Appendix

Proof of Equation (3-4)

Assume $\delta = 0$, and $D_m = D(p, x)$, the Hamiltonian of the optimal pricing problem given by equations (3-1) is:

$$H\ (x, p, \lambda) = (p - c + \lambda)\, g, \tag{1}$$

where λ should satisfy the differential equation:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}, \qquad \lambda(T) = 0. \tag{2}$$

The first-order condition is

$$\frac{\partial H}{\partial p} = g + [p - c + \lambda]\frac{\partial g}{\partial p} = 0. \tag{3}$$

The optimal pricing trajectory is the solution to the following equation:

$$p = \frac{-\,g}{\frac{\partial g}{\partial p}} + c - \lambda. \tag{4}$$

The second-order condition for a maximum is

$$\frac{\partial^2 H}{\partial p^2} = \left[\,2 - \frac{g\,\frac{\partial^2 p}{\partial p^2}}{\left(\frac{\partial g}{\partial p}\right)^2}\,\right]\frac{\partial g}{\partial p} \le 0. \tag{5}$$

Thus,

$$\left[\,2 - \frac{g\,\frac{\partial^2 p}{\partial p^2}}{\left(\frac{\partial g}{\partial p}\right)^2}\,\right] \ge 0. \tag{6}$$

Taking the derivative of (4) with respect to time, and substituting (2) for $\dot{\lambda}(t)$ and (4) for λ, we get

$$\dot{p}(t)\left[\,2 - \frac{g\,\frac{\partial^2 p}{\partial p^2}}{\left(\frac{\partial g}{\partial p}\right)^2}\,\right] = -\frac{2\,\frac{\partial g}{\partial x}\,g}{\frac{\partial g}{\partial p}} + \frac{\frac{\partial^2 p}{\partial p\,\partial x}\,g^2}{\left(\frac{\partial g}{\partial p}\right)^2}. \tag{7}$$

Therefore, the sign of $\dot{p}(t)$ will be equal to the sign of right side of (7). Without an imitation effect, $g = b\,(D_m - x)$. Equation (7) becomes

$$\dot{p}(t)\left[\,2 - \frac{g\,\frac{\partial^2 p}{\partial p^2}}{\left(\frac{\partial g}{\partial p}\right)^2}\,\right] = \frac{[\,b\,(D_m - x)]}{\left(\frac{\partial D_m}{\partial p}\right)^2}\left[-2\,\frac{\partial D_m}{\partial p}\left(\frac{\partial D_m}{\partial x} - 1\right) + \frac{\partial^2 D_m}{\partial x\,\partial p}\,(D_m - x)\,\right]. \tag{8}$$

If

$$\left[-2\,\frac{\partial D_m}{\partial p}\left(\frac{\partial D_m}{\partial x} - 1\right) + \frac{\partial^2 D_m}{\partial x\,\partial p}\,(D_m - x)\,\right] > 0, \tag{9}$$

then $\dot{p}(t) > 0$.

Equation (9) can be written as

$$\frac{\partial D_m}{\partial x} > 1 - \frac{(D_m - x)\,\frac{\partial^2 D_m}{\partial x\,\partial p}}{2\left|\frac{\partial D_m}{\partial p}\right|}. \tag{10}$$

An Example:

Assume the demand can be expressed as $D = \mathrm{Exp}\left(-\frac{p}{\frac{x}{a} + v}\right)$. Our simulation results show that given $x(t_0) = 0$, for $a = 0.05$, $v = 2$, equation (10) is satisfied from $t = 0$ to $t = 20$. For example, at $t = 10$, the optimal price is 5.5 and the installed base is 0.19. Then

$$\frac{\partial D_m}{\partial x} = 1.267 > 1 - \frac{(D_m - x) \cdot \frac{\partial^2 D_m}{\partial x \, \partial p}}{2 \left| \frac{\partial D_m}{\partial p} \right|} = 0.9065.$$

Therefore, the optimal pricing is increasing at $t = 10$.

Proof of Proposition 3-1

Let $x_i(t)$ and $p_i(t)$ be the installed base and price of firm i in a duopoly market; $x(t)$ and $p(t)$ be the installed base and price of firm 1 in a monopoly market; and $g(p, x)$ and $g_1(p_1, p_2, x_1, x_2)$ be the diffusion equations of firm 1 under monopoly (equation 3-2) and duopoly (equation 3-7) respectively. Assume $[x_i^*(t), p_i^*(t)]$ are the optimal trajectories of firm i given by (3-16) in a duopoly market where the two products are incompatible. Starting from $t=t_0$, where $x(t_0)= x_1(t_0)$. Consider the case where the monopolist continues to follow a growth path such that $x(t)= x_1^*(t)$. We will show that following this constrained growth path, which is not necessarily the optimal path for the monopolist, the profit of firm 1 under monopoly is higher than the profit it can obtain under incompatible duopoly. Since optimal pricing by the monopoly must produce even higher profit, the proposition will be proved.

Since the two products are partial substitutes, for $p_2^* < \infty$, we have

$$g(p_1^*, x_1^*) \geq g_1(p_1^*, p_2^*, x_1^*, x_2^*). \tag{11}$$

Let p' be the pricing path such that

$$g(p', x_1^*) = g_1(p_1^*, p_2^*, x_1^*, x_2^*). \tag{12}$$

Since g is decreasing in p, from equations(11) and (12), we must have

$$p' \geq p_1^*. \tag{13}$$

When the two product are not compatible, by following the optimal duopolistic pricing strategy which satisfy equation(3-16), firm 1's present value of future profits over the planning horizon is

$$\pi_1^* = \int_0^T e^{-\delta t} \{ [p_1^* - c_1(x_1^*)] \, g_1(p_1^*, p_2^*, x_1^*, x_2^*) \} dt. \tag{14}$$

Let

$$\pi'_m = \int_0^T e^{-\delta t} \{ [p' - c_1(x_1^*)] \cdot g(p', x_1^*) \} dt. \tag{15}$$

From equations (12) and (13), we know

$$\pi'_m \geq \pi_1^*. \tag{16}$$

Let π_m^* be firm1's present value of future discounted profits over the planning horizon in a monopoly market when firm 1 follows the optimal monopoly pricing strategy given by equation(3-3). Since p' is not necessarily the optimal pricing path for firm 1 in a monopoly market, we can conclude

$$\pi_m^* \geq \pi'_m. \tag{17}$$

Therefore

$$\pi_m^* \geq \pi_1^*. \tag{18}$$

Proof of Corollary 3-1

Assume $[x^*(t), p^*(t)]$ are the optimal trajectory of firm i (and j) under the Nash game in a symmetric duopoly market where the two products are incompatible. Starting from $t=t_0$, where the initial installed bases are the same for the compatible and incompatible duopolists. Consider the case where the compatible duopolists continue to follow a growth path such that

$$x_i(t)_{[\text{compatible}]} = x_j(t)_{[\text{compatible}]} = x^*(t). \tag{19}$$

We will show that following this constrained growth path, the profits of the compatible duopolists are higher than that of the incompatible duopolists. Since the optimal pricing policies for the compatible duopolists should produce even higher profits than the profits they can have under the constrained growth path given by equation(19), our proposition will be proved.

The diffusion equations under compatible and incompatible duopolists differ from each other by having different dynamic demand:

$$g_i\,(p_1, p_2, x_1, x_2\,)_{\,[\text{compatible}]} = B_i\,(x_1, x_2\,)\,(D_i(p_1, p_2, x_1, x_2\,)_{[\text{compatible}]} - x_i\,) \tag{20}$$

$$g_i\,(p_1, p_2, x_1, x_2\,)_{\,[\text{incompatible}]} = B_i\,(x_1, x_2\,)\,(D_i(p_1, p_2, x_1, x_2\,)_{[\text{incompatible}]} - x_i\,) \tag{21}$$

Since the duopolists are symmetric, we have

$$(D_i(p_1, p_2, x_1, x_2\,)_{[\text{compatible}]} > (D_i(p_1, p_2, x_1, x_2\,)_{[\text{incompatible}]}. \tag{22}$$

Thus,

$$g_i\,(p^*, p^*, x^*, x^*\,)_{\,[\text{compatible}]} > g_i\,(p^*, p^*, x^*, x^*\,)_{\,[\text{incompatible}]}. \tag{23}$$

Let p' be a pricing path such that at any given time t, $t_0 \leq t \leq T$,

$$g_i\,(p', p', x^*, x^*\,)_{\,[\text{compatible}]} = g_i\,(p^*, p^*, x^*, x^*\,)_{\,[\text{incompatible}]}. \tag{24}$$

From equation (3-10), we know

$$p' > p^*. \tag{25}$$

When the two product are incompatible, by following the optimal duopolistic pricing strategy given by equation(3-16), firm i's present value of future profits over the planning horizon is

$$\pi_i^*\,[\text{incompatible}] = \int_0^T e^{-\delta t}\,\{\,[p^* - c_i\,(x^*)\,]\,g_i\,(p^*, p^*, x^*, x^*\,)_{\,[\text{incompatible}]}\,\}\,dt \tag{26}$$

Let

$$\pi_i{}'_{\,[\text{compatible}]} = \int_0^T e^{-\delta t}\,\{\,[p' - c_i\,(x^*)\,]\;g_i\,(p', p', x^*, x^*\,)_{\,[\text{compatible}]}\,\}\,dt. \tag{27}$$

From equations (24) and (25), we know

$$\pi_i{}'_{\,[\text{compatible}]} > \pi_i^*{}_{\,[\text{incompatible}]} \tag{28}$$

Let $\pi_i^*{}_{[\text{compatible}]}$ be the present value of future profits over the planning horizon in a duopoly market where the two products are compatible on the assumption that firm i follows the optimal duopolistic pricing strategy solved from equation(3-16). Since p' is not necessarily the optimal pricing path for firm i in a market where the two products are compatible, we must have

$$\pi_i^*{}_{\,[\text{compatible}]} \geq \pi_i{}'_{\,[\text{compatible}]} \tag{29}$$

Therefore, we can conclude

$$\pi_i^*{}_{\,[\text{compatible}]} > \pi_i^*{}_{\,[\text{incompatible}]} \tag{30}$$

<u>Proof of Propositions 4-1 and 4-2:</u>

Let's rewrite equations (4-3) -(4-6) here:

$$E_i = \frac{(B_i + B_j)\;\alpha_j}{(B_j + B_i x_i - B_j x_j)\;(\alpha_i + \alpha_j)} \tag{31}$$

$$E_j = \frac{(B_i + B_j)\;\alpha_i}{(B_i + B_j x_j - B_i x_i)\;(\alpha_i + \alpha_j)} \tag{32}$$

$$\bar{p}_i = \frac{\ln E_i}{\alpha_i} = \frac{1}{\alpha_i}\,Ln\,\left\{\frac{(B_i + B_j)\alpha_j}{(B_j + B_i\,x_i - B_j\,x_j)(\alpha_i + \alpha_j)}\right\} \tag{33}$$

$$\bar{p}_j = \frac{1}{\alpha_j}\,Ln\,\left\{\frac{B_j + \dfrac{(B_i\,\alpha_i - B_j\,\alpha_j)\mathrm{Exp}(-\alpha_i\,p_i)}{(\alpha_i + \alpha_j)}}{B_i\,\mathrm{Exp}(-\alpha_i\,p_i) - B_i\,x_i + B_j\,x_j}\right\} \tag{34}$$

We prove Propositions 4-1 and 4-2 by proving the following conjectures.

Conjecture 1: $\dot{x}_i \leq \dot{x}_j$ **if and only if** $p_j \leq \bar{p}_j$

If $\dot{x}_i \leq \dot{x}_j$, then (35)

$$B_i (D_i - x_i) \leq B_j (D_j - x_j) \quad (36)$$

For the density function (4-1), the dynamic demand can be derived from equations (2-6) and (2-7):

$$D_i = \mathrm{Exp}(-\alpha_i p_i) - \frac{\alpha_i \mathrm{Exp}(-\alpha_i p_i - \alpha_j p_j)}{\alpha_i + \alpha_j} \quad (37)$$

$$D_j = \mathrm{Exp}(-\alpha_j p_j) - \frac{\alpha_j \mathrm{Exp}(-\alpha_j p_j - \alpha_i p_i)}{\alpha_j + \alpha_i} \quad (38)$$

Thus, equation (36) can be written as

$$B_i \mathrm{Exp}(-\alpha_i p_i) - B_i x_i + B_j x_j \leq \left(B_j + \frac{(B_i \alpha_i - B_j \alpha_j)\mathrm{Exp}(-\alpha_i p_i)}{(\alpha_i + \alpha_j)}\right) \mathrm{Exp}(-\alpha_j p_j) \quad (39)$$

From equation(37), we know that the left side of equation(39) is positive, thus equation(39) can be written as

$$1 \leq \frac{B_j + \frac{(B_i \alpha_i - B_j \alpha_j)\mathrm{Exp}(-\alpha_i p_i)}{(\alpha_i + \alpha_j)}}{B_i \mathrm{Exp}(-\alpha_i p_i) - B_i x_i + B_j x_j} \mathrm{Exp}(-\alpha_j p_j) \quad (40)$$

or,

$$p_j \leq \frac{1}{\alpha_j} Ln \left\{ \frac{B_j + \frac{(B_i \alpha_i - B_j \alpha_j)\mathrm{Exp}(-\alpha_i p_i)}{(\alpha_i + \alpha_j)}}{B_i \mathrm{Exp}(-\alpha_i p_i) - B_i x_i + B_j x_j} \right\} = \bar{p}_j \quad (41)$$

Thus, we have proved that if $\dot{x}_i \leq \dot{x}_j$, then $p_j \leq \bar{p}_j$. It is also easy to see that if $p_j \leq \bar{p}_j$, then $\frac{dx_i}{dt} \leq \frac{dx_j}{dt}$.

Conjecture 2: **If** $p_i \leq \bar{p}_i$ **, then** $\bar{p}_j \leq 0$

When $\bar{p}_j = 0$, equation (34) becomes

$$B_j + \frac{(B_i \alpha_i - B_j \alpha_j)\mathrm{Exp}(-\alpha_i p_i)}{(\alpha_i + \alpha_j)} = B_i \mathrm{Exp}(-\alpha_i p_i) - B_i x_i + B_j x_j \quad (42)$$

Solve p_i from equation(42), we get

$$p_i \,(\bar{p}_j = 0) = \frac{1}{\alpha_i} Ln \left\{ \frac{(B_i + B_j)\alpha_j}{(B_j + B_i x_i - B_j x_j)(\alpha_i + \alpha_j)} \right\} = \bar{p}_i \quad (43)$$

It is easy to see from equation (34) that $\bar{p}_j$ is a decreasing function of p_i:

$$\frac{\partial \bar{p}_j}{\partial p_i} < 0 \quad (44)$$

From equations (43) and (44), we know that, if $p_i \leq \bar{p}_i$, then $\bar{p}_j \leq 0$.

Conjecture 3: **If** $E_i > E_j$ **then** $\bar{p}_i > 0$

If $E_i > E_j$, we know from equations (31) and (32)

$$B_i \alpha_j + B_j x_j \alpha_j + B_j x_j \alpha_i - B_j \alpha_i - B_i x_i \alpha_i - B_i x_i \alpha_j > 0 \quad (45)$$

Let

$$L1 = (B_i + B_j)\alpha_j \quad (46)$$

$$L2 = B_j + B_i x_i - B_j x_j)(\alpha_i + \alpha_j) \quad (47)$$

we have

$$L1 - L2 = B_i\alpha_j + B_j x_j \alpha_j + B_j x_j \alpha_i - B_j \alpha_i - B_i x_i \alpha_i - B_i x_i \alpha_j > 0 \quad (48)$$

Thus

$$\bar{p}_i = \frac{ln\, E_i}{\alpha_i} = \frac{1}{\alpha_i} Ln \left\{ \frac{L1}{L2} \right\} > 0 \quad (49)$$

Conjecture 3 is proved.

Conjecture 1 implies that the necessary and sufficient condition for firm *j* to have a higher sales rate is to set it's price below $\overline{p}_j$. However, Conjecture 2 tells us that as long as p_i is lower than $\overline{p}_i$, $\overline{p}_j$ will be negative. Furthermore, Conjecture 3 indicates that as long as Ei is greater than Ej, firm *i* can always find a positive $p_i \le \overline{p}_i$. By combining these 3 conjectures, we have proved that, if $Ei > Ej$, as long as $p_i \le \overline{p}_i$, firm *i* will always have a higher sales rate whatever firm *j*'s (positive)price. However, if $p_i > \overline{p}_i$, then firm *j* is able to have a higher sales rate by setting a positive price $p_j < \overline{p}_j$. Thus, we have proved Propositions 4-1 and 4-2.

Proof of Corollary 4-1:

When $E_i = E_j$,

we have
$$L1 - L2 = 0 \tag{50}$$

thus
$$\overline{p}_i = \frac{1}{\alpha_i} Ln \left\{ \frac{L1}{L2} \right\} = 0 \tag{51}$$

Firm *i* can not find a positive $p_i < \overline{p}_i$. In this case neither firm has market power, the lower priced firm will have a higher sales rate.

Proof of Corollary 4-2:

From $\overline{p}_i = \dfrac{ln\, E_i}{\alpha_i}$ we get

$$\frac{d\,\overline{p}_i}{dx_i} = \frac{1}{(\alpha_i)^2} \left\{ \frac{\alpha_i \dfrac{\partial E_i}{\partial x_i}}{E_i} - \frac{d\alpha_i}{dx_i} ln\, E_i \right\}$$
$$= \frac{1}{E_i\, \alpha_i} \left\{ \frac{\partial B_i}{\partial x_i} M - B_i (B_i + B_j)\ \alpha_j\ (\alpha_i + \alpha_j) - \frac{d\alpha_i}{dx_i} L \right\} \tag{52}$$

where

$$M = \alpha_j B_j (\alpha_i + \alpha_j)(1 - x_i - x_j) > 0, \tag{53}$$
$$L = (B_i + B_j)\ \alpha_j (B_j + B_i x_i - B_j x_j) + \frac{E_i}{\alpha_i} ln\, E_i > 0 \tag{54}$$

Therefore, we have

$$\frac{\partial \overline{p}_i}{\partial x_i} > 0 \qquad \text{if and only if}$$
$$\frac{\partial B_i}{\partial x_i} M - B_i (B_i + B_j)\ \alpha_j\ (\alpha_i + \alpha_j) - \frac{d\alpha_i}{dx_i} L > 0 \tag{55}$$

For any B_i with $\dfrac{\partial B_i}{\partial x_i} \ge 0$, condition (55) will hold if

$$\frac{d\alpha_i}{dx_i} < - \frac{B_i\ (B_i + B_j)\ \ (\alpha_i + \alpha_j)\alpha_j}{L} \tag{56}$$