] stating that all envy free equilibria are efficient holds for NP_{2D} as well. However, before we state the result, we need to extend the notion of envy free equilibria to NP_{2D}; we will then prove, via Lemma 4.2, that the set of envy-free equilibria are all efficient.

The notion of envy free equilibria in [8] can be thought of as *restricting* the set of bid vectors that are Nash equilibria to those that also generate *envy free prices* [13], *i.e.*, a price for each slot such that no bidder envies the allocation of another bidder at this price. We use exactly this idea to define envy free equilibria for outcome M in the NP_{2D} auction: A vector of bids leading to outcome M in NP_{2D} is an envy free equilibrium if for any i and j $(1 \leq i, j \leq n)$

$$\theta_i(v'_i - p_i) \geq \theta_j(v'_i - p_j)$$

where p_i and p_j are the prices bidders i and j are currently paying for slots i and j respectively. (Recall that $\theta_i = 0$ for $i > k$.)

LEMMA 4.2. *If* $v'_a > v'_b$ *for bidders* a *and* b, *and* $\theta_p > \theta_q$ *for slots* p *and* q, *then any allocation* A *that assigns bidder* a *to slot* q *and bidder* b *to slot* p *is not envy-free.*

PROOF. Assume for sake of contradiction that A is envy-free and suppose that the prices for slots p and q are p_p and p_q respectively. For A being envy-free we must have $\theta_p(v'_a - p_p) \leq \theta_q(v'_a - p_q)$ and $\theta_p(v'_b - p_p) \geq \theta_q(v'_b - p_q)$. Subtracting the second inequality from the first one we get $\theta_p(v'_a - v'_b) \leq \theta_q(v'_a - v'_b)$, but this last inequality contradicts $v'_a > v'_b$ and $\theta_q < \theta_p$. □

The theorem below follows immediately from this lemma, since it implies that in any envy-free allocation (not necessarily even an equilibrium) of NP_{2D} with outcome M, the bidders must be allocated to the slots in decreasing order of their M-values.

THEOREM 4.3. *Suppose the efficient outcome is* M. *Then any envy free equilibrium of* NP_{2D} *with outcome* M *is efficient.*

4.2 Revenue

When the equilibrium outcome in NP_{2D} is S, the revenue is high, in the following sense:

THEOREM 4.4. *Any good equilibrium of* NP_{2D} *with outcome* S *has at least the same revenue as* VCG_{2D}.

Note that here NP_{2D} does better than GSP_{2D}: the revenue of GSP_{2D} can be as small as half the VCG_{2D} revenue when the equilibrium outcome is S and the efficient outcome is either S or M (Theorems 3.2 and 3.4). However, while NP_{2D} leads to better revenue guarantees when the outcome is S, the same is not true for M: unlike GSP_{2D}, where every envy free equilibrium with outcome M revenue-dominates VCG_{2D}, the revenue in an envy free equilibrium of NP_{2D} can be arbitrarily smaller than that of VCG_{2D}, as the following example shows.

EXAMPLE 4.1. *Suppose that there are two slots with* $\theta_1 = 1$ *and* $\theta_2 = 1 - \epsilon/3$. *There are two bidders with values* $(v_1, v'_1) = (3, 3)$ *and* $(v_2, v'_2) = (4, 2)$. *The efficient outcome is* M. *If the bidders bid* $(b_1, b'_1) = (3, 3)$ *and* $(b_2, b'_2) = (2\epsilon, \epsilon)$, *the outcome is* M*; it is an envy-free equilibrium; and the revenue is* ϵ. *However, revenue of* VCG_{2D} *on this example is* 2.

That is, we cannot obtain a result similar to the previous revenue results bounding the revenue loss with respect to VCG_{2D} by a multiplicative constant.

However, as the next three theorems will show, the situation is not quite as bleak as the previous example might suggest: first, Theorem 4.5 shows that the revenue of NP_{2D} in any envy-free equilibrium with outcome M is at least the VCG_M revenue. Second, and more importantly, Proposition 4.1 shows that there always exists an equilibrium of

NP_{2D} with this revenue— note that this is not the case with GSP_{2D}, where there exist values such that every equilibrium of GSP_{2D} has revenue strictly less than the revenue of VCG_M (Example 3.1). The revenue comparison with VCG_M is important for the following reason— the VCG_M revenue can be thought of as a proxy for the GSP_M *revenue*, since there always exists an equilibrium of GSP_M with this revenue [8], and further, this is a "likely" equilibrium in the sense that if bidders update their bids according to reasonable greedy bidding strategies, the bids converge to this equilibrium of GSP_M[6]. Therefore, unlike GSP_{2D}, there always exists an equilibrium of NP_{2D} with revenue at least as much as in GSP_M, *i.e.*, the transition to the richer outcome space does not lead to revenue loss.

Finally, Theorem 4.6 shows that the NP_{2D} auction also retains all the high revenue equilibria with outcome M of GSP_{2D}: the reason for the nonexistence of a multiplicative bound with respect to VCG_{2D} is simply that NP_{2D} has a larger set of equilibria, some of which have poor revenue; however, no high revenue M-equilibria of GSP_{2D} are lost in using the NP_{2D} auction.

THEOREM 4.5. *Every envy-free outcome (not necessarily an equilibrium) of NP_{2D} with outcome* M *has revenue at least as much as VCG_M.*

We point out that the proof of this result is independent of how payments p_i are calculated. In fact, any allocation and pricing (even not restricted to our two-dimensional setting) which is envy-free and efficient satisfies the conditions needed for the above proof, and hence has revenue at least as much as VCG_M.

PROPOSITION 4.1. *There always exists a good equilibrium of NP_{2D} with revenue greater than or equal to that of VCG_M.*

The revenue of the equilibria constructed in Proposition 4.1 are at least $\sum_{i=1}^{k} \theta_i v'_{[i+1]}$. Therefore, assuming bidders do not play weakly dominated strategies in GSP_M(specifically, bidders do not overstate their values), no equilibrium of GSP_M can have revenue higher than the equilibrium of NP_{2D} constructed in Proposition 4.1.

Finally, we show that NP_{2D} retains all the high revenue M-equilibria of GSP_{2D}.

THEOREM 4.6. *Every equilibrium of GSP_{2D} with outcome* M *is an equilibrium with outcome* M *of NP_{2D} with equal revenue.*

That is, while there is no multiplicative bound on the revenue of an envy-free equilibrium of NP_{2D} with outcome M, all the high revenue M-equilibria of GSP_{2D}, which dominate the VCG_{2D} revenue, are also equilibria of NP_{2D}.

4.2.1 Revenue Non-monotonicity

We point out an interesting property of the NP_{2D} auction: when bids go up, the revenue can actually decrease. The following example illustrates this non-monotonicity in revenue as a function of the bids:

EXAMPLE 4.2. *Suppose there are two slots with $\theta_1 = \theta_2 = 1$ and three bidders with bids $(10, 0)$, $(9, 9)$ and $(2, 2)$. The outcome is* M*, and the prices for bidders* 2 *and* 3 *are* 8 *and* 1 *respectively, so the revenue is* 9*. However, if the third bidder increases her bid to $(3, 3)$, the outcome remains* M*, but the payments change to* 7 *and* 1 *for bidder* 2 *and* 3 *respectively. Therefore, the revenue decreases to* 8*.*

When the bids are such that $b_{\max} = \sum \theta_i b'_i$, the total revenue is exactly $\sum \theta_i b'_i$ irrespective of which outcome is chosen. If the tie is broken in favor of outcome M, every bidder must pay exactly his bid, since for any bid below this the outcome will switch to S. Now suppose all bidders bid $b'_i + \epsilon$ (and don't change their S- bids). The outcome remains M, but every bidders payment decreases, since the minimum amount needed to maintain outcome M *given the other bids* has decreased. So the revenue decreases even though the bids increase.

Note that this revenue non-monotonicity occurs in the VCG auction as well, for the same reason. However, GSP_{2D} does not have this property in the sense that when the bids increase the total revenue cannot decrease. Revenue monotonicity is often considered a desirable property in practice, and could influence the choice between which of the two auctions, GSP_{2D} or NP_{2D} is actually used in practice.

5. DISCUSSION

In this paper, we designed two expressive GSP-like auctions for exclusivity-based valuations, and showed that they have good revenue and efficiency properties in equilibria. While the NP_{2D} auction has, roughly speaking, better worst-case efficiency properties, and does better than GSP_{2D} in several cases in terms of revenue, it does have envy-free equilibria with poor revenues, and shares VCG's revenue non-monotonicity problem. Choosing between the two auctions will require an empirical assessment of the marketplace parameters as well as an understanding of bidder valuations and behavior, to predict which equilibria are actually likely to arise in practice. (Note, as an aside, that there is no way to infer S-values from the current GSP_M auction, since multiple ads are always shown.)

There are many directions for further work. The first obvious direction is to combine the more expressive bidding language with more complex cascade-like models for CTRs, and analyze a model that incorporates both attention and conversion based externalities. Second, the bidding language we use can be thought of as a one way to succinctly represent a general decreasing k-dimensional vector valuation, where an advertiser specifies the first entry in the vector (v_i), and uses v'_i as a proxy for all the remaining $k-1$ entries. An alternative language (also discussed in [16]) which also solicits only two bids from an advertiser, is one where an advertiser specifies that he has value v_i provided no more than n_i advertisers are shown in all (and zero if any more are shown). Which of these is a better representation of actual advertiser valuations, and which is it possible to design better mechanisms for? In general, the question of designing succinct mechanisms that achieve high efficiency in the presence of underlying high-dimensional valuations is an interesting open question.

While the auctions we design have pleasant properties with respect to revenue and efficiency in their equilibria, the comparison of these GSP-like auctions to VCG does not remain quite as starkly positive as in the original one-dimensional setting. Specifically, unlike [8, 20], where all envy-free equilibria of GSP_M are efficient and have at least the revenue of VCG_M, both the NP_{2D} and GSP_{2D} auctions suffer from losses in efficiency and revenue with respect to VCG_{2D}: neither auction need always have an efficient equilibrium, or one that guarantees at least as much revenue as VCG_{2D}. In fact, other research [1, 4] suggests as well that the GSP_M auction does not always have desirable properties under more complex valuation models or more sophisticated models of bidder behavior. Our results suggest that while GSP turns out to have excellent properties for the simplest model of advertiser valuations, this is very possibly no more than a fortunate coincidence that does not extend to more complex valuations. Thus, rather than continuing to build on the GSP auction, it might be necessary to approach the design of more expressive auctions for advertising on the Internet from a clean slate.

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7. APPENDIX

Lemma 7.1. *Overstating* M*-value is weakly dominated in* NP_{2D}*. In other words, for a bidder participating* NP_{2D}*, any strategy that sometimes overstates* M*-value is dominated by a strategy that never overstates* M*-value.*

Proof. Fix a vector of bids by all the bidders other than i. First assume the outcome is M. If bidder i overstates her value, she will either be moved to a higher slot with some price higher than her value and consequently negative utility, or will stay in her current place for the same price. Therefore, she cannot benefit from overstating her value in this case. Now, suppose the outcome is S. In this case, if bidder i changes the outcome to M by overstating her value, she has to pay more than her value because of the pricing policy. Therefore, she cannot benefit from overstating her M-value in this case either. □

PROOF (THEOREM 3.1). Assume for the sake of contradiction that (b, b') is an equilibrium vector of bids in GSP_{2D} with outcome M and recall that indices $1, \ldots, n$ are such that $b'_1 \geq \ldots \geq b'_n$.

First, note that by Lemma 3.1, since (b, b') is an equilibrium, $v_{\max} > \sum_{i=1}^{k} \theta_i v'_i \geq \sum_{i=1}^{k} \theta_i b'_{i+1}$. Therefore, if the bidder with highest S value, max, bids his true value, the outcome will change to S. Consequently, max must be in $\{1, \ldots, k\}$, since otherwise, he has utility 0 in the outcome M while he can make his utility positive by bidding truthfully.

Next, we show that even if max $\in \{1, \ldots, k\}$, he still has an incentive to deviate and bid truthfully to change the outcome to S. More precisely, we show that the (following lower bound on the) payoff of max in outcome S is more than his payoff in outcome M:

$$v_{\max} - (\theta_1 b'_2 + \ldots + \theta_k b'_{k+1}) > \theta_j (v'_{\max} - b'_{j+1}),$$

where index j is such that $j = \max$. That is, we want to show

$$v_j - (\theta_1 b'_2 + \ldots + \theta_k b'_{k+1}) > \theta_j (v'_j - b'_{j+1}).$$

Rearranging, and using the fact that max $\in \{1, \ldots, k\}$, that is, $j \leq k$, it suffices to prove:

$$v_j > (\theta_1 b'_2 + \ldots + \theta_{j-1} b'_j + \theta_j v'_j + \theta_{j+1} b'_{j+2} + \ldots + \theta_k b'_{k+1}).$$

To show this, we start with:

$$v_{\max} > \theta_1 v'_{[1]} + \ldots + \theta_k v'_{[k]}.$$

Since $v'_{[1]} \geq \ldots \geq v'_{[n]}$, and θ_i's are decreasing, we get:

$$v_{\max} > \theta_1 v'_1 + \ldots + \theta_k v'_k.$$

By Lemma 3.1, we can replace v'_i by b'_{i+1} for all $i \neq j$, which gives us

$$v_j > (\theta_1 b'_2 + \ldots + \theta_{j-1} b'_j + \theta_j v'_j + \theta_{j+1} b'_{j+2} + \ldots + \theta_k b'_{k+1}),$$

which is what we needed to show. □

PROOF (PROPOSITION 3.1). For brevity and clarity, we describe the proof when bidder max is bidder [1], *i.e.*, has the highest M-value (the proof for $[j]$ is very similar). Since the outcome of GSP_{2D} is S, she must prefer outcome S to every slot; specifically, to the first slot in outcome M. Therefore,

$$v_{\max} - \sum_{i=1}^{k} \theta_i b'_{i+1} \geq \theta_1 (v'_{[1]} - b'_2).$$

Since losers bid at least their true values $b'_i \geq v'_{[i+1]}$, so we get

$$v_{\max} \geq \theta_1 v'_{[1]} + \sum_{i=2}^{k} \theta_i v'_{[i+2]}.$$

Our goal is to show $3v_{\max} \geq \sum_{i=1}^{k} \theta_i v'_{[i]}$ which follows from the above inequality because $v_{\max} \geq \theta_1 v'_{[1]}$, $v_{\max} \geq \theta_2 v'_{[2]}$, and $\theta_1 v'_{[1]} + \sum_{i=2}^{k} \theta_i v'_{[i+2]} \geq \sum_{i=3}^{k} \theta_i v'_{[i]}$.

The efficiency of an envy-free equilibrium of GSP_{2D} follows directly from the arguments in [20]. □

PROOF (THEOREM 4.1). We show that the vector of bids where all bidders except max $= [j]$ bid truthfully and max bids $(v_{\max}, 0)$, is an equilibrium of NP_{2D}. First, note that these bids lead to outcome S because

$$v_{\max} > \sum_{i=1}^{k} \theta_i v'_{[i]} \geq \sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]} = \Gamma.$$

Next, we show that no bidder $i \neq$ max has an incentive to deviate: If loser i wants to change the outcome to M by increasing her current bid to $b'_i > v'_i$, we show that her payment becomes larger than her value. By the pricing rule we have $\theta_i p_i \geq v_{\max} - \sum_{l \neq i} \theta_l b'_l$; furthermore, since the initial outcome is S we have $v_{\max} - \sum_{l \neq i} \theta_l b'_l > \theta_i v'_i$. Therefore, $\theta_i p_i > \theta_i v'_i$.

It only remains to show bidder max $= j$ has no incentive to deviate, *i.e.*, she prefers outcome S to any position in outcome M; but this is directly implied by Lemma 4.1. Therefore, the vector of bids in which every bidder except max bids truthfully and max bids $(v_{\max}, 0)$ is a good equilibrium of NP_{2D}.

We show in Lemma 7.1 in the appendix that bidding above one's true value is a weakly dominated strategy in NP_{2D}. Suppose all bidders play undominated strategies, then no bidder is bidding higher than her true value. If max bids truthfully and every other bidder bids less equal her true value, the outcome will be S (since the efficient outcome is S). By Lemma 4.1, we know that bidder max prefers this outcome to winning any slot in outcome M. Therefore, for any outcome M, the deviation to truthfulness is always profitable for max. □

PROOF (THEOREM 4.2). Since the equilibrium with outcome S is a good equilibrium, the winner is max (since otherwise max would be a loser and hence must bid at least her true S-value leading to negative utility for the winner, which contradicts the equilibrium assumption). So, the welfare in this equilibrium is $v_{\max}$.

Since the efficient outcome is M we have $\sum_{i=1}^{k} \theta_i v'_{[i]} \geq v_{\max}$. Since the NP_{2D} equilibrium is a good equilibrium we have $v_{\max} \geq \sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]}$, where j is the rank of bidder max among all M-values, *i.e.* max $= [j]$. Therefore, the loss in efficiency of any good equilibrium with outcome S of NP_{2D} is at most $\sum_{i=j}^{k} \theta_i (v'_{[i]} - v'_{[i+1]}) = \theta_j v'_{[j]} - \sum_{i=j}^{k} (\theta_i - \theta_{i+1}) v'_{[i+1]} \leq \theta_j v'_{[j]}$. Finally, since $v_{\max} = v_{[j]} \geq v'_{[j]} \geq \theta_j v'_{[j]}$,

$$2v_{\max} \geq v_{\max} + \theta_j v'_{[j]} \geq (\sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]}) + (\sum_{i=j}^{k} \theta_i (v'_{[i]} - v'_{[i+1]})) = \sum_{i=1}^{k} \theta_i v'_{[i]}.$$

□

PROOF (THEOREM 4.4). If the outcome of VCG_{2D} is S, then since all losers bid at least their true values, the payment of the winner in NP_{2D} is at least as much as the winner's payment in VCG_{2D}. Therefore, the revenue of NP_{2D} is greater than or equal to the revenue of VCG_{2D}.

If the outcome of VCG_{2D} is M, we consider two cases, depending on whether $v_{\max^2} - \sum_{i=1, i\neq j}^{k} \theta_i v'_{[i]}$ is the dominant term of the payment for bidder max $= [j]$ in VCG_{2D}, or $\sum_{i=j}^{k} (\theta_i - \theta_{i+1}) v'_{[i+1]}$ is the dominant term. In both cases, by individual rationality, we use $\sum_{i=1, i\neq j}^{k} \theta_i v'_{[i]}$ as an upper-bound on the sum of the payments of all other bidders in VCG_{2D}. In the first case, the revenue of VCG_{2D} is upper-bounded by $v_{\max^2}$, and in the second, it is upper-bounded by $\sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]}$. But, by the NP_{2D} pricing rule, we know that the revenue of any good equilibrium of NP_{2D} with outcome S is at least $\max(v_{\max^2}, \sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]})$. □

Proof (Theorem 4.5). Pick an arbitrary envy-free allocation of NP_{2D} with outcome M and suppose that the price of the i-th slot is p_i. By theorem 4.3 we know $v'_i = v'_{[i]}$. Moreover, the bidder occupying the i-th slot must not envy the bidder occupying the $i-1$-st slot, *i.e.*, $\theta_{i-1}(v_i - p_{i-1}) \leq \theta_i(v_i - p_i)$. By writing this inequality for every i $(2 \leq i \leq k+1)$, then multiplying both sides of the inequality for bidder i by a factor i, and finally summing up all inequalities (and canceling out repeated terms) we get

$$\sum_{i=1}^{k} i(\theta_i - \theta_{i+1}) v'_{i+1} \leq \sum_{i=1}^{k} \theta_i p_i.$$

The LHS of the above inequality is revenue of VCG_M while the RHS is the sum of payments of all bidders in NP_{2D}, or in other words, the revenue of NP_{2D}. □

Proof (Proposition 4.1). If the efficient outcome is S, we proved in Theorem 4.1 the good equilibrium with outcome S exists. The revenue of such equilibrium is at least $\sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]} \geq \sum_{i=1}^{k} \theta_i v'_{[i+1]} \geq \sum_{i=1}^{k} i(\theta_i - \theta_{i+1}) v'_{[i+1]}$,which is greater than or equal to the revenue of VCG_M.

Now suppose that the efficient outcome is M, or equivalently for NP_{2D}, the outcome with truthful bids is M. Consider the utility maximizing deviation for bidder max keeping the bids of the remaining bidders fixed, with respect to NP_{2D} pricing and allocation. If the deviation leads to outcome S, then max bidding $(\infty, 0)$ and all other bidders bidding truthfully is an equilibrium with outcome S since no bidder $i \neq$ max can profitably deviate. If the deviation leads to outcome M, then just increase $b_{\max}$ to just ϵ below the threshold where the outcome of NP_{2D} switches to S, *i.e.*, set $b_{\max} = \sum_{i=1}^{k} \theta_i b'_i - \epsilon$ where b' is the vector of bids in which every bidder except max is bidding truthfully, and max is M-bidding according to her profitable deviation. Setting $b_{\max}$ to this value does not affect the price of max herself after deviation, therefore, max does not have any profitable deviation with the current vector of bids. For any bidder $i \neq$ max, decreasing M-bid changes the outcome to S non-profitably. Moreover, since bidder i is currently M-bidding v'_i, increasing the M-bid cannot be profitable either, since it can only result in negative utility for her. In either case, we have constructed an equilibrium of NP_{2D}. Both of these bid vectors have revenue at least $\sum_{i=1}^{j-1} \theta_i v'_{[i]} + \sum_{i=j}^{k} \theta_i v'_{[i+1]} \geq \sum_{i=1}^{k} \theta_i v'_{[i+1]} \geq \sum_{i=1}^{k} i(\theta_i - \theta_{i+1}) v'_{[i+1]}$, which is at least as much as revenue of VCG_M. □

Proof (Theorem 4.6). Suppose (b_i, b'_i) is an equilibrium vector of bids with outcome M in GSP_{2D}. Since $\sum_{i=1}^{k} \theta_i b'_i \geq \sum_{i=1}^{k} \theta_i b'_{i+1}$, the outcome of NP_{2D} for the same vector of bids is M too. First we argue that no bidder wants to deviate to obtain a different slot in M. Since $b_{\max} \leq \sum_{i=1}^{k} \theta_i b'_{i+1}$, we have $\theta_j b'_{j+1} \geq b_{\max} - \sum_{i\neq j}^{k} \theta_i b'_{i+1} \geq b_{\max} - \sum_{i=1}^{k} \theta_i b'_i + \theta_j b'_j$, so the dominant term for the price of the bidder in slot j (for $1 \leq j \leq k$) is $\theta_j b'_{j+1}$. Therefore, the outcome of NP_{2D} has the same allocation and pricing as GSP_{2D}. Since deviating to slot l was not profitable for any bidder j in GSP_{2D}, it is not profitable in NP_{2D} either (note that the price upon deviation in NP_{2D} for any slot is at least as much as the price in GSP_{2D} for the same deviation).

Next, we show that deviating and changing the outcome to S cannot be profitable for any bidder. The price in NP_{2D} for bidder j for outcome S is at least $\sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}$. The price for outcome S for bidder j in GSP_{2D} is at most $\sum_{i=1}^{k} \theta_i b'_{i+1} \leq \sum_{i=1}^{j-1} \theta_i b'_i + \sum_{i=j}^{k} \theta_i b'_{i+1}$. Therefore, since bidder j has no incentive deviate to outcome S in GSP_{2D}, she has no incentive to deviate to outcome S in NP_{2D} either. □