

# FISCAL HEDGING WITH NOMINAL ASSETS

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Received Date; Received in Revised Form Date; Accepted Date

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## Abstract

We analyze optimal fiscal and monetary policy in an economy with distortionary labor income taxes, nominal rigidities and nominal debt of various maturities. Optimal policy prescribes the exclusive use of long term debt. Such debt mitigates the distortions associated with hedging fiscal shocks by allowing the government to allocate them efficiently across states and periods.

*Keywords:* Optimal fiscal and monetary policy, fiscal hedging

*JEL classification:* Z12, Z24, Z36

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†We appreciate comments of seminar participants at Boston University, Carnegie Mellon University, NYU, Rochester University and the FRB of Chicago.

## 1. Introduction

Governments have traditionally financed deficits by selling *nominal* bonds of varied maturities. A long standing policy question concerns the optimal management of such liabilities. Various contributors have posited a role for short term nominal debt. Campbell (1995) argues that a cost-minimizing government should respond to a steeply sloped nominal yield curve by shortening the maturity structure since high yield spreads tend to predict high expected bond returns in the future. Barro (1997) emphasizes tax smoothing considerations. He asserts that governments can reduce their risk exposure and better smooth taxes by shortening the maturity structure when the inflation process becomes more volatile and persistent. Barro characterizes the reduction in the average maturity of US Federal bonds between 1946 and 1976 as an optimal response to changes in the inflation process. Both lines of argument treat the processes for inflation and nominal interest rates exogenously.

In this paper, we explore optimal maturity management in a fully specified general equilibrium model. We identify a motive for issuing long term nominal debt and give calibrated examples in which there is exclusive use of the longest term nominal debt available. In these examples, the management of nominal interest rates departs from the Friedman rule. An adverse fiscal shock is followed by increases in current and future short term nominal interest rates, with increases in the latter concentrated in future adverse shock states. When a spell of adverse fiscal shocks begins, the yield curve takes a corresponding humped shape, with the hump occurring at the longest traded debt maturity. It reverts to a lower level and a flatter shape when this spell ends or when the debt outstanding at the beginning of the spell has matured. Optimal policy implies that long term nominal debt is riskier than short term debt. However, the volatility of long term debt returns is deliberate and managed so as to hedge the fiscal risk the government faces. The risk premium on this debt resembles an insurance premium paid by the government; it does not provide a motive for shortening the maturity structure.

Since our focus is the management of the nominal maturity structure, we consider an economy in which only non-contingent nominal debt is traded. This assumption implies that the government must hedge fiscal shocks indirectly through contemporaneous inflations or variations to the nominal term structure. We introduce two nominal rigidities that enrich the government's policy problem.<sup>1</sup> First, we assume that some firms set their prices before the realization of the current state. This rigidity implies that contemporaneous innovations to inflation are associated with costly misallocations of production across firms. The government must trade such distortions off against the hedging benefits that inflation innovations provide. Second, we assume that households face a cash-in-advance constraint applied to some goods (cash goods), but not others (credit goods). Variations in the nominal term structure imply positive short term nominal interest rates after some histories and, hence, misallocations of consumption across cash and credit goods. The government must trade the hedging benefits of these variations off against the consumption distortions they induce.

Absent any restrictions on short selling by the government, an allocation in the neighborhood of the optimal complete markets one can be implemented by taking arbitrarily large positions in debt markets at different maturities and making arbitrarily small innovations to nominal interest rates and inflation. In this paper, we assume that households are anonymous in debt markets and cannot borrow. This precludes short-selling by the government and, hence, such extreme asset market positions. We use a simple example with shocks at a single point in time to isolate an advantage of long term debt in this case: such debt allows the government to reduce and postpone the costly positive nominal interest rates used in hedging. As noted, calibrated numerical examples confirm that the government relies almost exclusively on the longest term debt available. This debt permits the maximum postponement of positive nominal interest rates and their concentration in states where they can contribute to the hedging of multiple past shocks. Such postponement and concentration effects underpin a gradual upward response of short term nominal interest rates during spells of adverse fiscal shocks.

The literature on optimal fiscal and monetary policy has made various assumptions about the asset structure confronting the government. Our paper is closest to Siu (2004). We follow him in restricting the

<sup>1</sup>Absent these rigidities all hedging could be achieved through contemporaneous adjustments in inflation and nominal debt would effectively function as a real contingent claim. See, for example, Chari et al (1991).

1 government to the use of nominal debt and incorporating frictions that render state-contingent inflations  
 2 distortionary. In contrast to Siu, we allow the government to trade nominal debt of more than one period  
 3 maturity. Thus, we are able to consider the optimal maturity structure. Additionally, in our model the  
 4 government can influence the price of outstanding nominal bonds via current and future nominal interest rate  
 5 policy. This opens up a second channel for hedging fiscal shocks that is absent in Siu's earlier contribution.

6 The plan for the paper is as follows. Sections II and III describe the environment and characterize  
 7 competitive allocations. Section IV gives the Ramsey problem for our economy and contrasts it with those  
 8 obtained under alternative asset market structures. Section V identifies a motive for using long term debt in a  
 9 simple example, while Section VI provides a general recursive formulation. Section VII uses this formulation  
 10 to obtain optimal policy in calibrated economies.

## 11 2. A model with sticky prices

12 The economy is inhabited by infinitely-lived households, firms and a government. Let  $s_t \in S = \{\widehat{s}_i\}_{i=1}^N$   
 13 denote a period  $t$  shock and  $s^t \in S^{t+1}$  a  $t$ -period history of shocks. We assume that  $s_0$  is distributed  
 14 according to  $\pi^0$  and that subsequently shocks evolve according to a Markov process with transition  $\pi$ . The  
 15 implied probability distribution over shock histories  $s^t$  is denoted  $\pi^t$ .

### 16 2.1. Households

17 **Preferences** Households have preferences over stochastic sequences of cash goods  $\{c_{1t}\}_{t=0}^\infty$ , credit goods  
 18  $\{c_{2t}\}_{t=0}^\infty$  and labor  $\{l_t\}_{t=0}^\infty$  of the form:

$$19 \quad E\left[\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}, l_t)\right], \tag{1}$$

20 where  $U : \mathbb{R}_+^2 \times [0, T] \rightarrow \mathbb{R}$  is twice continuously differentiable on the interior of its domain, strictly concave,  
 21 strictly increasing in its first two arguments and decreasing in its third argument. We assume that  $U$   
 22 satisfies the Inada conditions for  $j = 1, 2$  and each  $(c_i, l)$ ,  $i \neq j$ ,  $\lim_{c_j \rightarrow 0} \frac{\partial U}{\partial c_j}(c_1, c_2, l) = \infty$  and for each  
 23  $(c_1, c_2)$ ,  $\lim_{l \rightarrow T} \frac{\partial U}{\partial l}(c_1, c_2, l) = -\infty$ . Finally, we assume that  $U$  is homothetic in  $(c_1, c_2)$  and weakly separable  
 24 in  $l$ . Let  $U_{jt}$ ,  $j = 1, 2, l$  denote the derivatives of  $U$  with respect to each of its arguments at date  $t$  and let  
 25  $U_{jkt}$ ,  $j, k = 1, 2, l$  denote its second derivatives at  $t$ .

26 **Trading** Each household enters period  $t$  with a portfolio of money  $M_t \geq 0$  and nominal (zero coupon)  
 27 bonds  $\{B_t^k\}_{k=1}^K \in \mathbb{R}_+^K$ , where the superscript  $k$  denotes the maturity of the bond and  $K$  is the maximal  
 28 maturity traded. The shock  $s_t$  is then realized. Asset market trading occurs in two rounds during the course  
 29 of the day. The first *liquidity trading round* occurs in the morning immediately after the shock realization.  
 30 In this, households are able to liquidate their bond holdings in light of their post-shock cash needs. On the  
 31 other side of the market, the government (one may think of it as the Fed) responds to these needs by trading  
 32 bonds for money. The household's budget constraint in this trading round is

$$33 \quad A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_t^{k+1}(s^{t-1}) \geq \widetilde{M}_t(s^t) + \sum_{k=1}^K Q_t^k(s^t) \widetilde{B}_t^k(s^t), \tag{2}$$

34 where  $Q_t^k$  is the nominal price of the  $k$ -th maturity bond,  $A_t = B_t^1 + M_t$  and  $\widetilde{M}_t$  and  $\{\widetilde{B}_t^k\}_{k=1}^K$  denote the  
 35 portfolio of money and bonds purchased by households. Households then shop for cash and credit goods,  
 36 exert effort in production and receive after-tax wage and dividend income. Since money is required for cash  
 37 goods consumption, households face the cash-in-advance constraint:

$$38 \quad P_t(s^t) c_{1t}(s^t) \leq \widetilde{M}_t(s^t). \tag{3}$$

1 In the afternoon, asset markets reopen allowing households to settle credit balances accrued whilst shopping  
 2 and invest income. This second *hedging trading round* also allows the government (one may now think of  
 3 it as the Treasury) to finance its budget deficit and purchase a portfolio that hedges itself against future  
 4 shocks. Define:  $\tilde{A}_t(s^t) \equiv \tilde{B}_t^1(s^t) + \{\tilde{M}_t(s^t) - P_t(s^t)c_{1t}(s^t)\} - P_t(s^t)c_{2t}(s^t) + (1 - \tau_t(s^t))I_t(s^t)$ . Here  $P_t$  is  
 5 the period  $t$  price level,  $\tau_t$  is the income tax rate and  $I_t$  is the household's nominal income. The latter is  
 6 given by  $I_t = W_t l_t + \int_0^1 \Pi_{i,t} di$ , where  $W_t$  is the nominal wage and  $\Pi_{i,t}$  is the nominal profit of intermediate  
 7 goods firm  $i$  at date  $t$ .<sup>2</sup> The household's budget constraint in the hedging trading round is:

$$8 \quad \tilde{A}_t(s^t) + \sum_{k=2}^K \tilde{Q}_t^k(s^t) \tilde{B}_t^k(s^t) \geq A_{t+1}(s^t) + \sum_{k=2}^K \tilde{Q}_t^k(s^t) B_{t+1}^k(s^{t-1}), \quad (4)$$

9 where  $\tilde{Q}_t^k$  denotes the price of a  $k$ -maturity bond in this round.

10 Following Chari and Kehoe (1993), we assume that household participation in bond markets is anony-  
 11 mous, so that bonds issued by households are unenforceable and no one is willing to buy them.<sup>3</sup> Formally,  
 12 we assume, for all  $t$ ,  $s^t$  and  $k$ ,

$$13 \quad B_t^k(s^{t-1}) \geq 0, \quad \tilde{B}_t^k(s^t) \geq 0. \quad (5)$$

14 This constraint precludes lending by the government to households in the bond market and we will refer to it  
 15 as a *no lending constraint*. Both the repayment of government loans and the payment of taxes are transfers  
 16 to the government. Ramsey models typically assume that the first is lump sum, while the second is not. This  
 17 distinction is arbitrary. In practice, costs associated with enforcing repayments and monitoring household  
 18 effort and productivity are likely to render loan repayments contingent on observed income or consumption.  
 19 Hence, they will distort household decisions just as taxes do. We do not explicitly model such costs, rather  
 20 we simply rule government loans out.

21 Households maximize (1) subject to the constraints for all  $i$ ,  $t$ ,  $c_{it} \geq 0$ ,  $l_t \in [0, T]$  and (2)-(5).

## 22 2.2. Final goods firms

23 Final goods firms produce output  $Y_t$  from intermediate goods  $Y_{it}$  using the technology:  $Y_t = [\int_0^1 Y_{it}^{\frac{1}{\mu}} di]^\mu$ ,  
 24  $\mu > 1$ . Intermediate goods are produced by sticky price firms who set their price  $P_{st}$  before  $s_t$  is realized,  
 25 and flexible price ones who set their price  $P_{ft}$  after  $s_t$  is learned. Letting  $\rho$  denote the fraction of sticky  
 26 price firms and assuming symmetry across each type of intermediate good firm, the total output of final  
 27 goods firms is given by:  $Y_t = [(1 - \rho)Y_{ft}^{1/\mu} + \rho Y_{st}^{1/\mu}]^\mu$ , where  $Y_{ft}$  and  $Y_{st}$  are, respectively, the amount of  
 28 flexible and sticky price intermediate good used. Final goods firms are competitive and choose quantities of  
 29 intermediate goods to maximize their profits:

$$30 \quad \sup_{Y_{ft}(s^t), Y_{st}(s^t)} P_t(s^t) \left[ (1 - \rho)Y_{ft}(s^t)^{\frac{1}{\mu}} + \rho Y_{st}(s^t)^{\frac{1}{\mu}} \right]^\mu - (1 - \rho)P_{ft}(s^t)Y_{ft}(s^t) - \rho P_{st}(s^{t-1})Y_{st}(s^t). \quad (6)$$

## 31 2.3. Intermediate goods

Intermediate goods are produced with labor according to the technology:  $Y_{it} = \theta_t L_{it}^\alpha$ , where  $\theta_t(s^t) =$   
 $\theta(s_t)$ ,  $\theta : S \rightarrow \mathbb{R}_+$ , is a productivity shock. Substituting this and the demand curves stemming from (6) into  
 its objective, a flexible price intermediate goods firm chooses its price  $P_{ft}(s^t)$  to solve:

$$\sup_{P_{ft}(s^t)} P_{ft}(s^t) \left( P_{ft}(s^t)/P_t(s^t) \right)^{\frac{-\mu}{\mu-1}} Y_t(s^t) - W_t(s^t) \left\{ \left( P_{ft}(s^t)/P_t(s^t) \right)^{\frac{-\mu}{\mu-1}} Y_t(s^t)/\theta_t(s^t) \right\}^{\frac{1}{\alpha}}.$$

<sup>2</sup> $\Pi_{i,t}$  is paid out as a dividend to the household. Without loss of generality, we omit a detailed description of the stock market and assume instead that households own a diversified and non-tradeable portfolio of shares.

<sup>3</sup>On the other hand, we do allow households to borrow from local stores to finance credit good consumption.

In contrast, a sticky price firm chooses its price  $P_{st}(s^{t-1})$  before  $s_t$  is determined, so as to solve:

$$\sup_{P_{st}(s^{t-1})} E_{s^{t-1}} \left[ (1 - \tau_t) U_{2t} / P_t \left( P_{st} (P_{st} / P_t)^{\frac{-\mu}{\mu-1}} Y_t - W_t \left\{ (P_{st} / P_t)^{\frac{-\mu}{\mu-1}} Y_t / \theta_t \right\}^{\frac{1}{\alpha}} \right) \right].$$

#### 2.4. Government

The government faces a stochastic process for government spending  $\{G_t\}_{t=0}^{\infty}$  of the form  $G_t(s^t) = G(s_t)$ , where  $G : S \rightarrow \mathbb{R}_+$ . The government finances its spending by levying taxes on labor and trading non-contingent nominal bonds. Its budget constraint in the liquidity trading round is  $A_{gt}(s^{t-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_{gt}^{k+1}(s^{t-1}) \leq \widetilde{M}_t(s^t) + \sum_{k=1}^K Q_t^k(s^t) \widetilde{B}_{gt}^k(s^t)$ , where we use a  $g$  subscript to distinguish elements of the government's portfolio and  $A_{gt} = B_{gt}^1 + M_t$ . Its budget constraint in the hedging trading round is:  $\widetilde{A}_{gt}(s^t) + \sum_{k=2}^K \widetilde{Q}_t^k(s^t) \widetilde{B}_{gt}^k(s^t) \leq A_{gt+1}(s^t) + \sum_{k=2}^K \widetilde{Q}_t^k(s^t) B_{gt+1}^k(s^t)$ , where  $\widetilde{A}_{gt}(s^t) = \widetilde{M}_t(s^t) + \widetilde{B}_{gt}^1(s^t) - \tau_t(s^t) I_t(s^t) + P_t(s^t) G(s_t)$ .

#### 2.5. Competitive equilibria and allocations

Define an *allocation* and an  $s^{t-1}$ -*continuation allocation* to be sequences  $e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^{\infty}$  and  $e^\infty(s^{t-1}) = \{c_{1t+r}(s^{t-1}, \cdot), c_{2t+r}(s^{t-1}, \cdot), L_{ft}(s^{t-1}, \cdot), L_{st}(s^{t-1}, \cdot)\}_{r=0}^{\infty}$ .

**Definition 1**  $\{c_{1t}, c_{2t}, l_t, L_{ft}, L_{st}, \tau_t, W_t, P_{st+1}, P_{ft}, P_t, \{Q_t^k\}_{k=1}^K, \{\widetilde{Q}_t^k\}_{k=1}^K, \{B_t^k\}_{k=1}^K, \{B_{gt}^k\}_{k=1}^K, M_t, \{\widetilde{B}_t^k\}_{k=1}^K, \{\widetilde{B}_{gt}^k\}_{k=1}^K, \widetilde{M}_t\}_{t=0}^{\infty}$  is a **competitive equilibrium** at  $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$  if each  $c_{it} \geq 0$ ,  $l_t \in [0, T]$  and

1.  $\{c_{1t}, c_{2t}, l_t, \{B_t^k\}_{k=1}^K, M_t, \{\widetilde{B}_t^k\}_{k=1}^K, \widetilde{M}_t\}_{t=0}^{\infty}$  solves the household's problem given  $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$ ;
2. the sequence of input amounts  $\{L_{ft}^\alpha\}_{t=0}^{\infty}$  and  $\{L_{st}^\alpha\}_{t=0}^{\infty}$  solve the final goods firm's problem; the price sequences  $\{P_{ft}\}_{t=0}^{\infty}$  and  $\{P_{s,t+1}\}_{t=0}^{\infty}$  solve the intermediate firms' problems;
3. the government's budget constraints hold at each date;
4. the labor, bonds and goods markets clears:  $\forall t, s^t, l_t = (1 - \rho)L_{ft} + \rho L_{st}$ ,  $B_t^k = B_{gt}^k$ ,  $\widetilde{B}_t^k = \widetilde{B}_{gt}^k$ ,  $c_{1t} + c_{2t} + G_t = \theta_t [(1 - \rho)L_{ft}^{\alpha/\mu} + \rho L_{st}^{\alpha/\mu}]^\mu$ ;
5. the no lending constraints hold:  $\forall t, s^t, k, B_{g,t}^k(s^{t-1}) \geq 0, \widetilde{B}_{g,t}^k(s^t) \geq 0$ .

$e^\infty$  is a competitive allocation if it is part of a competitive equilibrium.

### 3. Characterizing competitive allocations

Proposition 2 below provides necessary and sufficient conditions for competitive allocations. We precede the proposition with a discussion of those conditions that are new to this paper, referring the reader to Siu (2004) for details of others that are more standard.

#### 3.1. Implementability constraints

Implementability constraints are implied by household first order conditions; they are central elements of any Ramsey taxation model. We describe these conditions under our asset market structure and then contrast them with the corresponding constraints in earlier work.

1 **Primary Surplus Values** The government's  $t$ -th period primary surplus is  $\frac{1-Q_t^1}{Q_t^1} \frac{M_t}{P_t} + \tau_t \frac{I_t}{P_t} - G_t$ , where  
2  $\frac{1-Q_t^1}{Q_t^1} \frac{M_t}{P_t}$  is seigniorage. The household's first order conditions, the expression for profits from an interme-  
3 diate goods firm and the resource constraint imply that in a competitive equilibrium this surplus equals  
4  $\Lambda_t \equiv \frac{U_{1t}}{U_{2t}} c_{1t} + c_{2t} + \frac{U_{1t}}{U_{2t}} \Upsilon_t$ ,  $\Upsilon_t \equiv \frac{\mu}{\alpha} [(1-\rho)L_{ft} + \rho L_{ft}^{1-\frac{\alpha}{\mu}} L_{st+j}^{\frac{\alpha}{\mu}}]$ .<sup>4</sup> The household's stochastic discount  
5 factor (SDF) between the  $t$ -th liquidity and the  $t+j$ -th hedging round is  $\beta \frac{U_{2t+j}}{U_{1t}}$ . Since surpluses are  
6 financed in the hedging round, it follows that the equilibrium value of the continuation primary surplus  
7 stream at  $s^t$  is  $\xi_t(s^t) \equiv E_{s^t}[\sum_{j=0}^{\infty} \beta^{t+j} \frac{U_{2t+j}}{U_{1t}} \Lambda_{t+j}]$ . We label  $\xi_t$  the period  $t$  *primary surplus value*. Let  
8  $\Xi_t(s^{t-1}) \equiv (\xi_t(s^{t-1}, \hat{s}_1), \dots, \xi_t(s^{t-1}, \hat{s}_N))'$ .

9 **Liability Values** The government's real liability value at the  $s^t$ -th liquidity round is  $\frac{A_t(s^{t-1})}{P_t(s^t)} + \sum_{k=1}^{K-1} \frac{Q_t^k(s^t)}{P_t(s^t)} B_t^{k+1}(s^{t-1})$ .  
10 It is convenient to rewrite this as:  $\sum_{k=1}^K \frac{P_{st}(s^{t-1})}{P_t(s^t)} Q_t^{k-1}(s^t) b_t^k(s^{t-1})$ , where the *portfolio weights*  $b_t^k(s^{t-1}) =$   
11  $\{b_t^k(s^{t-1})\}_{k=1}^K$  are given by  $b_t^1(s^{t-1}) = \frac{A_t(s^{t-1})}{P_{st}(s^{t-1})}$  and  $b_t^k(s^{t-1}) = \frac{B_t^k(s^{t-1})}{P_{st}(s^{t-1})}$ ,  $k > 1$  and  $\frac{P_{st}(s^{t-1})}{P_t(s^t)} Q_t^{k-1}(s^t)$  is in-  
12 terpreted as a price for  $b_t^k(s^{t-1})$ . The  $\frac{P_{st}(s^{t-1})}{P_t(s^t)}$  term in these prices may be interpreted as a current inflation  
13 shock. We normalize  $Q_t^0$  to 1.

14 **Inflation shocks** In a competitive equilibrium,  $\frac{P_{st}(s^{t-1})}{P_t(s^t)} = N_t(s^t)$ , where  $N_t(s^t) \equiv ((1-\rho)L_{ft}^{\frac{\alpha}{\mu}}(s^t) +$   
15  $\rho L_{st}^{\frac{\alpha}{\mu}}(s^t))^{\mu} / L_{st}^{\alpha}(s^t))^{\frac{\mu-1}{\mu}}$ .

**Nominal bond prices** The household's nominal SDF between the  $t$  and  $t+k$ -th liquidity rounds is  
 $\prod_{j=0}^{k-1} \frac{U_{2t+j}}{U_{1t+j}} \prod_{j=1}^k \frac{N_{t+j} U_{1t+j}}{E_{t+j-1}[N_{t+j} U_{1t+j-1}]}$ .<sup>5</sup> This implies that equilibrium bond prices in the period  $t$  liquidity  
round are given by:  $Q_t^k = D_t^k$ ,<sup>6</sup> where  $D_t^0(s^t) = 1$  and for  $k \geq 1$ ,

$$D_t^k(s^t) \equiv \sum_{s^{t+k}} \left[ \prod_{j=0}^{k-1} \frac{U_{2t+j}(s^{t+j})}{U_{1t+j}(s^{t+j})} \prod_{j=1}^k \left\{ \frac{N_{t+j}(s^{t+j}) U_{1t+j}(s^{t+j})}{E_{s^{t+j-1}}[N_{t+j} U_{1t+j}]} \right\} \pi^k(s^{t+k} | s^t) \right].$$

16 Reorganizing this formula, the (equilibrium) bond price  $Q_t^k(s^t)$ ,  $k \geq 1$ , is the expected product of cash-credit  
17 good marginal rates of substitution (MRS's)  $\prod_{j=0}^{k-1} \frac{U_{2t+j}(s^{t+j})}{U_{1t+j}(s^{t+j})}$  under the "distorted" probabilities, for  $k \geq 1$ ,

$$18 \quad \tilde{\pi}^k(s^{t+k} | s^t) = \prod_{j=1}^k \left\{ \frac{N_{t+j}(s^{t+j}) U_{1t+j}(s^{t+j})}{E_{s^{t+j-1}}[N_{t+j} U_{1t+j}]} \right\} \pi^k(s^{t+k} | s^t). \quad (7)$$

19  $\tilde{\pi}^k(\cdot | s^t)$  weight states in which inflation shocks are relatively modest and cash goods relatively scare more  
20 heavily. Note that these formula imply that the gross one period nominal rate of interest equals  $U_{1t}(s^t)/U_{2t}(s^t)$ .  
21 Departures of  $U_{1t}(s^t)/U_{2t}(s^t)$  from unity represent distortionary liquidity premia or cash-credit wedges. The  
22 fact that, by the above formulas, nominal bond prices depart from unity only in so far as a distortionary  
23 wedge is introduced after some history plays a major role in the subsequent analysis. Let  $\Psi_t(s^{t-1})$  be the  
24  $N \times K$  pricing matrix with  $(i, k)$ -th element  $\psi_t^{i,k}(s^{t-1}) = \frac{P_{st}(s^{t-1})}{P_t(s^t, \hat{s}_i)} Q_t^{k-1}(s^t, \hat{s}_i) = N_t(s^{t-1}, \hat{s}_i) D_t^{k-1}(s^{t-1}, \hat{s}_i)$ .

<sup>4</sup>We formally prove this and other statements made in the current section in the appendix.

<sup>5</sup>This formula follows directly from the household's and firm's first order conditions.

<sup>6</sup>The inability of households to borrow on bond markets introduces potential indeterminacy in bond prices. We resolve this by assuming throughout that bond prices are set so that households never wish to borrow. This assumption does not restrict the set of competitive allocations; it ensures non-negative interest rates at all times.

**The complete constraints** Using this notation, the *implementability constraints*<sup>7</sup> are  $\forall t, s^{t-1}$ ,

$$\underbrace{\Psi_t(s^{t-1})b_t(s^{t-1})}_{\text{liability value}} = \underbrace{\Xi_t(s^{t-1})}_{\text{primary surplus}}. \quad (8)$$

The preceding discussion implies that the left and right hand sides of (8) correspond to government liability and primary surplus values; (8) asserts that these values must be equal after all histories.

At date 0, the portfolio weights  $b_t$  are predetermined, whereas at dates  $t > 0$  they are chosen as part of a competitive equilibrium. In the latter case, they are measurable with respect to information at date  $t-1$  and (8) places cross state restrictions on the process for  $\xi_t$ . Although the portfolio weights are  $s^{t-1}$ -measurable, variations in the price level or the nominal term structure allow the government to adjust the value of its liabilities in response to and, hence, hedge contemporaneous shocks. The  $N_t(s^t)$  and  $D_t^k(s^t)$  terms in the pricing formulas capture the corresponding variations in allocations and the distortions associated with hedging. If events at  $t$  induce flexible price firms to alter their prices relative to their previously expected level, then  $N_t(s^t)$  departs from 1 and an inefficient allocation of production across firms will occur. If the price of the  $k$ -th maturity outstanding bonds falls, then  $D_t^k(s^t)$  also departs from 1 and the short run nominal interest rate must exceed zero either now or in some future state. This results in a misallocation of consumption across cash and credit goods as households seek to economize on their use of cash.

### 3.2. Comparison with Existing Models

Key differences between our model and others become apparent in the implementability constraints.

**Complete contingent claims** Lucas and Stokey (1983) assume that the government has access to a complete set of contingent claims markets. In this case, a portfolio purchased at  $s^{t-1}$  can be represented by an  $N \times 1$  vector  $b_t(s^{t-1}) = \{b_t^k(s^{t-1})\}_{k=1}^N$  where  $k$  indexes the state ( $s^{t-1}, \hat{s}_k$ ) in which the claim pays out. The corresponding price matrices  $\Psi_t(s^{t-1})$  then equal the  $N$ -dimensional identity matrix and the implementability constraints reduce to  $b_t(s^{t-1}) = \Xi_t(s^{t-1})$ , where  $\{\Xi_t\}$  are the primary surplus values for this case. Except at date 0 when  $b_0$  is fixed, the government's portfolio can be chosen to satisfy the current implementability constraint and so, for  $t > 0$ , these constraints do not restrict allocations.

**Real non-contingent debt** Aiyagari et al (2002), Angeletos (2002) and Buera-Nicolini (2004) assume that only non-contingent real debt is traded. A portfolio at  $s^{t-1}$  is then a vector  $b_t(s^{t-1}) = (b_t^1(s^{t-1}), \dots, b_t^K(s^{t-1}))'$  with  $b_t^k(s^{t-1})$  the quantity of real non-contingent debt maturing at  $t+k-1$ . The implementability constraints in this case are  $\Psi_t^{\text{real}}(s^{t-1})b_t(s^{t-1}) = \Xi_t(s^{t-1})$ , where the  $N \times K$ -pricing matrix  $\Psi_t^{\text{real}}(s^{t-1})$  is comprised of expected intertemporal MRS's, i.e. has  $(i, k)$ -th element  $E \left[ \frac{\beta^{k-1} U'_{t+k-1}}{U'_t} | s^{t-1}, \hat{s}_i \right]$ , where  $U'_t$  is the marginal utility of consumption at  $t$ . Angeletos focusses on the case  $K = N$  and shows that for a dense subset of allocations implementable in the complete markets economy each  $\Psi_t^{\text{real}}(s^{t-1})$  is invertible. For an allocation in this set with primary surplus matrices  $\{\Xi_t\}$ , the implementability constraints in the real non-contingent debt economy after date 0 are satisfied at the portfolios:  $b_t(s^{t-1}) = \Psi_t^{\text{real}}(s^{t-1})^{-1} \Xi_t(s^{t-1})$ . In particular, the optimal complete markets allocation is generically implementable.

Aiyagari et al focus on the restrictive case  $K = 1$ . In this,  $\Psi_t(s^{t-1})$  reduces to an  $N \times 1$  vector of ones, the elements of  $\Xi_t$  are constrained to be  $s^{t-1}$ -measurable and no hedging is possible.

**Nominal non-contingent debt** Siu (2004) considers a special case of our nominal debt economy in which only one period nominal debt is traded. In this case,  $b_t(s^{t-1})$  is just a number and  $\Psi_t(s^{t-1})$  reduces to the  $N \times 1$  vector  $(N_t(s^{t-1}, \hat{s}_1), \dots, N_t(s^{t-1}, \hat{s}_N))'$ . In contrast to Aiyagari et al (2002), the assumption

<sup>7</sup>The label implementability constraint is sometimes reserved for the date 0 version of (8), with date  $t > 0$  versions variously labeled measurability constraints (Aiyagari et al, 2002), sticky price constraints (Siu, 2004) or intertemporal budget conditions (Angeletos, 2002).

1 of nominal debt allows for some hedging through current inflation shocks; in contrast to us, the assumption  
 2 of only one period debt shuts down any possibility of hedging through the nominal term structure.

### 3 3.3. No Lending and Financial Wealth-in-Advance Constraints

4 The *no lending constraints*  $b_t \geq 0$  ensure that the household's bond holdings are non-negative. The  
 5 *financial wealth-in-advance constraints* stem from the fact that households must use some fraction of their  
 6 total financial wealth in the liquidity round to obtain the money necessary for cash good expenditure.  
 7 Expressed in terms of allocations, these constraints are:

$$8 \quad \xi_t(s^t) \geq c_{1t}(s^t). \quad (9)$$

9 Proposition 2 formally characterizes competitive allocations; its proof is in the Appendix.

10 **Proposition 2**  $e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^\infty$  is a competitive allocation at  $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$  if there exists  
 11 a sequence of portfolio weights  $\{b_t\}_{t=0}^\infty$  with  $b_0^1 = \frac{M_0 + B_0^1}{P_{s0}}$  and  $b_0^k = \frac{B_0^k}{P_{s0}}$ ,  $k > 1$  such that  $\{b_t\}_{t=0}^\infty$  and  $e^\infty$   
 12 satisfy  $\forall i, t, s^t$ ,  $c_{it}(s^t) > 0$ ,  $(1 - \rho)L_{ft}(s^t) + \rho L_{st}(s^t) \in (0, T)$ , (8), no lending:  $\forall k, t, s^{t-1}$ ,  $b_t^k(s^{t-1}) \geq 0$ , (9)  
 13 and

14 (Transactions) for all  $t, s^t$ ,

$$15 \quad U_{1t}(s^t)/U_{2t}(s^t) \geq 1; \quad (10)$$

16 (Resource) for all  $t, s^t$ ,

$$17 \quad G(s_t) + c_{1t}(s^t) + c_{2t}(s^t) = \theta(s_t)[(1 - \rho)L_{ft}^{\frac{\alpha}{\mu}}(s^t) + \rho L_{st}^{\frac{\alpha}{\mu}}(s^t)]^\mu; \quad (11)$$

18 (Sticky price optimality) for all  $t > 0, s^{t-1}$ ,

$$19 \quad \sum_{s^t | s^{t-1}} \pi(s^t | s^{t-1}) U_{1t}(s^t) [L_{ft}(s^t)^{1 - \frac{\alpha}{\mu}} L_{st}(s^t)^{\frac{\alpha}{\mu}} - L_{st}(s^t)] = 0. \quad (12)$$

20 If  $e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^\infty$  is a competitive allocation at  $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$  with each  $c_{it} > 0$  and  
 21  $(1 - \rho)L_{ft} + \rho L_{st} \in (0, T)$ , then  $e^\infty$  satisfies (8), no lending and (9)-(12) for some sequence of portfolio  
 22 weights  $\{b_t\}_{t=0}^\infty$ , with  $b_0^1 = \frac{M_0 + B_0^1}{P_{s0}}$  and  $b_0^k = \frac{B_0^k}{P_{s0}}$ ,  $k > 1$ .

## 23 4. Ramsey problems for incomplete and complete markets economies

24 **The Ramsey problem with complete markets** We begin with a benchmark complete markets economy.  
 25 In this the government faces no restrictions on lending and can trade contingent claims. Otherwise, the  
 26 economy is as described previously. Given an initial portfolio  $\{b_0^k\}_{k=1}^K$ ,  $K = N$ , the corresponding Ramsey  
 27 problem is:

$$28 \quad \sup_{\{c_{1t}, c_{2t}, L_{ft}, L_{st}, \{b_t^k\}_{k=1}^K\}_{t=0}^\infty} E \left[ \sum_{t=0}^\infty \beta^t U(c_{1t}, c_{2t}, (1 - \rho)L_{ft} + \rho L_{st}) \right] \quad (13)$$

29 subject to  $\forall t, c_{1t}, c_{2t} \geq 0$  and  $(1 - \rho)L_{ft} + \rho L_{st} \in [0, T]$ , (10)-(12) and the implementability constraint  
 30  $b_0 = \Xi_0$ . This problem is considered by Siu (2004) and others. We state two key properties of its solution.

31 **Proposition 3** Under our assumed preferences, after period 0: 1) the Friedman rule holds and  $U_{1t} = U_{2t}$ ,  
 32 2) flexible price firms set their prices equal to those of sticky price firms and  $N_t = 1$ .

33 Thus, when markets are complete, optimal policy implies nominal yields equal to zero at all maturities  
 34 and an absence of inflation surprises, i.e.  $P_t/P_{t-1}$  is  $s^{t-1}$ -measurable,  $t \geq 1$ .

1 **The Ramsey problem with real non-contingent debt** When the government issues non-contingent  
 2 real debt, the Ramsey problem resembles Problem 13 except that the implementability constraints  $\Psi_t^{\text{real}}(s^{t-1})b_t(s^{t-1}) = \Xi_t(s^{t-1})$   
 3  $\Xi_t(s^{t-1})$  are imposed. As noted if  $K = N$  then, generically, the price matrices  $\Psi_t^{\text{real}}(s^{t-1})$  associated  
 4 with a solution to Problem 13 are invertible and this solution can be implemented with the portfolios  
 5  $b_t(s^{t-1}) = \Psi_t^{\text{real}}(s^{t-1})^{-1}\Xi_t(s^{t-1})$ , where  $\Psi_t^{\text{real}}(s^{t-1})$  and  $\Xi_t(s^{t-1})$  are evaluated at the solution allocation. As  
 6 Buera-Nicolini (2004) show, however, such portfolios are highly sensitive to the underlying parameters of the  
 7 economy and often feature very large positive and negative positions at different maturities.

8 **The Ramsey problem with nominal non-contingent debt** This Ramsey problem, the focus of our  
 9 analysis, imposes the implementability constraints (8) as well as the no lending and financial wealth-in-  
 10 advance constraints. Given a triple  $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$ , the initial portfolio weights are fixed at  $b_0^1 = \frac{M_0 + B_0^1}{P_{s0}}$   
 11 and  $b_0^k = \frac{B_0^k}{P_{s0}}$ ,  $k > 1$ .

12 The price matrices  $\Psi_t$  in this case are very different from their real non-contingent debt counterparts  $\Psi_t^{\text{real}}$   
 13 and the implementability constraints in the two environments have correspondingly different implications for  
 14 hedging and optimal portfolio choice. In particular,  $\Psi_t(s^{t-1})$  has  $(i, k)$ -th element  $\psi_t^{i,k}(s^{t-1}) = N_t(s^{t-1}, \hat{s}_i)$   
 15  $D_t^{k-1}(s^{t-1}, \hat{s}_i)$  and so, by Proposition 3, at an optimal complete markets allocation each  $\Psi_t(s^{t-1})$  equals the  
 16  $K \times N$  unit matrix. It then follows from (8) that such allocations are only implementable in economies with  
 17 non-contingent nominal debt in the very special case that their surplus values  $\xi_t(s^t)$  are  $s^{t-1}$ -measurable.  
 18 The logic is simple: optimal complete markets allocations (usually) require that fiscal shocks be hedged and  
 19 liability values varied, yet they preclude state-contingent variations in interest rates and inflation. The latter  
 20 are precisely the means by which hedging is achieved in an economy with non-contingent nominal debt. At  
 21 the optimal complete markets allocation all nominal assets (regardless of maturity) have the same risk free  
 22 return and no fiscal hedging is possible.

23 In nominal debt economies, optimal fiscal policy can be heuristically characterized as a policy that  
 24 achieves the maximal amount of hedging with the smallest and least costly deviations of the  $\Psi_t$ -matrices  
 25 from the unit matrix. Absent the no lending constraints, small, state-specific substitutions of cash for credit  
 26 good consumption at the optimal complete markets allocation can be used to perturb the  $\{\Psi_t\}$  so that  
 27 (8) holds. However, while allocations very close to the optimal complete markets one can be implemented  
 28 in non-contingent nominal debt economies, their implementation usually requires very large negative (and  
 29 positive) asset positions at some maturities. Such positions are needed to obtain sufficient hedging off of  
 30 the small variations in interest rates and inflation implied by the perturbation. Clearly, restrictions on short  
 31 selling in general, and our no lending constraints in particular, prevent the government from obtaining large  
 32 negative asset positions. In doing so they usually preclude allocations in a neighborhood of the optimal  
 33 complete markets one.

34 The reinstatement of the no lending constraints coupled with the desire to minimize distortionary devia-  
 35 tions of the  $\Psi_t$ -matrices from the unit matrix introduces a motive for issuing long term debt. Such debt gives  
 36 the government greater flexibility in using distortionary cash-credit wedges to perturb the pricing matrices  
 37 away from the unit matrix. As we describe below, it allows the government to defer such wedges.<sup>8</sup>

## 38 5. Optimal hedging with nominal incompleteness

39 This section uses an analytically tractable example to isolate the postponement motive for issuing long  
 40 term debt. To begin with we use the first order conditions from the nominal debt Ramsey problem to  
 41 formalize the tradeoff between the costs and (hedging) benefits of borrowing at each maturity. We then use  
 42 a variational argument to show how cash-credit substitutions later in the term of outstanding debt confer

<sup>8</sup>Nosbusch (2007) considers a three period model in which inflation is exogenous, there is no money and there are no liquidity premia associated with nominal bonds. Nominal debt coincides with real debt that has an exogenously given, random terminal payout. In Nosbusch's calibrations, when a no lending constraint is imposed, long term debt is optimal. It confers a hedging advantage that short term debt does not. The logic of our model is rather different: long term nominal debt allows the government to better manage the costly distortions necessary for hedging.

1 the same marginal hedging benefits, but have a more heavily discounted marginal cost. This is a force for  
 2 the postponement of nominal interest rate variations used in hedging; longer term nominal debt facilitates  
 3 such postponement. The section concludes with an analytically tractable example that makes this explicit.

4 **General Case** Let  $-\omega_t, t \geq 0$ , denote the Lagrange multipliers on the implementability constraints from  
 5 the Ramsey problem (with nominal debt). Define the cumulative multiplier  $\chi_t := \sum_{j=0}^t \omega_j / U_{1j}$ ;  $\chi_t$  is the  
 6 normalized shadow price of the government's continuation primary surplus stream.  $\omega_t$  represents a shock  
 7 to  $\chi_t$  and may be interpreted as a measure of the government's additional desire for funds in period  $t$ . We  
 8 leave the underlying source of shocks unspecified; they may perturb government spending, productivity or  
 9 both. Their exact source is irrelevant for the subsequent analysis.

10 **The costs and benefits of hedging** The first order conditions from the Ramsey problem reveal the  
 11 basic cost-hedging tradeoff that influences the choice of maturity structure. In particular, the first order  
 12 conditions for the portfolio weights coupled with the household's Euler equations and the definition of  $\chi_t$   
 13 yield the asset pricing equation:

$$14 \quad \frac{Q_t^k}{P_t} = \beta E_t \left[ \frac{Q_{t+1}^{k-1} U_{1t+1} \chi_{t+1}}{P_{t+1} U_{1t} \chi_t} \right] - \kappa_t^k, \quad (14)$$

15 where  $\kappa_t^k$  is the normalized multiplier from  $(k, t)$ -th no lending constraint and  $\beta \frac{U_{1t+1} \chi_{t+1}}{U_{1t} \chi_t}$  has a natural  
 16 interpretation as the government's SDF. Expanding (14) yields the CAPM-like equation:

$$17 \quad E_t[R_{t+1}^k] - R_{t+1}^f = -R_{t+1}^f \text{Cov}_t \left( R_{t+1}^k, \beta \frac{U_{1t+1} \chi_{t+1}}{U_{1t} \chi_t} \right) + \tilde{\kappa}_t^k. \quad (15)$$

18 Here  $R_{t+1}^k$  is the gross return on the  $k$ -th period nominal debt,  $R_{t+1}^f$  is the riskless rate implied by the  
 19 government's SDF (i.e.  $1/E_t[\beta \frac{U_{1t+1} \chi_{t+1}}{U_{1t} \chi_t}]$ ) and  $\tilde{\kappa}_t^k$  is a re-normalized no lending multiplier. The CAPM  
 20 equation (15) formalizes the tradeoff between the expected cost and the hedging benefits of borrowing at  
 21 a specific maturity. Larger cost premia ( $E_t[R_{t+1}^k] - R_{t+1}^f$ ) may be associated with an increased no lending  
 22 shadow price  $\tilde{\kappa}_t^k$  or larger hedging benefits (as captured by the negative covariance term in (15)). Thus, risk  
 23 premia on traded bonds, i.e. bonds with  $\tilde{\kappa}_t^k = 0$ , can be interpreted as government-paid insurance premia.  
 24 A high risk premium at a given maturity is not per se rationale for the government to avoid issuing debt of  
 25 that maturity.

26 **Cash-credit substitutions** Suppose that  $U(c_1, c_2, l) = (1 - \sigma) \log(c_1) + \sigma \log(c_2) + v(l)$  for some  
 27 smooth, concave, decreasing function  $v$ . This functional form simplifies the analysis by rendering the (renor-  
 28 malized) primary surplus values  $\xi_t U_{1t}$  independent of the consumption allocation. We can then focus on the  
 29 determination of liability values and the hedging properties of nominal debt. The first order conditions for  
 30  $c_{it}(s^t)$ ,  $i = 1, 2$  may now be combined to give:

$$31 \quad \beta^t [U_{1t}(s^t) - U_{2t}(s^t)] = -\beta^t \eta_t(s^t) [U_{11t}(s^t) + U_{22t}(s^t)] + \sum_{j=0}^{K-2} \beta^{t-j} \delta_{t,t-j}(s^t) \omega_{t-j}(s^{t-j}) \quad (16)$$

32 where  $\eta_t(s^t)$  is the multiplier on the  $s^t$ -th transactions constraint and  $\delta_{t,t-j}(s^t) > 0$  is the effect of a small  
 33 cash-credit substitution at  $t$  on liability values at  $t - j$ .<sup>9</sup> Equation (16) describes the costs and benefits  
 34 of a small substitution of cash for credit goods at  $t$ . The term on the left hand side is the utility cost of  
 35 the substitution, while those terms on the right hand side capture the shadow benefits from relaxing the

<sup>9</sup>More precisely,  $\delta_{t,t-j} := \frac{1}{U_{1t-j}} \left[ \frac{\partial \Omega_{t-j}}{\partial c_{1t}} - \frac{\partial \Omega_{t-j}}{\partial c_{2t}} \right] > 0$ , where for  $t - j \geq 0$ ,  $\Omega_{t-j}(s^t)$  with  $s^t = (s^{t-j-1}, \hat{s}^i, \dots, s_t)$  equals the  $i$ -th element of  $U_{1t-j}(s^{t-j-1}, \hat{s}^i) \cdot \Psi_{t-j}(s^{t-j-1}) b_{t-j}(s^{t-j-1})$  and for  $t - j < 0$ ,  $\Omega_{t-j} := 0$ .

1 transactions constraint and from hedging. Note that the hedging benefit, the final term on the right hand side,  
 2 is a weighted sum of the implementability shadow prices  $\{\omega_{t-j}\}_{j=0}^{K-2}$ . If  $\sum_{j=0}^{K-2} \beta^{t-j} \delta_{t,t-j}(s^{t-1}) \omega_{t-j}(s^{t-j}) >$   
 3 0, then  $U_{1t}(s^t) - U_{2t}(s^t) > 0$  and the  $s^t$ -th nominal interest is positive, otherwise, the nominal interest rate  
 4 is 0.

5 Two features of optimal interest rate policy become apparent in (16). First, nominal interest rates in  
 6 period  $t$  are used to adjust liability values and hedge shocks in multiple past periods as well as the present.  
 7 One implication of this is that the effect of a shock on nominal interest rates is propagated over time.  
 8 Another is that positive nominal interest rate rises are concentrated in states where they can best contribute  
 9 to the hedging of various prior shocks. In particular, if at  $s^t$  there have been multiple positive shocks to  
 10 the government's desire for additional funds (as captured by positive past  $\omega_{t-j}$  values), then, other things  
 11 equal, nominal interest rates are adjusted upwards to a greater extent. Second, the weight attached to  $\omega_{t-j}$   
 12 is scaled by  $\beta^{t-j}$ . This scaling implies that current nominal interest rates (at  $t$ ) are more sensitive to funding  
 13 need shocks further back in the past (at  $t-j$ , but within the maturity of the government's portfolio). The  
 14 logic behind this is straightforward. A perturbation to the cash-credit allocation at date  $t$ , confers a hedging  
 15 benefit at  $t-j$ . Since the nominal debt price is given by the (distorted) conditional expectation of the  
 16 product of cash-credit MRS's over the term of the debt, a perturbation at  $t$  is, potentially, as effective at  
 17 altering the price of outstanding debt (with maturity in excess of  $j+2$ ) as a perturbation at  $t-j$ . However,  
 18 the utility cost of the former perturbation is not born for  $j$  periods and is correspondingly discounted. This  
 19 is a force for the optimal postponement of the nominal interest rate adjustments used in the hedging of  
 20 shocks; the relative scaling in (16) captures this. It is also a force for the use of longer term debt; such debt  
 21 permits greater postponement. The following example makes this explicit.

22 **Example** We make two additional simplifying assumptions. First, we suppose that a shock  $s$  is drawn at a  
 23 single date 1. In the remainder of this section all variables dated  $t \geq 1$  will be indexed by this shock. Again,  
 24 the exact source of the shock is not important, we merely require that it introduces stochastic variation  
 25 into the implementability constraint shadow prices. Second, we suppose that there is no debt of maturity  
 26 greater than one outstanding in period 0 so that for all  $t = 1, \dots, K-2$ ,  $\delta_{t,0}(s^{t-1}) = 0$ . These assumptions  
 27 disentangle the effects of multiple shocks, they imply that variations in nominal interest rates in periods  
 28  $t \geq 1$  are used solely to hedge the period 1 shock and that their variation has no implications for asset prices  
 29 prior to period 1. We retain the utility function  $U(c_1, c_2, l) = (1-\sigma) \log c_1 + \sigma \log c_2 + v(l)$ .

30 The first order condition for  $b_1^k$  is:

$$31 \quad 0 = - \sum_{s \in S} \psi_1^k(s) \omega_1(s) \pi(s) + \kappa^k, \quad (17)$$

32 where  $\psi_1^k(s) = N_1(s) \prod_{j=1}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)}$  gives  $b_1^k$ 's price in state  $s$  and the first term on the right hand side  
 33 of (17) captures the marginal hedging benefits associated with  $b_1^k$ . We can partition  $S$  into two subsets  
 34  $S' = \{s \in S | \omega_1(s) > 0\}$  and  $S'' = \{s \in S | \omega_1(s) \leq 0\}$ . Apart from knife-edge cases, there is stochastic  
 35 variation in the government's ex post desire for funds and both  $S'$  and  $S''$  are non-empty. Manipulation of  
 36 (17) and the first order conditions for consumption reveals first that it is optimal to use *only* the longest  
 37 term debt and, second, for all  $s' \in S'$ ,  $s'' \in S''$  and  $t \leq K$ ,  $\frac{U_{2t}(s')}{U_{1t}(s')} > 1 = \frac{U_{2t}(s'')}{U_{1t}(s'')}$ . Intuitively, hedging is  
 38 largely achieved through increases in nominal interest rates in periods that follow a positive shock to the  
 39 government's ex post desire for funds ( $s \in S'$ ) and that are within the term of the government's debt at  
 40 date 1. For  $s' \in S'$ , there is a positive marginal hedging benefit from raising nominal interest rates at all  $t$   
 41 such that debt of maturity  $k = t+1$  was sold at date 0. In particular, if debt of maturity  $K$  was sold then  
 42 nominal interest rates over periods  $t = 1, \dots, K-1$  should be raised. On the other hand if nominal interest  
 43 rates at dates  $t = 1, \dots, K-1$  are raised following  $s \in S'$ , so as to hedge the shock, then it is optimal to  
 44 use debt of maturity  $K$  to fully exploit the associated variations in liability values.

45 We now derive an explicit formula for nominal interest rates. In the current example, equation (16)  
 46 reduces to:

$$47 \quad \beta^{t-1} [U_{1t}(s) - U_{2t}(s)] = -\beta^{t-1} \eta_t(s) [U_{11t}(s) + U_{22t}(s)] - \delta_{t,1}(s) \omega_1(s), \quad (18)$$

1 where  $\delta_{1,1}(s) = \left[ \frac{U_{221}(s)}{U_{21}(s)} \right] \psi_1^K(s) b_1^K$  and for  $t > 1$ ,  $\delta_{t,1}(s) = \left[ \frac{U_{11t}(s)}{U_{1t}(s)} + \frac{U_{22t}(s)}{U_{2t}(s)} \right] \psi_1^K(s) b_1^K$  are the marginal  
 2 hedging benefits. This equation can be rearranged to give the following expression for cash-credit MRS's  
 3 (or, equivalently, nominal interest rates), for  $t = 2, \dots, K - 1$

$$4 \quad \frac{U_{1t}(s)}{U_{2t}(s)} = H \left( \max(0, \beta^{-(t-1)} \psi_1^K(s) b_1^K \omega_1(s)) \right), \quad (19)$$

5 where  $H(x) := \frac{1+x}{1-\frac{\sigma}{1-\sigma}x}$ . Since  $H$  is increasing on  $[0, (1-\sigma)/\sigma)$  and  $\beta^{-(t-1)}$  is increasing, (19) implies that  
 6 when  $\omega_1(s) > 0$  nominal interest rates are positive *and increasing in  $t$* , for  $t = 2, \dots, K - 1$ . Otherwise,  
 7 they are set to 0. Put differently, when the government receives a shock to its marginal desire for funds, the  
 8 increases in cash-credit MRS's necessary to devalue its liabilities are postponed until later in the term of the  
 9 debt. Intuitively, later increases confer the same hedging benefits, but their associated costs, as captured by  
 10 the first term in (18), are more heavily discounted.

11 The following results formalize these arguments; their proofs are supplied in the appendix.

12 **Lemma 4** (*Use of long term debt*) Suppose that the solution to the Ramsey problem is such that  $S' \neq \emptyset$ , and  
 13 for some  $s'$ ,  $\xi_1(s') \neq 0$ , then the government uses only the longest term debt. When  $\omega_1(s) > 0$ , the nominal  
 14 interest rate is greater than zero in periods  $t = 2, \dots, K - 1$ .

15 **Lemma 5** (*Postponement effect*) Suppose that the solution to the Ramsey problem is such that  $S' \neq \emptyset$  and  
 16 for some  $s'$ ,  $\xi_1(s') \neq 0$ . Then, for  $s \in S'$ ,  $Q_{t+1}^1(s) < Q_t^1(s)$ ,  $k = 1, \dots, K - 1$ . For  $t > K - 1$ ,  $Q_t^1(s) = 1$ .  
 17 For  $s \in S''$  such that  $\omega_1(s) < 0$ ,  $Q_t^1(s) = 1$  for all  $t$ .

18 **Implications for the yield curve and term premia** Lemma 5 has immediate implications for the yield  
 19 curve. Since all uncertainty is resolved at date 1, the expectations hypothesis holds from that date onwards  
 20 and  $Q_1^k = \prod_{j=1}^k Q_{1+j}^1$ . It follows that if  $\omega_1(s) \leq 0$ , then the date 1 yield curve remains at zero. On the other  
 21 hand, if  $\omega_1(s) > 0$ , the yield curve rises and steepens over the horizon  $k = 1, \dots, K - 1$ . Yields at maturities  
 22 greater than  $K - 1$  asymptote towards zero as the maturity increases. Thus, the yield curve is hump shaped,  
 23 with the hump occurring at maturity  $K - 1$ . As time passes and the debt outstanding at the time of the  
 24 shock matures, the hump occurs at a progressively shorter maturity, before disappearing at date  $K$ .

25 Lemma 5 and the earlier definition of the multiplier  $\chi_t$  imply that the covariance between the period 1  
 26 debt price and the government's SDF is decreasing in the debt's maturity. Equation (15) then suggests and  
 27 numerical results confirm that the period 0 expected return on debt is increasing in its maturity. Nonetheless,  
 28 the government chooses to issue more expensive long term debt. By issuing such debt the government is  
 29 able to use postponed nominal interest rate variations to hedge current shocks and so reduce the (welfare)  
 30 costs of hedging.

## 31 6. A recursive formulation

We now look for a recursive formulation of the Ramsey problem. This formulation must ensure that con-  
 tinuation choices attain the primary surplus and liability values implied by past implementability constraints.  
 Let  $\phi_t(s^{t-1}) = E_{s^{t-1}}[\xi_t(s^t)]$  and  $\gamma_t^k(s^{t-1}) = \tilde{E}_{s^{t-1}}[D_t^k(s^t)]$ , where the latter expectation is computed using  
 the distorted probabilities  $\tilde{\pi}^t(s^t|s^{t-1})$ . These variables may be interpreted as implicit "promises" made by  
 the government at  $t - 1$  concerning the expected value of its primary surplus stream and of specific bonds  
 within its portfolio at  $t$ . They evolve recursively according to *promise-keeping conditions*

$$\phi_t(s^{t-1}) = E_{s^{t-1}} [\Lambda_t(s^t) + \beta \phi_{t+1}(s^t)], \quad (20)$$

$$\gamma_t^k(s^{t-1}) = \tilde{E}_{s^{t-1}} \left[ \frac{U_{2t}(s^t)}{U_{1t}(s^t)} \gamma_{t+1}^{k-1}(s^t) \right], k = 1, \dots, K - 1 \text{ and } \gamma_t^0 = 1. \quad (21)$$

1 The implementability constraints may then be rewritten as:

$$2 \quad N_t(s^t)b_t^1(s^{t-1}) + \sum_{k=2}^K \left[ N_t(s^t) \frac{U_{2t}(s^t)}{U_{1t}(s^t)} \gamma_{t+1}^{k-2}(s^t) \right] b_t^k(s^{t-1}) = \Lambda_t(s^t) + \beta\phi_{t+1}(s^t), \quad (22)$$

3 while (9) can be recast as:

$$4 \quad \Lambda_t(s^t) + \beta\phi_{t+1}(s^t) \geq c_{1t}(s^t). \quad (23)$$

5 Our recursive approach to the Ramsey problem treats the variables  $x_t = \{s_{t-1}, \phi_t, \{\gamma_t^k\}_{k=1}^{K-2}\}$  as state  
6 variables that summarize relevant aspects of the past history of the economy and ensure that past constraints  
7 are satisfied. As with most Ramsey problems, the initial period of ours differs from subsequent ones. In  
8 the initial period, the government faces a fixed vector of portfolio weights  $b_0$  rather than a fixed vector of  
9 state variables  $x_0$ . In later periods, this is reversed: the government (effectively) enters period  $t \geq 1$  with  
10 a state vector  $x_t$  and, prior to the shock realization, chooses a vector of portfolio weights  $b_t$  and a family  
11 of shock-contingent current allocations  $\{c_{1t}, c_{2t}, L_{ft}, L_{st}\}$  and state vectors  $x_{t+1}$ . Thus, the continuation  
12 Ramsey problem is recursive in the state variables  $\{x_t\}$  and admits a dynamic programming formulation.  
13 We give this formulation below.<sup>10</sup>

14 Let  $\mathbf{X}$  denote the set of tuples  $\{s, \phi, \{\gamma^k\}_{k=2}^{K-1}\}$  that are attained by some continuation competitive  
15 allocation in its initial period. We collect recursive versions of the constraints that define a competitive  
16 allocation into a correspondence  $\Gamma$ . Given an inherited tuple of state variables, these constraints ensure  
17 that a current consumption-labor allocation and tuple of future state variables are consistent with the  
18 requirements of a competitive allocation.

19 **Definition 6** Let  $\Gamma(s, \phi, \{\gamma^k\}_{k=1}^{K-2})$  equal all tuples  $\{\{b^k\}_{k=1}^K, c_1, c_2, L_f, L_s, \phi', \{\gamma^{k'}\}_{k=1}^{K-2}\}$  that satisfy for  
20 each  $s'$ ,  $c_i(s') \geq 0$ ,  $i = 1, 2$ ,  $(1 - \rho)L_f(s') + \rho L_s(s') \in [0, L]$ , (10)-(12),  $b^k \geq 0$ , (20)-(22) and for each  $s'$   
21  $(s', \phi'(s'), \{\gamma^{k'}(s')\}_{k=1}^{K-2}) \in \mathbf{X}$ .

22 The correspondence  $\Gamma$  provides the constraint set for our dynamic programming problem:

$$23 \quad V(s, \phi, \{\gamma^k\}_{k=2}^{K-1}) = \sup_{\Gamma(s, \phi, \{\gamma^k\}_{k=2}^{K-1})} E_s[U(c_1, c_2, (1 - \rho)L_f + \rho L_s) + \beta V(s', \phi', \{\gamma^{k'}\}_{k=2}^{K-1})]. \quad (24)$$

24 Problem (24) can be solved numerically and its computed policy functions used to obtain the optimal  
25 continuation Ramsey allocation along with the supporting optimal fiscal and monetary policies at each  
26 initial state vector  $\{s_0, \phi_1, \{\gamma_1^k\}_{k=1}^{K-2}\}$ . We pursue this approach in Section 7..

## 27 7. A Calibrated Example

### 28 7.1. Numerical method and parameter values

29 **Numerical method** We solve the dynamic programming problem (24) numerically and then back out the  
30 implied optimal policies. The state space  $\mathbf{X}$  for these problems is endogenous and of dimension  $K$ . In our  
31 calculations we restrict the state space to be a  $K$ -dimensional rectangular set  $\tilde{X}$  and check that enlarging  
32  $\tilde{X}$  does not significantly alter the numerical results we report. The dynamic programming problem is solved  
33 by a value iteration. The main computational difficulty concerns the dimension of the state space which is  
34 increasing in the maximal debt maturity  $K$ . To enable us to solve problems with a maturity structure of  
35 reasonable length, we use Smolyak's algorithm to approximate the government's value function on a sparse  
36 grid fitted to  $\tilde{X}$ . For further details on Smolyak's algorithm see Krueger and Kubler (2004).

<sup>10</sup>LSY (2006) formally demonstrate the recursivity of the continuation Ramsey problem in these state variables.

1 **Baseline Calibration** To permit comparability of our results to those in Siu (2004) and Chari et al (1991),  
 2 we compute a baseline case with parameter values that are close to theirs. In this baseline case, we assume  
 3 preferences of the form:

$$4 \quad U(c_1, c_2, l) = \log [(1 - \sigma)c_1^\varphi + \sigma c_2^\varphi]^{\frac{1}{\varphi}} + \zeta \log(T - l). \quad (25)$$

5 The preference parameters  $\sigma$ ,  $\varphi$  and  $\beta$  are set to 0.58, 0.79 and 0.96;  $\zeta$  is chosen so that approximately 30%  
 6 of an agent's time is spent working. The values of  $\sigma$  and  $\varphi$  are similar to those used by Siu (2004) and  
 7 Chari et al (1991). We follow Siu (2004) and set the production parameters  $\alpha$ ,  $\mu$  and  $\rho$  to 1.0, 1.05 and 0.08  
 8 respectively. The government spending and productivity processes take on two values, denoted  $\underline{G}$ ,  $\overline{G}$ ,  $\underline{\theta}$  and  
 9  $\overline{\theta}$ . The government spending process has a mean of around 20% of GDP in a complete markets model with  
 10 a debt to GDP ratio of 60%. We set the standard deviation of both processes to be approximately 7% and  
 11 their autocorrelation coefficients to 0.95. These values are close to those estimated from data and conform  
 12 to the values used in Siu.

13 Our baseline case sets the maximal maturity  $K$  to 7. We contrast this with variations of the model in  
 14 which  $K$  is less than 7 and conjecture that all of the effects we identify would be quantitatively reinforced if  
 15  $K$  were raised above 7.

## 16 7.2. Results

### 17 7.2.1. Maturity structure

18 Let  $V_t^k$  denote the fraction of the government's portfolio value at each maturity, i.e.  $V_t^k := Q_t^{k-1} B_t^k / [A_t +$   
 19  $\sum_{k=1}^{K-1} Q_t^k B_t^{k+1}]$ . We compute sample averages for  $\{V_t^k\}_{k=1}^K$  over long 10,000 period samples. We find that  
 20 for our baseline case  $V_t^K$ ,  $K = 7$ , is of the order 99.1%. The other maturities collectively make up less than  
 21 0.9% of the value of the portfolio on average. We obtain similar results for other cases with fewer shocks  
 22 and different preferences and attribute these tiny holdings of lower maturity debt largely to numerical error.  
 23 We infer that holdings of debt less than the maximum are either zero or very small. In the remainder of this  
 24 section, we focus on the implications of optimal policy for nominal interest rates, inflation and debt holding  
 25 returns. We illustrate these implications with short simulations that highlight the consequences of particular  
 26 sequences of shocks and with sample moments from long simulations.

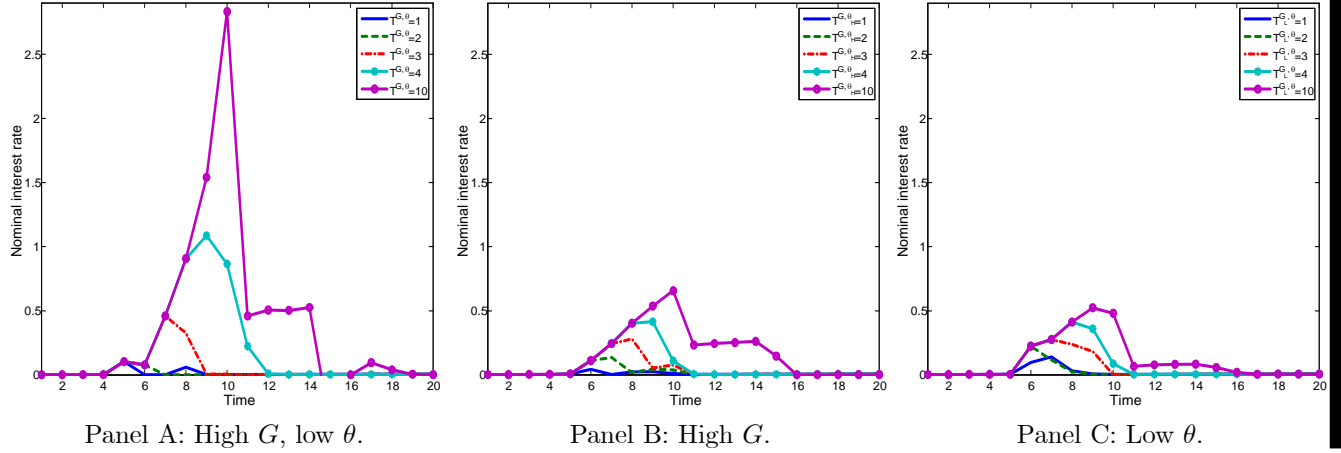
### 27 7.2.2. Short simulations

28 This section presents short simulations. In each, low spending and high productivity shocks are drawn  
 29 until period 4, then an *adverse fiscal spell* consisting of some combination of high spending and low pro-  
 30 ductivity occurs between periods 5 and  $4 + T^G$ , a reversion to low spending and high productivity shocks  
 31 follows. The government has an initial debt to output ratio of about 40%.

32 **Nominal interest rates** Figure 1 shows several sample paths for one period nominal interest rates.  
 33 In the first panel high spending and low productivity shocks are drawn between periods 5 and  $4 + T^G$ , in the  
 34 second high spending (and high productivity) and in the third low productivity (and low spending). The  
 35 different sample paths within each panel correspond to different adverse fiscal spell lengths  $T^G$ .

36 The pattern of sample paths across the figures is qualitatively similar. Within an adverse fiscal spell,  
 37 nominal interest rates rise until  $\min(t + T^G, t + K - 1)$ , i.e. until the spell ends or until the maturity date  
 38 of debt outstanding at  $t$  is reached. In the first case, the nominal interest rate quickly falls back to 0; in the  
 39 second (see the  $T^G = 10$  line in each panel), it falls to a positive number, falling back to 0 only when the  
 40 high spending spell finally comes to an end.

41 **Figure 1: Nominal interest rates**

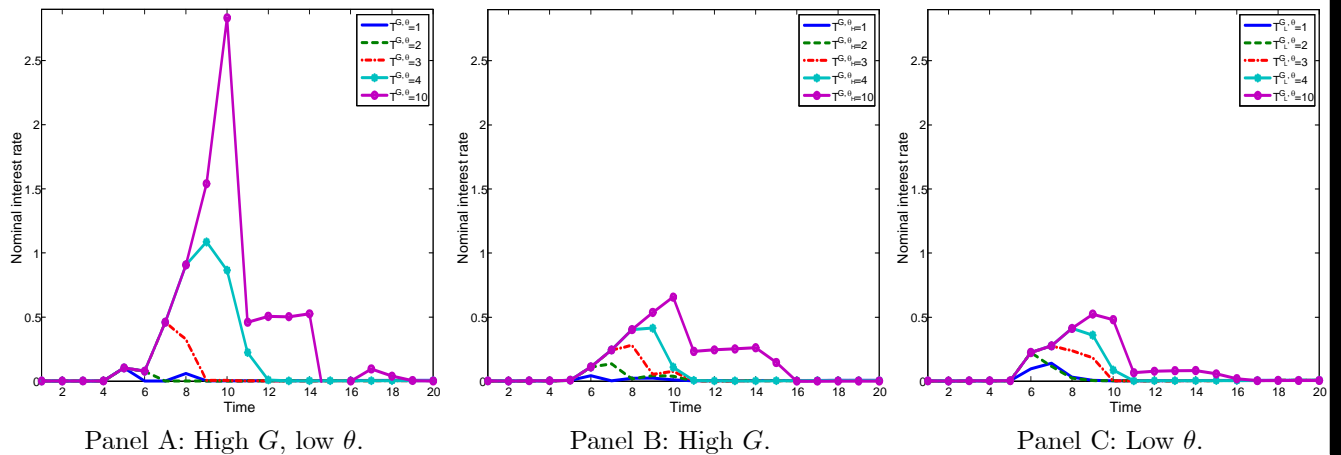


Higher nominal interest rates during the course of an adverse fiscal spell deliver capital losses to long term debt holders at the beginning of the spell and contribute to fiscal hedging. The particular pattern of interest rates across time and states reflects postponement and efforts to hedge shocks in multiple periods. The highest nominal interest rates occur later in the term of the debt so postponing their costs and after several consecutive high spending shocks when they can contribute to hedging in multiple periods.

Although the qualitative effects are similar across the panels, they are quantitatively larger in the first case, when the combination of high spending and low productivity contributes to a more severe adverse fiscal spell. Similar effects also occur in economies with shorter maximal debt maturities  $K = 3, \dots, 6$ , but they are quantitatively damped.

**Inflation** Figure 2 shows paths of realized and conditional expected inflation ( $E_{t-1} \left[ \frac{P_t}{P_{t-1}} \right]$ ) for adverse fiscal spells of length  $T^G = 10$ . As before, in the first panel high spending and low productivity shocks are drawn between periods 5 and  $4 + T^G$ , in the second high spending (and high productivity) and in the third low productivity (and low spending).

Figure 2: Inflation



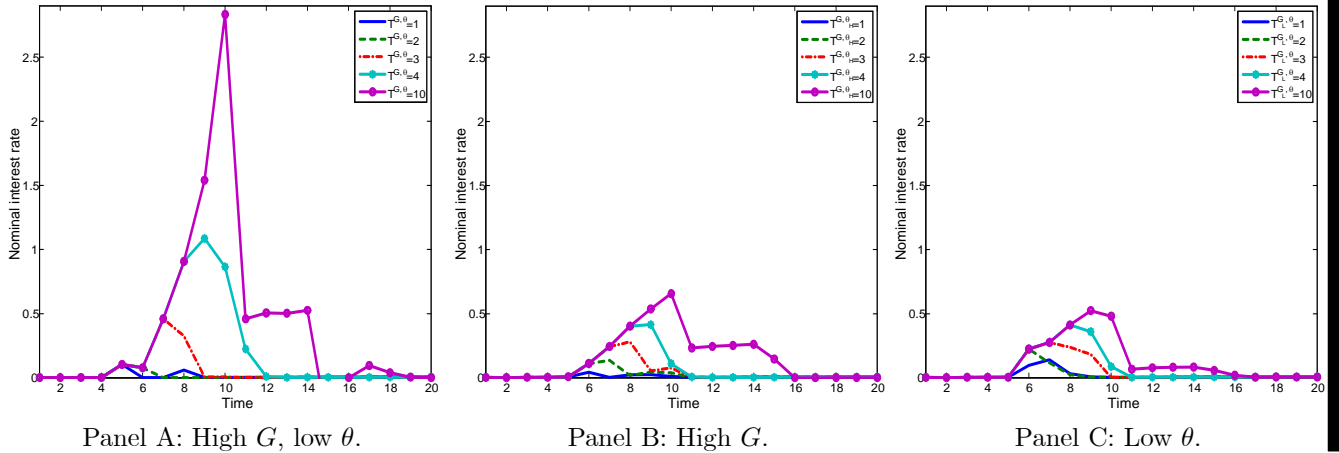
In each case, a small positive innovation in inflation occurs in period 5; this contributes to the devaluation of the government’s debt at the beginning of the adverse fiscal spell. It is followed by an increase in both realized *and* expected inflation over periods 7-10. The latter increase is associated with the rise in nominal

16  
17  
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1 interest rates at this time.<sup>11</sup> As the rise in nominal interest rates is attenuated after period 11, so too is that  
 2 in expected and realized inflation. When the adverse fiscal spell comes to an end, there is a small negative  
 3 inflation innovation.

4 **Holding returns** Collectively, the movements in nominal interest rates and inflation described above  
 5 facilitate the hedging of fiscal shocks by altering realized real holding returns. Figure 3 shows paths for  
 6 realized and conditional expected real holding returns on the government’s debt portfolio for each case and  
 7  $T^G = 10$ . When a low government spending shock persists realized real holding returns are slightly below  
 8 their expected level, when a high shock persists realized real holding returns are slightly above this level,  
 9 indicating a moderate degree of hedging at these times. More significant adjustments in realized holding  
 10 returns coincide with transitions from low to high and high to low government spending shock states. GIVE  
 11 NUMBERS.. the realized holding return falls from 4.1% to 2.4% at the onset of the high government  
 12 spending spell. The effect of this is to reduce the real value of the government’s liabilities by about 0.7% of  
 13 GDP. In the  $T^G = 1$  (resp.  $T^G = 10$ ) case, the realized real holding return jumps to 5.3% (resp. 5.8%) on  
 14 the reverse transition.

15 **Figure 3: Holding returns**

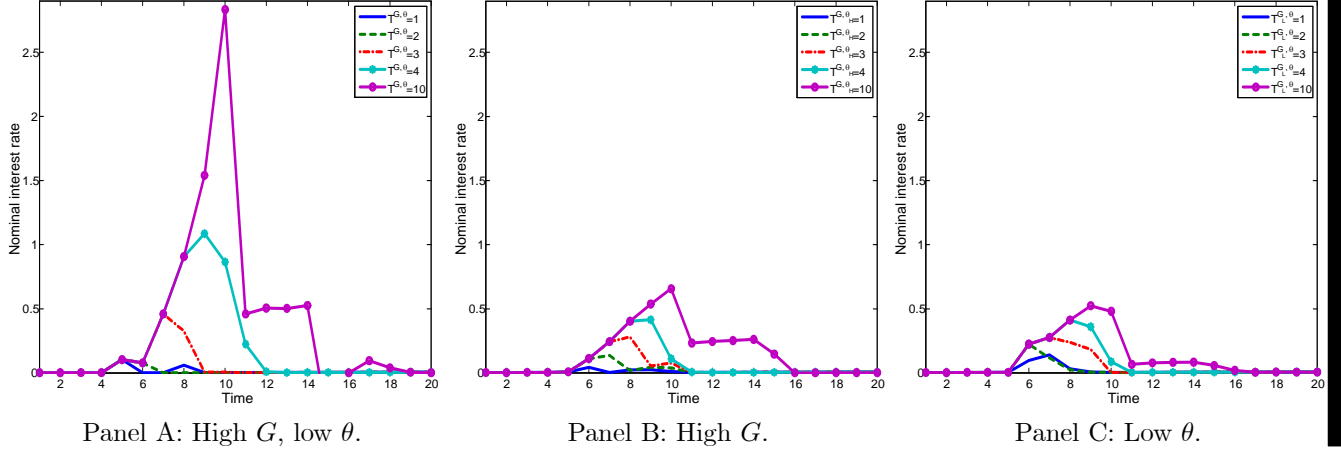


16

17 **Yield curves** Figure 4 plots the evolution of the yield curve as the economy is hit by a series of high  
 18 spending shocks. Initially, at  $t = 4$ , government spending is low and the yield curve (dotted line) is fairly flat  
 19 and close to zero. With the realization of the first high spending shock at  $t = 5$ , nominal interest rates rise  
 20 at all maturities (solid line). Consistent with the pattern of short run nominal interest rates and our earlier  
 21 simple example, the increase is greatest at the longest outstanding maturity  $K - 1 = 6$ . Thus, the yield  
 22 curve is hump shaped, tilting upwards over maturities  $k = 1$  to  $K - 1$  and downwards from  $K - 1$  onwards.  
 23 Over periods 6 to 10, the hump rises and passes to lower maturities. Once all initially outstanding debt has  
 24 matured in period 11, the yield curve falls back to a lower level and adopts a flatter shape (solid-circle line).

25 **Figure 4: Yield curves**

<sup>11</sup>A risk-adjusted Fisher equation holds:  $\beta E_t[U_{1t+1}/U_{1t}] E_t[P_t/P_{t+1}] Q_t^1 + \beta \text{Cov}_t[U_{1t+1}/U_{1t}, P_t/P_{t+1}] = 1$ .



7.2.3. Long simulations

In this section we report results from long simulations of various economies. Each simulation is of length  $T_{\text{sample}} = 20,000$ .

**The extent of fiscal hedging** Let  $\Delta VB_t$  denote the variation in the real value of the government’s portfolio across shock states at date  $t$ ,  $\Delta VB_t = \sum_{k=1}^K \left[ \frac{Q_t^{k-1}(\underline{G})}{P_t(\underline{G})} - \frac{Q_t^{k-1}(\bar{G})}{P_t(\bar{G})} \right] B_t^k$ . Table 1 shows long simulation sample averages of  $\Delta VB_t$  for economies with  $K = 1$  and  $K = 7$  under two different normalizations. The first normalization provides a measure of the extent of fiscal hedging relative to the size of the economy, variations in portfolio values are divided by the conditional expectation of output. The associated sample moments,  $\Delta VB/Y := \sum_{t=0}^{T_{\text{sample}}} \frac{\Delta VB_t}{E_{t-1}[Y_t]}$ , are given in the first row of the table with the next two rows breaking these variations down into components that come from nominal capital losses and from contemporaneous inflations. The second normalization gives a measure of the extent of fiscal hedging relative to the optimal amount in a complete markets economy with the same primary surplus value.<sup>12</sup> We denote sample moments under this normalization by  $\Delta VB/\Delta VB^C$ .

The results in Table 1 may be summarized as follows. First, our measures of fiscal hedging increase 5-fold as the maximal debt maturity rises from 1 to 7. When  $K = 1$ , variations in portfolio values are, on average, about 0.4% of GDP; when  $K = 7$ , these variations average about 2.1% of GDP. Second, as  $K$  rises the extent to which fiscal hedging is obtained from movements in debt prices rather than contemporaneous inflations increases. When  $K = 1$ , all hedging must necessarily come from contemporaneous innovations in the price level; when  $K$  rises to 7, over 80% of the variation in average portfolio values comes from changes in debt prices. Finally, the amount of fiscal hedging relative to the complete markets economy is quite small when  $K = 1$  ( $\Delta VB/\Delta VB^C = 4.7\%$ ), but is significantly greater when  $K = 7$ , ( $\Delta VB/\Delta VB^C = 24.4\%$ ).

Table 1: Financing Government Spending

	$K = 1$	$K = 7$
$\Delta VB/Y$	0.393	2.17
change in inflation	0.393	0.386
change in price of debt	0.000	1.78
$\Delta VB/\Delta VB^C$	4.70	24.4

<sup>12</sup>More precisely, the long simulation of a nominal debt model generates a sequence of expected primary surplus values  $\{\phi_t\}_{t=1}^{T_{\text{sample}}}$ . These values serve as state variables in recursive formulations of both the nominal debt and the complete markets models. We use the policy functions from the latter to compute complete markets variations in real portfolio values at each  $\phi_t$  generated along the sample path of a nominal debt economy. These portfolio variations are then used to normalize those from the nominal debt economy.

1 **Variability and persistence of inflation and interest rates** Table 2 reports standard deviations,  
 2 autocorrelations and correlations with government spending shocks for inflation and nominal interest rates  
 3 from long simulations of economies with  $K = 1, 3$  and  $7$ . The table indicates that all of these statistics  
 4 are increasing in the maximal debt maturity. Thus, fiscal hedging in economies with higher maximal debt  
 5 maturities leads to inflation and interest rate processes that are more volatile, persistent and correlated with  
 6 spending shocks.<sup>13</sup>

7 Table 2: Statistics from Long Simulations

	$K = 1$	$K = 3$	$K = 7$		
<i>inflation</i>					
st. deviation	<b>0.227</b>	<b>0.290</b>	<b>0.334</b>		
autocorrelation	<b>0.315</b>	<b>0.430</b>	<b>0.655</b>		
correlation with G-shock	<b>0.167</b>	<b>0.369</b>	<b>0.589</b>		
<i>1-period nom. interest rate</i>					
st. deviation	<b>0.168</b>	<b>0.181</b>	<b>0.364</b>		
autocorrelation	<b>0.081</b>	<b>0.152</b>	<b>0.713</b>		
correlation with G-shock	<b>-0.396</b>	<b>0.420</b>	<b>0.509</b>		

9 **Welfare** Increasing the maximal feasible debt maturity  $K$  provides positive, but small increases in wel-  
 10 fare. Let  $(b_0, s)$  be an initial state for the Ramsey problem with  $K = 1$  and denote the corresponding  
 11 optimal household allocation by  $\{c_{1t}(b_0, s), c_{2t}(b_0, s), l_t(b_0, s)\}_{t=0}^{\infty}$ . Define  $W(b_0, s)$  to be the optimal payoff  
 12 to household in an economy with  $K = 7$  if the initial state is  $(b_0, s)$  and let  $\Delta(b_0, s)$  be such that  $W(b_0, s) =$   
 13  $E_s \sum_{t=0}^{\infty} \beta^t U((1 + \Delta(b_0, s))c_{1t}(b_0, s), (1 + \Delta(b_0, s))c_{2t}(b_0, s), l_t(b_0, s))$ , i.e.  $\Delta(b_0, s)$  is the proportional increase  
 14 in the  $K = 1$  optimal consumption allocation (at  $(b_0, s)$ ) necessary to yield the same payoff as in the  $K = 7$   
 15 economy.  $\Delta \times 100\%$  varies between 0.02% and 0.1% with larger values at higher initial debt values.

16 **8. Conclusion**

17 We have explored optimal debt management and taxation when the government is restricted to using non-  
 18 contingent nominal debt of various maturities and is limited in its ability to lend. We identify a postponement  
 19 motive for using long term debt and find that the government relies almost exclusively on the longest term  
 20 debt available in calibrated examples. Other contributors have argued that the use of long term debt may  
 21 raise the government’s financing costs or expose it to unnecessary risk. Their arguments have implicitly  
 22 treated inflation and/ or the yield curve as parameters. In our model, which endogenizes all prices, the  
 23 holding return on long term nominal debt is more volatile than that on short term debt. However, this  
 24 volatility is deliberate and is used to hedge fiscal shocks. Higher risk premia on long term debt are the  
 25 analogues of insurance premia paid by the government and are not per se a rationale for shortening the  
 26 maturity structure.

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<sup>13</sup>In LSY (2006), we report results from economies with more volatile shocks and with a lower elasticity of substitution between cash and credit goods. When shocks are more volatile, optimal policy implies more volatility and inflation and nominal interest rates and greater correlation between these variables and the shocks. When the cash-credit elasticity is reduced, distortions to the cash-credit good margin are less costly and the government is prepared to distort this margin more. Nominal interest rates are again more volatile and more correlated with shocks.

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13 **Appendix**

PROOF OF PROPOSITION 1

NECESSITY Suppose  $\{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^{\infty}$  is a competitive allocation at  $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$  with  $\forall i, t, c_{it} \geq 0$  and  $(1 - \rho)L_{ft} + \rho L_{st} \in [0, T]$ . We show that it satisfies the conditions in the proposition. There is no loss of generality in assuming that at the equilibrium bond prices households have no desire to borrow. We assume the existence of optimal Lagrange multipliers on the households' constraints. Let  $\mu_t(s^t)$  denote the multiplier on the household's period  $t$  cash-in-advance constraint and let  $\lambda_t(s^t)$  and  $\tilde{\lambda}_t(s^t)$  denote the multipliers on the liquidity and hedging round budget constraints. The first order conditions for consumption and labor supply are:

$$c_{1t} : \quad \{\tilde{\lambda}_t(s^t) + \mu_t(s^t)\}P_t(s^t) = U_{1t}(s^t) \quad (26)$$

$$c_{2t} : \quad \tilde{\lambda}_t(s^t)P_t(s^t) = U_{2t}(s^t) \quad (27)$$

$$l_t : \quad \tilde{\lambda}_t(s^t)(1 - \tau_t(s^t))\frac{\partial I_t}{\partial l_t}(s^t) = -U_{lt}(s^t). \quad (28)$$

The first order conditions for money and bonds are:

$$\tilde{M}_t : \quad \lambda_t(s^t) = \mu_t(s^t) + \tilde{\lambda}_t(s^t) \quad (29)$$

$$M_{t+1} : \quad \tilde{\lambda}_t(s^t) = \beta E_{s^t}[\lambda_{t+1}] \quad (30)$$

$$\tilde{B}_t^k : \quad Q_t^k(s^t)\lambda_t(s^t) = \tilde{Q}_t^k(s^t)\tilde{\lambda}_t(s^t) \quad (31)$$

$$B_{t+1}^k : \quad \tilde{Q}_t^k(s^t)\tilde{\lambda}_t(s^t) = \beta E_{s^t}[Q_{t+1}^{k-1}\lambda_{t+1}] \quad (32)$$

14 We also have the transversality condition:  $\lim_{t \rightarrow \infty} \beta^t E[\lambda_t \{\tilde{A}_t + \sum_{k=0}^{K-1} Q_t^k \tilde{B}_t^{k+1}\}] = 0$ . Combining (26) and (27), we  
 15 obtain:  $\frac{U_{1t}}{U_{2t}} = \frac{\tilde{\lambda}_t + \mu_t}{\lambda_t} \geq 1$ . This establishes (10). Adding the household's and the government's hedging round budget  
 16 constraints and using the definition of firm profits gives (11).

17 The first order condition from the final goods firm implies  $\frac{P_{it}}{P_t} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{\mu-1}{\mu}}$ ,  $i = f, s$ . Thus, we have

$$18 \quad P_{st}(s^{t-1}) = (Y_t(s^t)/Y_{st}(s^t))^{\frac{\mu-1}{\mu}} P_t(s^t) = (Y_{ft}(s^t)/Y_{st}(s^t))^{\frac{\mu-1}{\mu}} P_{ft}(s^t). \quad (33)$$

19 The flexible price firm's first order conditions gives  $P_{f,t}(s^t) = \frac{\mu}{\alpha} \frac{W_t(s^t)}{\theta(s^t)} L_{f,t}(s^t)^{1-\alpha}$ . Combining this with (33) we  
 20 obtain  $P_{st}(s^{t-1}) = \left(\frac{Y_{ft}(s^t)}{Y_{st}(s^t)}\right)^{\frac{\mu-1}{\mu}} \frac{\mu}{\alpha} \frac{W_t(s^t)}{\theta(s^t)} L_{f,t}(s^t)^{1-\alpha}$ . As in Siu (2004), this last expression, the first order condition  
 21 from the sticky price firm's problem and the household's first order conditions imply (12).

Next take the household's hedging round budget constraint at  $t$ , multiply it by  $\tilde{\lambda}_t(s^t)$ , add  $\mu_t(s^t)\tilde{M}_t(s^t)$  and use the household's first order conditions to obtain:

$$\begin{aligned} \tilde{\lambda}_t(s^t) \sum_{k=1}^K \tilde{Q}_t^k(s^t) \tilde{B}_t^k(s^t) + \{\tilde{\lambda}_t(s^t) + \mu_t(s^t)\} \tilde{M}_t(s^t) &= U_{1t}(s^t) c_{1t}(s^t) + U_{2t}(s^t) c_{2t}(s^t) + U_{lt}(s^t) I_t(s^t) / W_t(s^t) \\ &+ \beta E_{s^t} \left[ \tilde{\lambda}_{t+1}(s^{t+1}) \sum_{k=1}^K \tilde{Q}_{t+1}^k(s^{t+1}) \tilde{B}_{t+1}^k(s^{t+1}) + \{\tilde{\lambda}_{t+1}(s^{t+1}) + \mu_{t+1}(s^{t+1})\} \tilde{M}_{t+1}(s^{t+1}) \right]. \end{aligned} \quad (34)$$

Using the expressions for profits from the intermediate goods firms problems,  $I_t(s^t)/W_t(s^t) = \Upsilon_t(s^t)$ . Iterating on (34) and using the household's first order and transversality conditions gives:  $[\sum_{k=1}^K Q_t^k(s^t) \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)} + \frac{\tilde{M}_t(s^t)}{P_t(s^t)}] = \xi_t(s^t)$ , where  $\xi_t(s^t)$  is defined as in the main text. The household's liquidity round budget constraint at  $t$  and the last equation imply:

$$A_t(s^{t-1})/P_t(s^t) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_t^{k+1}(s^{t-1})/P_t(s^t) = \xi_t(s^t). \quad (35)$$

Combining (33), the definition of  $N_t$  and the household's first order conditions gives  $\frac{P_t}{P_{t+1}} = \frac{1}{\beta} \frac{N_{t+1} U_{2t}}{E_t[N_{t+1} U_{1t+1}]}$ . Using this and the household's first order conditions again, we obtain:

$$Q_t^k = E_t \left[ \frac{U_{2t+k-1}}{U_{1t}} \prod_{j=0}^{k-2} \left\{ \frac{N_{t+j+1} U_{2t+j}}{E_{t+j}[N_{t+j+1} U_{1t+j+1}]} \right\} \right] = D_t^k. \quad (36)$$

Combining (33), (35), (36) and the definitions  $b_t^1(s^{t-1}) = A_t(s^{t-1})/P_{st}(s^{t-1})$  and,  $k > 1$ ,  $b_t^k(s^{t-1}) = B_t^k(s^{t-1})/P_{st}(s^{t-1})$ , we have the implementability constraints (8). The definitions of  $b_t^k$  and the fact that  $B_t^k \geq 0$  gives the no lending constraints. Finally, from (8), the non-negativity constraints on debt and the cash-in-advance constraint, we obtain:  $\xi_t(s^t) = \sum_{k=1}^K Q_t^k(s^t) \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)} + \frac{\tilde{M}_t(s^t)}{P_t(s^t)} \geq c_{1t}(s^t)$ , and, hence, (9).

**SUFFICIENCY** We construct a candidate competitive equilibrium from an allocation and a portfolio weight sequence satisfying the conditions in the proposition. First we set prices. At date 0,  $P_{s0}$  is a parameter, while  $P_0$  and  $P_{f0}$  are set to  $P_0(s^0) = P_{s0} (Y_{s0}(s^0)/Y_0(s^0))^{\frac{\mu-1}{\mu}}$  and  $P_{f0}(s^0) = (Y_{s0}(s^0)/Y_{f0}(s^0))^{\frac{\mu-1}{\mu}}$  respectively. For  $t > 0$ , set the relative sticky price to:

$$P_{st}/P_{t-1} = \beta/U_{2t-1} E_{t-1}[(Y_t/Y_{st})^{\frac{\mu-1}{\mu}} U_{1t}], \quad (37)$$

the gross (final goods) rate of inflation to:

$$P_t(s^t)/P_{t-1}(s^{t-1}) = P_{s,t}(s^{t-1})/P_{t-1}(s^{t-1}) (Y_{s,t}(s^t)/Y_t(s^t))^{\frac{\mu-1}{\mu}}. \quad (38)$$

and the flexible price to  $P_{ft}(s^t) = P_{st}(s^{t-1}) (Y_{st}(s^t)/Y_{ft}(s^t))^{\frac{\mu-1}{\mu}}$ . These conditions allow us to recursively recover all goods prices. For  $k > 0$  and  $t \geq 0$ , set the asset prices  $Q_t^k$  from the period  $t$  liquidity round budget constraint to:

$$Q_t^k(s^t) = D_t^k(s^t). \quad (39)$$

Also, for  $k > 0$  and  $t \geq 0$ , set the asset prices from the period  $t$  hedging round budget constraint to be  $\tilde{Q}_t^k(s^t) = \frac{U_{1t}(s^t)}{U_{2t}(s^t)} D_t^k(s^t)$ . For  $t > 0$ , we set the portfolios purchased by households in the hedging round as follows. The level of debt of  $k > 1$  maturity is fixed at  $B_t^k(s^{t-1}) = b_t^k(s^{t-1}) P_{st}(s^{t-1})$ . Using the no lending constraint,  $B_t^k(s^{t-1}) \geq 0$ . Also by this constraint,  $b_t^1(s^{t-1}) \geq 0$ , and we can choose  $M_t(s^{t-1}) \geq 0$  and  $B_t^1(s^{t-1}) \geq 0$  so that  $M_t(s^{t-1}) + B_t^1(s^{t-1}) = b_t^1(s^{t-1}) P_{st}(s^{t-1})$ . Let  $A_t(s^{t-1}) = b_t^1(s^{t-1}) P_{st}(s^{t-1})$ . Next we turn to the portfolios purchased in the liquidity round. For  $t \geq 0$ , the money supply is set to  $\tilde{M}_t(s^t) = P_t(s^t) c_{1t}(s^t)$ . From the implementability constraints (8), the above definitions of goods prices, asset prices and portfolios and the financial wealth-in-advance constraints (9), we have:  $\frac{A_t(s^{t-1})}{P_t(s^t)} + \sum_{k=1}^{K-1} Q_t^k(s^t) \frac{B_t^{k+1}(s^{t-1})}{P_t(s^t)} = \frac{\xi_t(s^t)}{U_{1t}(s^t)} \geq c_{1t}(s^t) = \frac{\tilde{M}_t(s^t)}{P_t(s^t)}$ . It follows that, at each date  $t$ , we can choose a non-negative debt portfolio  $\{\tilde{B}_t^k(s^t)\}_{k=1}^K \in \mathbb{R}_+^K$  so that the liquidity round budget constraints hold with equality.

1 Hence, the no lending, liquidity round budget and cash-in-advance constraints are satisfied. The government's debt  
2 holdings are set equal to the household's holdings of bonds.

3 We verify the household's first order conditions. Set the real wage to  $\frac{W_t}{P_t} = \frac{\alpha}{\mu} \theta_t L_{ft}^{\alpha-1} \left( \frac{Y_t}{Y_{ft}} \right)^{\frac{\mu-1}{\mu}}$ , the income tax  
4 rate to  $(1 - \tau_t) = -\frac{U_{1t}}{U_{2t}} \frac{P_t}{W_t}$  and the Lagrange multipliers to  $\lambda_t P_t = U_{1t} \geq 0$ ,  $\tilde{\lambda}_t P_t = U_{2t} \geq 0$  and  $\mu_t P_t = U_{1t} - U_{2t} \geq 0$ .  
5 It is then immediate that  $\lambda_t = \mu_t + \tilde{\lambda}_t$ ,  $\{\tilde{\lambda}_t + \mu_t\} P_t = U_{1t}$ ,  $\tilde{\lambda}_t P_t = U_{2t}$  and  $\tilde{\lambda}_t (1 - \tau_t) \partial I_t / \partial l_t = -U_{1t}$ . Also, (37) and  
6 (38) imply  $U_{2t} = \beta E_t \left( \frac{P_t}{P_{t+1}} U_{1t+1} \right)$ , so that  $\lambda_t = \beta E_{st} [\lambda_{t+1}]$ . Finally, the definitions of  $Q_t^k$ ,  $\tilde{Q}_{t+1}^k$  and the multipliers  
7 gives  $Q_t^k \lambda_t = \tilde{Q}_{t+1}^k \tilde{\lambda}_t$  and  $\tilde{Q}_{t+1}^k \tilde{\lambda}_t = \beta E_{st} [Q_{t+1}^{k-1} \lambda_{t+1}]$ .

8 Next we verify the household's hedging round budget constraints. Combining (8), (38) and (39) gives:

$$9 \quad \xi_t(s^t) = \left[ \frac{A_t(s^{t-1})}{P_t(s^t)} + \sum_{k=1}^{K-1} Q_t^k(s^t) \frac{B_t^{k+1}(s^{t-1})}{P_t(s^t)} \right]. \quad (40)$$

10 Hence, using the liquidity round budget constraint and the definitions of  $Q_t^k(s^t)$  and  $\tilde{Q}_{t+1}^k(s^t)$  and dividing by  $U_{2t}(s^t)$ ,  
11 we have  $\frac{U_{1t}(s^t) \xi_t(s^t)}{U_{2t}(s^t)} = \frac{U_{1t}(s^t)}{U_{2t}(s^t)} \frac{\tilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^K \tilde{Q}_t^k(s^t) \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)}$ . Adding  $\frac{U_{2t}(s^t) - U_{1t}(s^t)}{U_{2t}(s^t)} \frac{\tilde{M}_t(s^t)}{P_t(s^t)}$  to each side of this equation,  
12 using the definition of  $\xi_t(s^t)$  and  $\tau_t(s^t)$  and  $\frac{\tilde{M}_t(s^t)}{P_t(s^t)} = c_{1t}(s^t)$  yields  $\frac{\tilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^K \tilde{Q}_t^k(s^t) \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)} = c_{1t}(s^t) + c_{2t}(s^t) -$   
13  $(1 - \tau_t(s^t)) I_t(s^t) + \frac{\beta}{U_{2t}(s^t)} E_{st} [U_{1t+1} \xi_{t+1}]$ . Then, using (40) at  $t + 1$ , the definitions of  $Q_{t+1}^k$ ,  $\tilde{Q}_t^k$  and  $P_{t+1}$  and the  
14 condition  $U_{2t} - \beta E_{t+1} \left[ \frac{P_t}{P_{t+1}} U_{1t+1} \right] = 0$ , we obtain:

$$15 \quad \frac{\tilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^K \tilde{Q}_t^k(s^t) \frac{\tilde{B}_t^k(s^t)}{P_t(s^t)} = c_{1t}(s^t) + c_{2t}(s^t) - (1 - \tau_t(s^t)) I_t(s^t) + \frac{A_{t+1}(s^t)}{P_t(s^t)} + \sum_{k=2}^{K-1} \tilde{Q}_t^k(s^t) \frac{B_{t+1}^k(s^t)}{P_t(s^t)} \quad (41)$$

16 The hedging round budget constraint at  $t$  then follows from (41) and the definition of  $\tilde{A}_t(s^t)$ .

17 By (8) and the interiority of the allocation,  $U_{10} \xi_0$  is finite. Using the definition of  $\xi_t$ , we have for all  $T$ ,  $E[U_{10} \xi_0]$   
18  $= E[\sum_{t=0}^{\infty} \beta^t \{U_{1t} c_{1t} + U_{2t} c_{2t} + U_{1t} \Upsilon_t\}] = E[\sum_{t=0}^T \beta^t \{U_{1t} c_{1t} + U_{2t} c_{2t} + U_{1t} \Upsilon_t\}] + \beta^{T+1} E[U_{1T+1} \xi_{T+1}]$ . Taking lim-  
19 its and using the period  $T + 1$  measurability constraint then gives:  $\lim_{T \rightarrow \infty} \beta^{T+1} E[U_{1T+1} \xi_{T+1}] = \lim_{T \rightarrow \infty} \beta^{T+1}$   
20  $E[U_{1T+1} (\frac{A_{T+1}}{P_{T+1}} + \sum_{k=1}^{K-1} Q_{T+1}^k \frac{B_{T+1}^{k+1}}{P_{T+1}})] = 0$  which confirms the transversality condition. Hence, the allocation is feasi-  
21 ble and optimal for households at the derived prices and tax rates. The household's budget constraints, the resource  
22 constraint and the definitions of  $A_{gt}$  and  $B_{gt}$  ensure that the government's budget constraints are satisfied. It is easy  
23 to verify that the derived choices of firms satisfy their first order conditions and are optimal. ■

24 PROOF OF LEMMA 4: LSY (2006) show that a solution to the government's problem exists. Let  $\{b_1^{k*}\}_{k=1}^K$  denote  
25 an optimal portfolio in period 1. Since, for some  $s'$ ,  $\xi_1(s') \neq 0$ , (8) at  $t = 1$  implies  $b_1^{k*} > 0$  for some  $k$ . Let  $\hat{k}$  denote  
26 the smallest  $k$  such that for all  $k > \hat{k}$ ,  $b_1^{k*} = 0$ . Suppose  $\hat{k} < K$ . Then, for  $t \geq \max\{1, \hat{k}\}$ , the first order condition  
27 for  $c_{it}$  reduces to  $0 = U_{it} + \eta_t [U_{1it} - U_{2it}] - \varsigma_t$ , where  $\varsigma_t$  is the multiplier on the resource constraint. If  $U_{1t} > U_{2t}$ ,  
28 then  $\eta_t = 0$  and this first order condition implies that each  $U_{it} = \varsigma_t$ . Hence,  $U_{1t} = U_{2t}$ . It then follows from the  
29 implementability constraint that the optimal allocation can be implemented with a portfolio in which  $b_1^K = b_1^{k*}$  and  
30  $b_1^k = 0$ . All other portfolio weights remain the same.

31 The first order condition for  $b_1^k$  is:

$$32 \quad 0 = - \sum_{s \in S} \omega_1(s) N_1(s) \prod_{j=1}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) + \kappa^k, \quad (42)$$

33 where  $\kappa^k \geq 0$  is the multiplier on the corresponding no lending constraint. It follows from the argument in the  
34 previous paragraph, that  $\kappa^K = 0$ . Partition  $S$  into  $S' = \{s \in S | \omega_1(s) > 0\}$  and  $S'' = \{s \in S | \omega_1(s) \leq 0\}$ . (42) and  
35  $\kappa^K = 0$  then imply that either A) for all  $s$ ,  $\omega_1(s) = 0$  and  $S' = \emptyset$  or B) both  $S'$  and  $S''$  are non-empty. We assume  
36 the latter. (In fact, Case A holds only in knife edge cases). Suppose  $b_1^{k*} > 0$ , some  $k = 2, \dots, K - 1$ , then  $-\sum_{s \in S}$   
37  $\omega_1(s) N_1(s) \prod_{j=1}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) = 0$ . The combined first order condition for  $c_{1t}$  and  $c_{2t}$ ,  $t = 2, \dots, K - 1$ , (18) implies  
38 that  $U_{1t}(s) - U_{2t}(s) > 0$  iff  $\omega_1(s) > 0$ , and  $U_{1t}(s) - U_{2t}(s) = 0$  otherwise. Hence,  $-\sum_{s \in S} \omega_1(s) N_1(s) \prod_{j=1}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)}$   
39  $\prod_{j=k}^{K-1} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) > 0$ . But this contradicts (42) at  $k = K$ . Thus,  $b_1^{k*} = 0$  for  $k = 2, \dots, K - 1$ . By a similar  
40 argument  $b_1^{1*} = 0$  as well. ■

1      PROOF OF LEMMA 5 From the proof of Lemma 4, if  $\omega_1(s) \leq 0$  or  $t \geq K$ , then  $U_{1t}(s) = U_{2t}(s)$  and  $Q_t^1(s) = 1$ ; if  
 2  $\omega_1(s) > 0$  and  $t = 2, \dots, K-1$ , then  $U_{1t}(s) - U_{2t}(s) > 0$  and  $Q_t^1(s) < 1$ . Also,  $b_1^k = 0$ ,  $k = 1, \dots, K$  at the optimal  
 3 allocation. When  $\omega_1(s) > 0$  and  $t = 2, \dots, K-1$ , the first order conditions for  $c_{1t}(s)$  and  $c_{2t}(s)$  and the equalities  
 4  $U_{11t}(s)/U_{1t}(s) = -U_{1t}(s)/(1-\sigma)$ ,  $U_{22t}(s)/U_{2t}(s) = -U_{2t}(s)/\sigma$  imply

$$5 \quad \frac{\frac{U_{1t}(s)}{U_{2t}(s)} - 1}{\frac{\sigma}{1-\sigma} \frac{U_{1t}(s)}{U_{2t}(s)} + 1} = \beta^{-(t-1)} \frac{\omega_1(s)}{\sigma} b_1^K \prod_{j=1}^{K-1} \left[ \frac{U_{2j}(s)}{U_{1j}(s)} \right] N_1. \quad (43)$$

Since  $\beta \in (0, 1)$  and  $\omega_1 > 0$ , we have  $\frac{U_{1t+1}(s)}{U_{2t+1}(s)} > \frac{U_{1t}(s)}{U_{2t}(s)}$  and, hence,  $1 > Q_t^1(s) > Q_{t+1}^1(s)$ , for  $t = 2, \dots, K-2$ . If  
 $Q_1^1(s) = 1$ , we are finished. If not, the first order conditions for  $c_{11}(s)$  and  $c_{22}(s)$  and (43) give:

$$\frac{\frac{U_{11}(s)}{U_{21}(s)} - 1}{\frac{\sigma}{1-\sigma} \frac{U_{11}(s)}{U_{21}(s)} + 1} < \frac{\omega_1(s)}{\sigma} b_1^K \prod_{j=1}^{K-1} \left[ \frac{U_{2j}(s)}{U_{1j}(s)} \right] N_1 < \frac{\frac{U_{1t}(s)}{U_{2t}(s)} - 1}{\frac{\sigma}{1-\sigma} \frac{U_{1t}(s)}{U_{2t}(s)} + 1}, \quad t = 2, \dots, K-2.$$

6 Thus,  $Q_{t+1}^1(s) < Q_t^1(s)$  for  $t = 1, \dots, K-2$ . ■