Quiz 1

The first letter of your LAST name

____

First Name Last Name

CAD / CAM / CAE Summer 08

Yokohama National University

QUIZ 1

Date: 7/23/2008
Time: 11:15 – 12:00 (45 min.)
Format: Closed book, closed notes
Weight: 10% of total grade

Note: You have 45 min. Be careful about the time allocation.
Try not to leave any problems totally blank so that
you can receive partial credit. Good luck!

Q1-1 (25 pts)	Q1-2 (25 pts)	Q1-3 (25 pts)	Q1-4 (25 pts)	Total (100 pts)

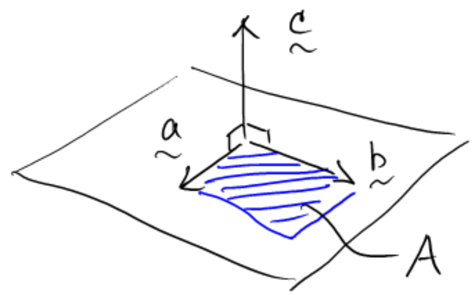
Q1-1 What is the definition of a cross product of two vectors?

(25 pts)

② outer/vector/cross product

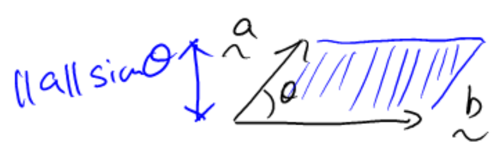
$$\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{c}$$

$$\underset{\sim}{a} \perp \underset{\sim}{c} \ \& \ \underset{\sim}{b} \perp \underset{\sim}{c}$$



$$\|c\| = \|a\| \|b\| \sin\theta$$

measures the signed area of



$$\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\hat{\underset{\sim}{i}}$: the unit vector in the x-axis

$\underset{\sim}{j}$: y-axis

$\underset{\sim}{k}$: z-axis

$$= \hat{\underset{\sim}{i}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} + \hat{\underset{\sim}{j}} \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} + \hat{\underset{\sim}{k}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (a_y \cdot b_z - a_z b_y) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (\qquad) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (\qquad)$$

Q1-2 (25 pts)

Find the distance from the origin to plane $x+y+z=13$. Show all the derivation steps for full credit.

The given equation is in implicit form:

$$ax+by+cz=d$$

From this equation, we know 2 things:

① unit normal vector of the plane $\hat{\underset{\sim}{n}}$

$$\hat{\underset{\sim}{n}} = \left[\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \right]$$

② shortest distance from the origin to plane ℓ

$$\ell = \frac{d}{\sqrt{a^2+b^2+c^2}}$$

Rewrite the given equation:

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = \frac{13}{\sqrt{3}}$$

$\therefore$ shortest distance ℓ from origin to plane $= \boxed{\frac{13}{\sqrt{3}}}$

Q1-3 Given two planes, $n_1 \bullet p = d_1$ and $n_2 \bullet p = d_2$, find the intersection of the two planes. Show all the derivation steps for full credit.

(25 pts)

Case 1 If They are the same plane.
All points on plane1 are also on plane2

Case2 Two planes are parallel
They will never intersect

Case3 Two planes are not parallel
⇒ intersection line equation : $\underset{\sim}{p} = \underset{\sim}{p_0} + \underset{\sim}{v}t$
⇒ $\underset{\sim}{v} = \hat{\underset{\sim}{n}}_1 \times \hat{\underset{\sim}{n}}_2$
⇒ Find p_0

Method 1

Since $\hat{\underset{\sim}{n}}_1$ and $\hat{\underset{\sim}{n}}_2$ are not parallel, position of p_0 can be found using combination of $\hat{\underset{\sim}{n}}_1$ and $\hat{\underset{\sim}{n}}_2$

$\boxed{p_0 = c_1\hat{\underset{\sim}{n}}_1 + c_2\hat{\underset{\sim}{n}}_2}$ where c_1 and c_2 are constants

⇒ substitute $\underset{\sim}{p}$ in plane equations

$\hat{\underset{\sim}{n}}_1 \circ (c_1\hat{\underset{\sim}{n}}_1 + c_2\hat{\underset{\sim}{n}}_2 + (\hat{\underset{\sim}{n}}_1 \times \hat{\underset{\sim}{n}}_2)t = d_1 \Rightarrow c_1\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_1 + c_2\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2 = d_1$ —— ①

$\hat{\underset{\sim}{n}}_2 \circ (c_1\hat{\underset{\sim}{n}}_1 + c_1\hat{\underset{\sim}{n}}_2 + (\hat{\underset{\sim}{n}}_1 \times \hat{\underset{\sim}{n}}_2)t = d_2 \Rightarrow c_1\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2 + c_2\hat{\underset{\sim}{n}}_2 \circ \hat{\underset{\sim}{n}}_2 = d_2$ —— ②

⇒ Solve for c_1 and c_2

[① × $(\hat{\underset{\sim}{n}}_2 \circ \hat{\underset{\sim}{n}}_2)$] − [② × $(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)$]

$$\boxed{c_1 = \frac{d_1(\hat{\underset{\sim}{n}}_2 \circ \hat{\underset{\sim}{n}}_2) - d_2(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)}{(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_1)(\hat{\underset{\sim}{n}}_2 \circ \hat{\underset{\sim}{n}}_2) - (\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)^2}}$$

[① × $(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)$] − [② × $(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_1)$]

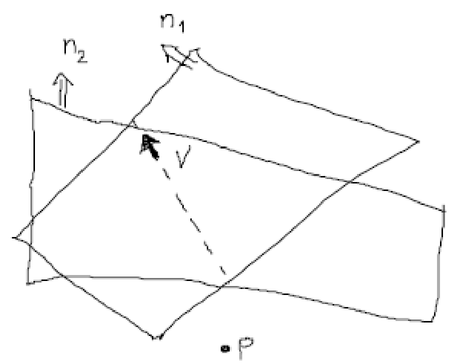
$$\boxed{c_2 = \frac{d_1(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2) - d_2(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_1)}{(\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)^2 - (\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_1)(\hat{\underset{\sim}{n}}_2 \circ \hat{\underset{\sim}{n}}_2)}}$$

Method 2

Select a point P, in 3D space

Project to plane 2 with $\hat{\underset{\sim}{n}}_2$: $P_2 = s\hat{\underset{\sim}{n}}_2$

Project P_2 to intersection line : $P_0 = P_2 + t(\hat{\underset{\sim}{n}}_2 \times \underset{\sim}{V})$



$$\Rightarrow \hat{\underset{\sim}{n}}_2 \circ \underset{\sim}{P}_2 = d_2 \quad \Rightarrow \quad \hat{\underset{\sim}{n}}_2 \circ (s\hat{\underset{\sim}{n}}_2) = d_2$$

$$s = \frac{d_2}{\cancel{\hat{\underset{\sim}{n}}_2 \circ \hat{\underset{\sim}{n}}_2}\, 1} = d_2 \quad \Rightarrow \quad P_2 = d_2 \hat{\underset{\sim}{n}}_2$$

$$\Rightarrow \hat{\underset{\sim}{n}}_1 \circ \underset{\sim}{P}_0 = d_1$$

$$\hat{\underset{\sim}{n}}_1 \circ (P_2 + t(\hat{\underset{\sim}{n}}_2 \times V)) = d_1$$

$$\hat{\underset{\sim}{n}}_1 \circ (d_2 \hat{\underset{\sim}{n}}_2 + t(\hat{\underset{\sim}{n}}_2 \times \underset{\sim}{V})) = d_1$$

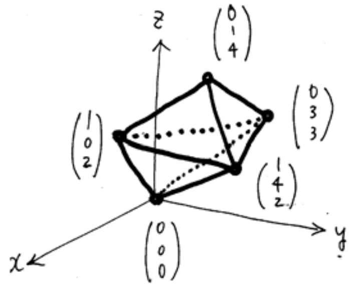
$$d_2 (\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2) + t\left(\hat{\underset{\sim}{n}}_1 \circ (\hat{\underset{\sim}{n}}_2 \times \underset{\sim}{V})\right) = d_1$$

$$t = \frac{d_1 - d_2 (\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)}{\hat{\underset{\sim}{n}}_1 \circ (\hat{\underset{\sim}{n}}_2 \times \underset{\sim}{V})}$$

$$\boxed{P_0 = d_2 \hat{\underset{\sim}{n}}_2 + \left[\frac{d_1 - d_2 (\hat{\underset{\sim}{n}}_1 \circ \hat{\underset{\sim}{n}}_2)}{\hat{\underset{\sim}{n}}_1 \circ (\hat{\underset{\sim}{n}}_2 \times v)}\right] (\hat{\underset{\sim}{n}}_2 \times \ )}$$

Q1-4 Find the volume of the polyhedron. Show all the derivation steps for full credit.

(25 pts)



\# Method 1 Divide polyhedron into 2 tets

① Tet ABCE

$$V_1 = \frac{1}{6}\left(\vec{AC} \cdot (\vec{AE} \times \vec{AB})\right)$$

$$= \frac{1}{6}\begin{vmatrix} 0 & 3 & 3 \\ 1 & 0 & 2 \\ 1 & 4 & 2 \end{vmatrix} = \frac{18-6}{6} = 2$$

② Tet BCDE

$$V_2 = \frac{1}{6}\left(\vec{BD} \cdot (\vec{BC} \times \vec{BE})\right)$$

$$= \frac{1}{6}\begin{vmatrix} -1 & -3 & 2 \\ -1 & -1 & 1 \\ 0 & -4 & 0 \end{vmatrix} = \frac{8-4}{6} = \frac{2}{3}$$

$$\boxed{\therefore \text{total volume} = 2 + \frac{2}{3} = \frac{8}{3}}$$

\# Method 2

$$\text{Volume} = \frac{1}{3}\left| \sum_K A_K d_K \right|$$

A_K = area of the K^{th} face and

d_K = the signed distance from origin to K^{th} face

There are 6 faces

$$\left.\begin{array}{l} ①\ ACB \Rightarrow V_{ACB} = 0 \quad (d_1 = 0) \\ ②\ ABE \Rightarrow V_{ABE} = 0 \quad (d_2 = 0) \\ ③\ AEC \Rightarrow V_{AEC} = 0 \quad (d_3 = 0) \end{array}\right\} \text{point through origin}$$

④ DEB $\Rightarrow V_{DEB} = \frac{1}{3}|A_4 d_4|$

$$A_4 = \frac{1}{2}|DE \times DB| = \frac{1}{2}\left|\begin{matrix} i & j & k \\ 1 & -1 & -2 \\ 1 & 3 & -2 \end{matrix}\right| = \frac{1}{2}|8i + 4j| = \frac{1}{2}\left(\sqrt{8^2+4^2}\right) = \frac{\sqrt{80}}{2}$$

$$\hat{n}_4 = \frac{8}{\sqrt{80}} i + \frac{4}{\sqrt{80}} k$$

$$d_4 = D \cdot \hat{\underset{\sim}{n}}_4 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 8/\sqrt{80} \\ 0 \\ 4/\sqrt{80} \end{pmatrix} = \frac{16}{\sqrt{80}}$$

$$V_{DEB} = \frac{1}{3}\left(\frac{\sqrt{80}}{2}\right)\left(\frac{16}{\sqrt{80}}\right) = \frac{8}{3}$$

⑤ DBC $\Rightarrow V_{DBC} = \frac{1}{3}|A_5 d_5|$

$$A_5 = \frac{1}{2}|DB \times DC| = \frac{1}{2}\left|\begin{matrix} i & j & k \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{matrix}\right| = \frac{1}{2}|i + j + 2k| = \frac{1}{2}\left(\sqrt{1+1+2^2}\right) = \frac{\sqrt{6}}{2}$$

$$\hat{\underset{\sim}{n}}_5 = \frac{i}{\sqrt{6}} + \frac{j}{\sqrt{6}} + \frac{2k}{\sqrt{6}}$$

$$d_5 = D \cdot \hat{\underset{\sim}{n}}_5 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} = \frac{9}{\sqrt{6}}$$

$$V_{DBC} = \frac{1}{3}\left(\frac{\sqrt{6}}{2}\right)\left(\frac{9}{\sqrt{6}}\right) = \frac{9}{6}$$

⑥ DCE $\Rightarrow V_{DCE} = \frac{1}{3}|A_6 d_6|$

$$A_6 = \frac{1}{2}|DC \times DE| = \frac{1}{2}\left|\begin{matrix} i & j & k \\ 0 & 2 & -1 \\ 1 & -1 & -2 \end{matrix}\right| = \frac{1}{2}|-5i - j - 2k| = \frac{1}{2}\left(\sqrt{25+1+4}\right) = \frac{\sqrt{30}}{2}$$

$$\hat{\underset{\sim}{n}}_6 = \frac{-5i}{\sqrt{30}} - \frac{j}{\sqrt{30}} - \frac{2k}{\sqrt{30}}$$

$$d_6 = D \cdot \hat{\underset{\sim}{n}}_6 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -5/\sqrt{30} \\ -1/\sqrt{30} \\ -2/\sqrt{30} \end{pmatrix} = -\frac{9}{\sqrt{30}}$$

$$V_{DCE} = \frac{1}{3}\left(\frac{\sqrt{30}}{2}\right)\left(\frac{-9}{\sqrt{30}}\right) = \frac{-9}{6}$$

$$\boxed{V_{total} = \frac{8}{3} + \frac{9}{6} - \frac{9}{6} = \frac{8}{3}}$$