Optimal Taxation with Endogenously Incomplete Debt Markets

Christopher Sleet
Department of Economics
University of Iowa
Iowa City, IA 52242, USA.

Şevin Yeltekin
MEDS, KSM
Northwestern University
Evanston, IL 60208, USA.

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Abstract
Empirical analyses of labor tax and public debt processes provide prima facie evidence for imperfect government insurance. This paper considers a model in which the government’s inability to commit to future policies or to report truthfully its spending needs renders government debt markets endogenously incomplete. A method for solving for optimal fiscal policy under these constraints is developed. Such policy is found to be intermediate between that implied by the complete insurance (Ramsey) model and a model with exogenously incomplete debt markets. In contrast to optimal Ramsey policy, optimal policy in this model is consistent with a variety of stylized fiscal policy facts such as the high persistence of labor tax rates and debt levels and the positive covariance between government spending and the value of government debt sales.


1 Introduction
This paper considers the optimal design of fiscal policy under two sets of restrictions. The first set is exogenous; it describes the technology by which the government can extract resources from agents. We follow the conventional Ramsey approach and suppose that resources can be obtained by levying linear taxes or selling state contingent debt. We also assume that the government cannot lend. The second set of restrictions stem from incentive problems on the side of the government which we assume can neither commit to repaying its debt nor to truthfully revealing private information about its spending needs. These frictions impede the government’s ability to use asset markets to hedge fiscal shocks. They endogenously restrict the set of asset trades the government can make and this, in turn, has implications for the optimal setting of taxes. To analyze fiscal

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policy design in such settings, we embed the government’s policy problem into a repeated game. We provide an equilibrium concept that extends Chari and Kehoe’s (1993a,b) sustainable equilibrium to environments with private government information. We then give necessary and sufficient conditions for an allocation to be an equilibrium outcome of this game. These conditions are recursive and we obtain a dynamic programming method for finding optimal equilibrium allocations that exploits this recursivity. We back out the supporting fiscal policies from these allocations and analyse optimal fiscal policy in this limited commitment-private information environment.

Our immediate motivation is a contrast between the benchmark Ramsey model of fiscal policy (as developed by Lucas and Stokey (1983)) and the data. The former implies that fiscal policy variables should depend only upon the current realization of the shocks perturbing the economy and, consequently, should inherit their stochastic properties from these shocks. In contrast, empirical evidence on labor tax rates and the public debt suggest that these variables exhibit considerable persistence, much more than that for government spending and other candidate shock processes.1 To paraphrase Aiyagari, Marcet, Sargent and Seppälä (AMSS) (2002), the empirical labor tax rate process is smooth in the sense of being highly persistent, rather than smooth in the sense of having a small variance.

The data are suggestive of considerable intertemporal, but limited interstate smoothing of taxes. Thus, they provide prima facie evidence for incomplete government insurance. The papers of AMSS, Marcet and Scott (2001) and Scott (1999), which assume exogenously incomplete government debt markets, corroborate this view and suggest that a limited ability to hedge against fiscal shocks may have significant implications for the design and conduct of fiscal policy. Given this, it becomes important to understand why this ability is limited and in what circumstances it might be more or less restricted.

Many commentators have informally suggested that moral hazard problems of one sort or another might underpin incomplete government insurance (e.g. Bohn (1990), Calvo and Guidotti (1990), Missale (1999)). The private information and limited commitment frictions that we incorporate into our model formalize these ideas. Both are linked to familiar time consistency considerations. The repayment of debt requires the levying of distortionary taxes. Ex post the government, and all households, would be better off if the debt were cancelled, but if such cancellation is anticipated ex ante, the government will be unable to sell any debt in the first place. Our model gives the government two channels via which it can avoid making debt repayments. The first is an outright repudiation of the debt. The second is more subtle; the government may exploit the private information it has over its spending needs and the state contingency of debt repayments to obtain a reduction in the latter. If, in order to smooth taxes, it has sold more claims against low relative to high spending needs states, the government can reduce its debt repayment by claiming its spending needs are high when they are really low.2

We call allocations that can be supported as equilibrium outcomes of our game “sustainable incentive-compatible competitive allocations” (SICCA’s). Our main focus is upon SICCA’s that are optimal from the

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1Marcet and Scott (2001) show that even after capital, which is absent from the Lucas and Stokey model, is incorporated the empirical processes for fiscal variables remain too persistent relative to those implied by the Ramsey model.

2The private information friction is less familiar than the limited commitment one. We provide an extended motivation for it in Section 3.
government’s point of view. We show that optimal SICCA’s are recursive in the value of the government’s debt. The limited commitment constraint translates into an upper bound on equilibrium debt values. Above this upper bound, the government can not be given incentives to repay its debt. It is the fiscal policy analogue of the endogenous solvency constraints that Alvarez and Jermann (2000) find in a model of households who are unable to commit ex ante to making debt repayments. There is also a lower debt value limit that stems from our assumption that the government cannot lend. In making this last assumption, we follow Chari and Kehoe (1993a). We elaborate on its role and its justification in Section 3.

Our recursive method allows us to jointly solve for the debt value limits and for the government’s optimal equilibrium payoff as a function of its current debt value. The method is related to the approach of Abreu, Pearce and Stacchetti (APS) (1990) and to the recent extensions of this approach to macroeconomic policy games provided by Chang (1998), Phelan and Stacchetti (2001) and Sleet (2001). We show that the government’s value function satisfies a Bellman equation on the set of debt values that lie between the limits. The policy functions from this dynamic programming problem can be used to recursively construct optimal allocations and, hence, optimal fiscal policy. This policy has the following characteristics. Away from the debt value limits, it exhibits considerable intertemporal tax smoothing, a moderate degree of state contingency in debt returns and considerable persistence in both taxes and debt. These features are consistent with the empirical analyses of Bizer and Durlauf (1990), Huang and Lin (1993) and Kingston (1987) (on taxes) and Marcet and Scott (2001) (on debt). Close to the limits, there is much more volatility in tax rates.

The limited commitment, no lending and incentive compatibility frictions interact in interesting ways. In a model with only the incentive compatibility friction, tax rates and the excess burden of taxation tend to drift upwards over time. Sleet (2003) shows that, under certain assumptions on preferences, this drift continues until the government is maximally indebted. At this point, it maximizes and uses all of its tax revenues to service debt. The severity of this outcome raises natural questions about the government’s ability to commit to implementing it ex ante. The endogenous upper debt value limit that stems from the commitment friction arrests the drift before this severe outcome is attained. Moreover, both the limited commitment and no lending frictions aggravate the incentive problem, especially when the government’s debt value is close to the debt value limits. This contributes to the greater tax rate volatility in these regions.

In a model with only the no lending and commitment frictions, such as that of Chari and Kehoe (1993a), debt value limits are also present. As Chari and Kehoe show, there is a reduced scope for fiscal hedging and a motive for intertemporal smoothing of taxes in the neighborhood of these limits. The addition of the incentive constraints further restricts the government’s ability to hedge fiscal shocks and creates a motive for intertemporal tax smoothing across the whole debt value domain. Overall, numerical calculations indicate that the three frictions result in a constrained optimal fiscal policy with properties somewhere between those implied by the Ramsey model and a model with non-contingent debt and exogenously set debt limits. Corroborative evidence reported by Marcet and Scott (2001) suggests that empirical fiscal policies are also somewhere between these benchmarks.

Our model is related to two recent contributions, Athey, Atkeson and Kehoe (AAK) (2003) and Sleet (2001), that have considered optimal monetary policy under private government information. In AAK’s model the private information concerns the government’s preferences for inflation and the government implements
policy contingent on the history of reports that it has made concerning its attitude towards inflation. A key result of AAK is that optimal monetary policy is in fact static and does not respond to past reports. Although our model shares some of the same structure as AAK’s, we find that optimal fiscal policy is not static.

Finally, Angeletos (2002) presents an alternative decentralization of benchmark Ramsey allocations that relies upon non-contingent debt of varied maturities and a state contingent fiscal policy. It is important to emphasize that even though explicitly state contingent debt is absent from this model, the decentralization proposed by Angeletos does not immunize the government from the frictions analyzed in this paper. In particular, if a government is privately informed about its spending needs, implementation of the Ramsey allocation under this alternative arrangement would still require it to condition its policies on this information. This would create an opportunity and an incentive for such a government to misrepresent its spending needs in order to justify alternative policy actions.

The outline for the remainder of the paper is as follows. Section 2 describes the benchmark environment with full commitment and without private information. Section 3 then introduces the incentive compatibility, limited commitment and no lending frictions and reformulates the model in game-theoretic terms. The next section provides a recursive formulation of the optimal SICCA that is amenable to computation. In particular, this section shows how limited commitment constraints can be recast as debt value limits. Section 5 gives a partial theoretical characterization of optimal SICCA’s, while Section 6 provides illustrative numerical calculations.

2 The benchmark environment

The benchmark Ramsey environment is characterized by complete information and full commitment on the part of the government. The economy is inhabited by a government and a continuum of identical households, all assumed to be infinitely lived. Taste shocks to the government’s and, under some interpretations, society’s preference for public goods are the underlying source of uncertainty. Denote the associated shock process by $\theta^\infty = \{\theta_t\}_{t=0}^\infty$, with each $\theta_t \in \Theta \equiv \{\hat{\theta}_1, \cdots, \hat{\theta}_N\}$, $\hat{\theta}_{i+1} > \hat{\theta}_i$. Assume that each $\theta_t$ is distributed i.i.d. with probability distribution $P = \{P(\hat{\theta}_i)\}_{i=1}^N \in \mathbb{R}_+^N$. Denote histories of shocks $(\theta_0, \cdots, \theta_t) \in \Theta_{t+1}$ by $\theta^t$. A household allocation gives the consumption $c_t$, and labor supply $l_t$ of a household at each $t$ and conditional on all possible $\theta^t$. An allocation augments this with a sequence of functions $g^\infty = \{g_t\}_{t=0}^\infty$ that gives government spending at each date and after all histories.

Definition 1 A household allocation is a collection of functions $e^{h\infty} = \{c_t, l_t\}_{t=0}^\infty$ with, for each $t$, $c_t : \Theta^t \to \mathbb{R}_+$ and $l_t : \Theta^t \to [0, T]$. A household allocation is interior if for all $\theta^t$, $c_t(\theta^t) > 0$ and $l_t(\theta^t) \in (0, T)$.

An allocation is a collection of functions $e^{\infty} = \{c_t, l_t, g_t\}_{t=0}^\infty$ with $\{c_t, l_t\}_{t=0}^\infty$ a household allocation and for each $t$, $g_t : \Theta^t \to \mathbb{R}_+$. An allocation $\{c_t, l_t, g_t\}_{t=0}^\infty$ is interior if $\{c_t, l_t\}_{t=0}^\infty$ is an interior household allocation. Let $\Xi$ denote the set of interior allocations.
Households value household allocations according to $V^h(e^{h\infty}) = E \left[ \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \right]$, with $\beta \in (0, 1)$, and $U(c, l) = u(c) + y(l)$, where

$$u(c) = \begin{cases} \frac{c^{1-p}}{1-p} & \rho \in (0, 1) \\ \ln c & \rho = 1 \end{cases}, \quad y(l) = \begin{cases} \frac{(T-l)^{1-\gamma}}{1-\gamma} & \gamma \in (0, 1) \\ \ln(T-l) & \gamma = 1 \end{cases}. \quad (1)$$

The arguments given below can be easily extended to the case $\gamma = 0$. To economize on space they are omitted. There is a linear production technology that converts one unit of labor supply into one unit of output. In each period, households can trade claims contingent on next period’s shock realization.

A fiscal policy is a collection of functions $x^{\infty} \equiv \{ s_t, S_t, \tau_t, q_t \}_{t=0}^{\infty}$. Here, $s_t : \Theta^t \mapsto [0, 1]$, $S_t : \Theta^t \mapsto \mathbb{R}_+$ and $\tau_t : \Theta^t \mapsto (-\infty, 1)$ denote, respectively, a tax on claim payouts, a lump sum transfer and a labor income tax set by the government at $t$ as a function of the history of shocks. $q_t : \Theta^{t+1} \mapsto \mathbb{R}_+$ is a pricing kernel, set by the government at $t$ after each $\theta^t$, for one period ahead claims contingent on $\theta^{t+1}$. The government supplies claims on demand at these prices. Let $Q^s_t = \prod_{s=1}^{t} [q_{t+s-1}/(1-s_{t+s})]$, $s \geq 1$, $Q^s_0 = 1$, denote the after-tax price of a unit of consumption at date $t+s$ in terms of date $t$ consumption. Let $b_{t+1}(\theta^t, \theta^{t+1})$ denote the quantity of $\theta_{t+1}$-contingent claims purchased by the household at $t$ after $\theta^t$, and define $a_t(\theta^t) \equiv (1-s_t(\theta^t))b_t(\theta^t) + S_t(\theta^t)$. Finally, define the household’s natural debt limit at $t$ after history $\theta^t$ as:

$$A_t(\theta^t) = \sum_{s=0}^{\infty} \sum_{\theta^s \in \Theta^s} Q^s_t(\theta^t, \theta^s) \left[ (1 - \tau_{t+s}(\theta^t, \theta^s))T + S_{t+s}(\theta^t, \theta^s) \right].$$

$A_t(\theta^t)$ gives the maximal amount a household can afford to repay after $\theta^t$. We assume that the fiscal policy satisfies $-(1-s_0(\theta^0))b_0(\theta^0) < A_0(\theta^0) < \infty$ and is such that $\{A_t\}_{t=0}^{\infty}$ is well defined and real-valued.

A household plan is a pair $\{e^{h\infty}, b^{\infty}\}$, where $b^{\infty} = \{b_{t+1}\}_{t=0}^{\infty}$ denotes a sequence of claim holdings. Given an initial portfolio of claims $b_0$ and a fiscal policy $x^{\infty}$, the household chooses such a plan to solve:

$$\sup_{\{e^{h\infty}, b^{\infty}\}} E \left[ \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \right] \quad (2)$$

subject to:

$$\forall t, \theta^t : \quad a_t(\theta^t) \geq c_t(\theta^t) - (1-\tau_t(\theta^t))l_t(\theta^t) + \sum_{\theta \in \Theta} q_t(\theta^t, \theta)b_{t+1}(\theta^t, \theta),$$

$$\forall t, \theta^t, \theta^{t+1} : \quad (1-s_t(\theta^t))b_{t+1}(\theta^t, \theta^{t+1}) \geq -A_{t+1}(\theta^t, \theta^{t+1}).$$

\textbf{Definition 2} A household plan $\{e^{h\infty}, b^{\infty}\}$, fiscal policy $x^{\infty}$, sequence of government spending functions $g^{\infty}$, and an initial household portfolio $b_0$ is a \textbf{competitive equilibrium}, if i) $\{e^{h\infty}, b^{\infty}\}$ solves the household’s problem given $x^{\infty}$ and $b_0$, ii) $x^{\infty}, \{e^{h\infty}, b^{\infty}\}$, and $g^{\infty}$ satisfy the government’s budget constraints\(^3\):

$$\forall t, \theta^t : a_t(\theta^t) + g_t(\theta^t) \leq \tau_t(\theta^t)l_t(\theta^t) + \sum_{\theta \in \Theta} q_t(\theta^t, \theta)b_{t+1}(\theta^t, \theta). \quad (3)$$

An allocation $e^{\infty}$ is said to be competitive if it is part of a competitive equilibrium.

\(^3\)We do not separately impose a No Ponzi condition on the government. Instead, we rely on the transversality condition from the household’s problem to ensure: $\lim_{t \to \infty} \sum_{\theta^{t+1} \in \Theta^{t+2}} Q_0(\theta^t, \theta^{t+1})b_{t+1}(\theta^t, \theta^{t+1}) \leq 0$. 

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Define $J(c, l) = \frac{\partial U}{\partial c}(c) c + \frac{\partial U}{\partial l}(c, l) l$ so that under the preferences in (1), $J(c, l) = c^{1-\rho} - \frac{l}{(1-\lambda)}$. In a competitive equilibrium, the budget constraints of the household and government and the household’s first order conditions imply $J(c_t, l_t) = c_t^{1-\rho}(c_t - (1 - \tau_t)l_t) = c_t^{1-\rho}(\tau_t l_t - g_t)$. Hence, $J(c_t, l_t)$ gives the government’s primary surplus in marginal utility of consumption terms. Let $J_c$ and $J_l$ denote the derivatives of $J$ with respect to consumption and labor supply.

2.1 Debt Values and Competitive Equilibria

Given a competitive equilibrium $\{e^{\infty}, b^{\infty}, x^{\infty}\}$, we can define the sequence of functions $\{\lambda_t\}_{t=0}^{\infty}$ pointwise as follows: $\lambda_0 = E[(1 - s_0)b_0c_0^{-\rho}]$ and for each $t \geq 0, \theta^t$, $\lambda_{t+1}(\theta^t) = E_{\theta^t}[(1 - s_{t+1})b_{t+1}c_{t+1}^{-\rho}]$. Thus, $\lambda_{t+1}(\theta^t)$ denotes the expected value of the household’s time $t + 1$ portfolio in marginal utility of consumption terms, conditional on information at $t$. We refer to $\lambda_t$ as a debt value. A particular competitive allocation may be part of many different competitive equilibria, each distinguished by a different sequence of debt holdings and lump sum transfers. This motivates the following definition.

Definition 3 A sequence of debt values $\{\lambda_t\}_{t=0}^{\infty}$, $\lambda_0 \in \mathbb{R}$ and, $t \geq 0, \lambda_{t+1} : \Theta^t \to \mathbb{R}$, is consistent with a competitive allocation $e^{\infty} = \{c_t, l_t, g_t\}_{t=0}^{\infty} \in \Xi$ if there exists a competitive equilibrium $\{e^{\infty}, b^{\infty}, x^{\infty}\}$ with taxes on claim payouts $\{s_t\}_{t=0}^{\infty}$ such that $\lambda_0 = E[(1 - s_0)b_0c_0^{-\rho}]$ and $\forall t \geq 0, \theta^t$, $\lambda_{t+1}(\theta^t) = E_{\theta^t}[(1 - s_{t+1})b_{t+1}c_{t+1}^{-\rho}]$.

The next proposition provides necessary and sufficient conditions for an interior allocation to be competitive. It formalizes the link between competitive allocations and debt values.

Proposition 1 $e^{\infty} = \{c_t, l_t, g_t\}_{t=0}^{\infty} \in \Xi$ is a competitive allocation if and only if there exists a sequence $\lambda^{\infty} = \{\lambda_t\}_{t=0}^{\infty}$ with $\lambda_0 \in \mathbb{R}$, and, for $t \geq 0, \lambda_{t+1} : \Theta^t \to \mathbb{R}$, such that $e^{\infty}$ and $\lambda^{\infty}$ satisfy:

1. (Recursive Implementability) $\lambda_0 \leq E[J(c_0, l_0) + \beta \lambda_1]$, and $\forall t \geq 1, \theta^{t-1}$

\[ \lambda_t(\theta^{t-1}) \leq E_{\theta^{t-1}}[J(c_t, l_t) + \beta \lambda_{t+1}], \]

\[ \lim_{t \to \infty} \beta^t E_{\theta^{t-1}}[\lambda_{t+1}] = 0, \text{ and} \]

2. (Resource constraints) $\forall t, \theta^t$, $l_t(\theta^t) - c_t(\theta^t) - g_t(\theta^t) \geq 0$.

Moreover, $\lambda^{\infty}$ is a consistent sequence of debt values for $e^{\infty}$.


Equation 4 provides a recursive version of the standard Ramsey implementability condition (see Chari and Kehoe (1999)). It is obtained by substituting prices and taxes out of the household’s budget constraint using the first order conditions from the household’s choice problem. Additionally, the inclusion of lump sum transfers relaxes (4) to an inequality. Given an allocation satisfying the conditions of the proposition, the household’s first order conditions can be used to pin down labor tax rates and after-tax asset prices. Other variables, in particular, debt holdings and lump sum transfers, are not uniquely determined.

\[ 4 \text{This relaxation ensures that the set of competitive allocations with a fixed initial debt value is convex, a fact used later.} \]
Let $\mathcal{E}(\lambda)$ denote the set of (interior) competitive allocations that are part of a competitive equilibrium with initial debt value equal to $\lambda$. Proposition 1 implies that competitive allocations are recursive. Specifically, if \( \{\lambda_t\}_{t=0}^{\infty} \) is consistent with \( e^\infty \in \Xi \), then after any \( \theta^t \), the continuation of \( e^\infty \), \( e^\infty|\theta^t = \{c_{t+r}(\theta^t, \cdot), l_{t+r}(\theta^t, \cdot), g_{t+r}(\theta^t, \cdot)\}_{r=1}^{\infty} \), is itself a competitive allocation belonging to \( \mathcal{E}(\lambda_{t+1}(\theta^t)) \). The following lemma establishes the existence of a natural debt value limit, \( \lambda^{\text{nat}} \): an upper bound on debt values. Later on we will impose additional frictions that tighten this limit.

**Lemma 1** Let \( \{\lambda_t\}_{t=0}^{\infty} \) be a sequence of debt values consistent with some competitive allocation, then

$$\forall t, \theta^t : \lambda_t(\theta^t) \leq \lambda^{\text{nat}} \equiv \sup_{l \in [0, T]} \frac{1}{1-\beta} J(l, l) < \infty.$$ 

For all $\lambda \in (-\infty, \lambda^{\text{nat}}]$, \( \mathcal{E}(\lambda) \neq \emptyset \).

**Proof:** See Sleet and Yeltekin (2004).

Define $\Gamma^C(\lambda)$ equal to all \( \{c, l, g, \lambda'\} \) such that:

$$\lambda \leq \sum_{\theta \in \Theta} [J(c(\theta), l(\theta)) + \beta \lambda'(\theta)];$$

$$l(\theta) - c(\theta) - g(\theta) \geq 0;$$

$$c(\theta) \geq 0, \quad g(\theta) \geq 0, \quad l(\theta) \in [0, T], \quad \lambda'(\theta) \in (-\infty, \lambda^{\text{nat}}].$$

Then, it is immediate that any competitive allocation \( \{c_t, l_t, g_t\}_{t=0}^{\infty} \) satisfies, for all $t$, $\theta^{t-1}$ and consistent debt sequence \( \{\lambda_t\}_{t=0}^{\infty}, \{c_t(\theta^{t-1}, \cdot), l_t(\theta^{t-1}, \cdot), g_t(\theta^{t-1}, \cdot), \lambda_{t+1}(\theta^{t-1}, \cdot)\} \in \Gamma^C(\lambda_t(\theta^{t-1})) \). Conversely, any allocation constructed by choosing $\lambda_0 \in (-\infty, \lambda^{\text{nat}}]$ and then successively selecting \( \{c_t(\theta^{t-1}, \cdot), l_t(\theta^{t-1}, \cdot), g_t(\theta^{t-1}, \cdot), \lambda_{t+1}(\theta^{t-1}, \cdot)\} \in \Gamma^C(\lambda_t(\theta^{t-1})) \) is a competitive allocation.

### 2.2 Government Preferences and the Ramsey Solution

The government values allocations according to:

$$V^g(e^\infty) = E \left[ \sum_{t=0}^{\infty} \beta^t W(c_t, l_t, g_t; \theta_t) \right], \quad \text{where} \quad W(c_t, l_t, g_t; \theta_t) = U(c_t, l_t) + \theta_t \frac{g_t}{1-\eta}^{1-\eta}, \quad (5)$$

and $\eta > 0$. As in many analyses of optimal taxation, the welfare of the representative household enters the government’s preferences. The second term in the above expression for $W$ may be interpreted as the utility private agents receive from the provision of public goods, or as the utility the government receives from its own consumption. In either case, the taste shock $\theta_t$ alters the weight placed on this second term in the government’s preferences. In this paper, we study policy that is optimal for the government. However, under the first interpretation, it is optimal for the household as well.

We assume that the government receives an initial debt value $\lambda_0$ and then selects a competitive allocation from $\mathcal{E}(\lambda_0)$ to solve:

$$V^R(\lambda_0) = \sup_{e^\infty \in \mathcal{E}(\lambda_0)} V^g(e^\infty).$$
We call this the Ramsey problem. To facilitate comparison with later sections, we provide a brief overview of this problem and of its solution. Proofs of the results given below follow from arguments in Zhu (1992), Chari and Kehoe (1999) and Marcet and Scott (2001) and are omitted.

Applying standard arguments, the Ramsey problem can be reformulated recursively. Specifically, $V^R : (-\infty, \lambda^\text{inf}] \to \mathbb{R}$ is the unique bounded, continuous solution to the functional equation:

$$V^R(\lambda) = \sup_{\{c,t,g,\lambda\}} \sum_{\theta \in \Theta} \left[ W(c(\theta),l(\theta),g(\theta);\theta) + \beta V^R(\lambda'(\theta)) \right] P(\theta).$$

(6)

Let $\{c^R,t^R,g^R,\lambda^R\}$ denote the (unique) policy functions that solve the optimizations in (6). Given an initial debt value, $\lambda_0$, these policy functions induce a Ramsey allocation, $e^R(\lambda_0)$, according to $c^R(\theta^t) = c^R(\lambda^R_{t-1}(\theta^t-1),\theta_t)$, $t^R(\theta^t) = t^R(\lambda^R_{t-1}(\theta^t-1),\theta_t)$, $g^R(\theta^t) = g^R(\lambda^R_{t-1}(\theta^t-1),\theta_t)$, where $\lambda^R_{t-1}(\theta^t) = \lambda^R(\lambda^R_{t-1}(\theta^t-1),\theta_t)$ and $\lambda^R_{t-1}(\theta^t) = \lambda_0$. Let $\mu^R : [0, \lambda^\text{inf}] \to \mathbb{R}_+$ denote the Lagrange multiplier on the recursive implementability constraint in this problem. We call an allocation $\{c_t,l_t,g_t\}_{t=0}^\infty$ static if there exist functions $c : \Theta \to \mathbb{R}_+$, $l : \Theta \to [0,T]$ and $g : \Theta \to \mathbb{R}_+$ such that $\forall t, \theta^{t-1}, \theta, c_t(\theta^{t-1}, \theta) = c(\theta), l_t(\theta^{t-1}, \theta) = l(\theta)$, and $g_t(\theta^{t-1}, \theta) = g(\theta)$. Otherwise, we call it dynamic.

**Proposition 2** The Ramsey allocation is static. For all $\lambda, \theta, \lambda^R(\lambda, \theta) = \lambda$

To ensure that the Ramsey competitive equilibrium associated with $e^R(\lambda_0)$ is uniquely determined, we assume that for all $t, \theta^t$, $s_t(\theta^t) = 0$. The household’s first order conditions can then be used to obtain the remaining variables that comprise this competitive equilibrium as functions of $\{c^R_t\}$ and $\{l^R_t\}$. The static nature of the Ramsey allocation then implies that after each $\theta^t$, the tax rate $\tau_t(\theta^t)$, quantity of claims outstanding, $b_t(\theta^t)$ and revenues raised from new claim sales $\sum_{\theta} g_t(\theta^t, \theta) b_{t+1}(\theta^t, \theta)$ are given by functions $\tau^R(\lambda_0, \theta_t)$, $b^R(\lambda_0, \theta_t)$ and $d^R(\lambda_0, \theta_t)$ respectively. The following proposition characterizes the implications of the Ramsey allocation for tax policy and debt management.

**Proposition 3** The functions $\tau^R$, $b^R$ and $d^R$ satisfy the following conditions.

1. $\tau^R(\lambda_0, \theta) = \frac{\mu^R(\lambda_0)}{\mu^R(\lambda_0) + \phi(\lambda_0)}$, where $\phi(l) = 1 + \gamma l/(T - l)$, or one plus the Frisch elasticity of labor supply.

2. If $\tilde{\theta} < \hat{\theta}$, then $b^R(\lambda_0, \tilde{\theta}) > b^R(\lambda_0, \hat{\theta})$.

3. If $\tilde{\theta} < \hat{\theta}$, then $d^R(\lambda_0, \tilde{\theta}) > d^R(\lambda_0, \hat{\theta})$.

The conventional Ramsey problem fixes $b_0$ rather than $\lambda_0$. In subsequent periods, however, the government can be modelled as acting as if it has inherited a debt value from the past. In contrast, we treat the first period symmetrically with later periods and think of the government as committed to a $\lambda_0$. 

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The text seems to be excerpted from a larger document discussing economic models, specifically focusing on the Ramsey problem and its implications. It references previous works by Zhu (1992), Chari and Kehoe (1999), and Marcet and Scott (2001). The propositions outline conditions for the Ramsey allocation, particularly in terms of tax rates, claims outstanding, and revenues, and they emphasize the dynamic versus static nature of the allocation.
\( \mu^R(\lambda_t(\theta^{t-1})) \) gives the multiplier on the implementability constraint after history \( \theta^{t-1} \). It can be interpreted as the excess burden of taxation. Since \( \lambda_t \) remains constant at \( \lambda_0 \), \( \mu^R(\lambda_t(\theta^{t-1})) \) remains constant as well. This is the key sense in which Ramsey policy “smooths”; its implications for taxes are captured in (7). Since \( \mu^R \) is constant, taxes only vary in so far as the Frisch elasticity of labor supply, \( \phi(l) \), varies across states. In contrast, if the government operated under a balanced budget rule, it would face a sequence of static implementability constraints of the form: \( J(c_t(\theta^t), l_t(\theta^t)) = 0 \). The multipliers on these constraints would fluctuate across states of nature and would impart additional volatility to tax rates beyond that implied by variations in \( \phi(l) \).

The second and third items in Proposition 3 describe two key aspects of optimal debt management. First, the government sells fewer claims against high \( \theta \) states and more against low \( \theta \) states. In this way it hedges \( \theta \) risk. Second, during high spending periods, the price of claims is low and this reduces the amount the government raises from selling the portfolio \( \{b^R(\lambda_0, \cdot)\} \).

3 Environments with private information and limited commitment

We now consider altering the environment of the previous section by introducing three extra frictions. First, we assume that the government has a limited ability to commit to implementing future policies. Second, we suppose that the government privately observes the taste shocks \( \{\theta_t\} \). Finally, we assume that the government can not lend, so that in equilibrium, \( \forall t, \theta^t, b_t(\theta^t) \geq 0 \). We formalize the first two frictions below and show that they lead to endogenous restrictions on the set of asset trading strategies available to the government. The third friction represents a direct exogenous restriction on asset trades. Before proceeding, we motivate the less familiar private government information and no lending frictions.

Throughout the remainder of the paper, we use the assumption that the \( \{\theta_t\} \) shocks are observed only by the government as a simple way of capturing the more general idea that the government has private information concerning its spending needs.\(^6\) We have in mind one of two situations. In the first, the government extracts resources for its own ends and its spending needs are imperfectly observable. In the second, private households are heterogenous in their taste for public goods. They may be well informed about their own tastes or the appropriate level of public good provision in their locality, but less informed than the government about the average need, as represented by \( \theta \). Alternatively, private households with different attitudes towards public goods provision may not know the extent to which the government, both in the current and in future periods, weights their varied preferences. Its weighting scheme may be subject to publicly unobservable shocks.

There may be many other potential sources of private government information. Such sources might include privately observed shocks to the technology for producing public goods, private information about the desirability of public investments (e.g. in military hardware or in wars themselves) and, if taxes are set before spending needs shocks are realized, private forecasts of spending needs shocks. Our formulation of private

\(^6\)We assume private information concerning the government’s spending needs rather than government spending itself. Although it is reasonable to assume that the former is less observable than the latter, several recent papers have documented how governments seek to obscure their spending levels through various forms of creative accounting, suggesting that even spending levels may not be perfectly observed. See, in particular, Milesi-Ferretti (2004).
government information seems the simplest amongst these. However, we conjecture that the incorporation of these alternative sources of private government information would lead to results qualitatively similar to ours.

It is worth noting that there is a tradition of analyzing privately informed policy makers in the monetary policy literature. Many of the situations considered in this literature parallel those described above. Specifically, Athey, Atkeson and Kehoe (2003), Barro (1986) and Cuckierman and Meltzer (1986) consider environments in which a central bank has private information about its preferences, while Canzoneri (1985) and Sleet (2001) consider ones in which the government has private information about the state of the economy.

We regard the no lending friction as a simple, if strong, way of capturing the costs and distortions associated with enforcing the repayment of loans to the government. Charik and Kehoe (1993a) motivate such a friction by supposing that private household asset trades are anonymous so that claims held against them are unenforceable. More generally, we think that transfers from households to the government, whether loan repayments or taxes, should be treated symmetrically. Unless loans from the government are collateralized, then, like taxes, loan repayments should depend upon the household’s current labor income and, hence, will be distortionary. Technically, the no lending constraint will reduce the payoff associated with the worst equilibrium in the game we construct below. This, in turn, will allow a larger set of fiscal policies to be supported in equilibrium.

### 3.1 The Policy Game

To analyze fiscal policy in the presence of the frictions described above, we embed the government’s problem into a policy game. The timing of moves in the game is illustrated in Figure 1. The government enters period $t$ with a portfolio of bonds $b_t \in \mathbb{R}_+^N$. These bonds are claims to a unit of consumption contingent on a taste shock report that the government makes later in the period. Before any shocks are realized, the government selects a menu of taxes and debt prices, $\delta_t = \{s_t, \bar{t}_t, \bar{S}_t, \bar{q}_t\}$, and a menu of government spending levels, $\bar{g}_t$. Here $s_t : \Theta \rightarrow [0, 1]$, $\bar{t}_t : \Theta \rightarrow (-\infty, 1)$, $\bar{S}_t : \Theta \rightarrow \mathbb{R}_+$, $\bar{q}_t : \Theta \rightarrow \mathbb{R}_N^+$ and $\bar{g}_t : \Theta \rightarrow \mathbb{R}_+$ give the current tax on debt, labor tax rate, lump sum subsidy, vector of bond prices and government spending level respectively, all contingent on the report made by the government. We denote the set of possible tax and debt price menus by $X$ and refer to a pair $\{\bar{x}_t, \bar{y}_t\}$ as a fiscal menu.

Next, the government privately observes the current shock $\theta_t$ and makes a report $\bar{\delta}_t$ concerning its value. It then implements the appropriate item from the fiscal menu, $\{s_t(\bar{\delta}_t), \bar{t}_t(\bar{\delta}_t), \bar{S}_t(\bar{\delta}_t), \bar{q}_t(\bar{\delta}_t), \bar{g}_t(\bar{\delta}_t)\}$. Having observed $\{\bar{x}_t, \bar{y}_t\}$ and the report $\bar{\delta}_t$, private agents make their choices $c_t$, $l_t$ and their bond purchases, $b_{t+1}$. The government stands by ready to supply any non-negative quantity of bonds at the prices it has previously set. These timing assumptions incorporate a degree of commitment on the part of the government; once it has chosen a fiscal menu, it is committed to making a selection from this menu after observing $\theta$. Within a period, the menu gives the government some discretion in responding to shocks. However, the government is unable, at time $t$, to commit to implementing future fiscal menus or to making specific shock-contingent reports. In particular, entering a period, the government may be tempted to implement a new menu with higher taxes on outstanding debt (an observable default), or, having observed the state, alter its report and misrepresent its true taste shock so as to obtain a reduction in its outstanding liabilities (a hidden default).

Define a $t$-period public history, $h_t$, to be a list of past policy actions taken and reports made by the
government: $h_t = \{b_0, \{\tilde{x}_t, \tilde{g}_t, \tilde{\delta}_t\}_{t=0}^{t-1}\}$, $t \geq 1$ and $h_0 = \{b_0\}$. Given an initial portfolio $b_0 \in \mathbb{R}_+^N$, let $H_t = \{b_0\} \times (X \times \mathbb{R}_+^N \times \Theta)^t$ denote the set of $t$-period public histories. Note that an element of $H_t$ describes the history of the economy at the beginning of period $t$ before the government acts. Private agents take their actions after the government so that their period $t$ behavior is conditioned on $h_{t+1}$. A $t$-period private government history, $h_{t+1}^g \in H_{t+1} = H_t \times \Theta$, augments $h_t$ with the most recent shock realization, $\theta_t$.

A government strategy is a collection of functions, $\sigma^g = \{\sigma^g_t\}_{t=0}^{\infty}$, that describes the government’s policy actions after each $h_t$. $\sigma^g$ has three components: a pair of fiscal menu functions $x^g_t : H_t \rightarrow X$, and $g^g_t : H_t \rightarrow \mathbb{R}_+^N$, and a report function $\delta^g_t : H^g_t \rightarrow \Theta$, with $\sigma^g_t(h_t, \theta_t) = (x^g_t(h_t), g^g_t(h_t), \delta^g_t(h_t, \theta_t))$. Thus, we restrict attention to government strategies that condition on public histories and the most recent realization of the private shock, but not on the entire history of private shocks. Let $h_t(\sigma^g, \theta^{t-1})$ be the $t$-period public history induced by $\sigma^g$ and the sequence of shocks $\theta^{t-1}$. Any strategy, together with the process for shocks, induces a probability distribution over public histories. Let $E^{\sigma^g}$ denote the expectations operator associated with this probability distribution.

An allocation profile $\sigma^h = \{\sigma^h_t\}_{t=0}^{\infty}$, with $\sigma^h_t = \{c^h_t, l^h_t, b^h_{t+1}\}$, describes the consumption, labor supply and bond holding choices of households contingent on the realization of public histories. Thus, $c^h_t : H_{t+1} \rightarrow \mathbb{R}_+$, $l^h_t : H_{t+1} \rightarrow \mathbb{R}_+$ and $b^h_{t+1} : H_{t+1} \rightarrow \mathbb{R}_+^N$. Let $\sigma^h|h_t = \{x^h_{t+1}(h_t, \cdot), \delta^h_{t+1}(h_{t+1}, \cdot)\}_{t=0}^{\infty}$ and $\sigma^h|h_{t+1} = \{c^h_{t+1}(h_{t+1}, \cdot), l^h_{t+1}(h_{t+1}, \cdot), b^h_{t+1}(h_{t+1}, \cdot)\}_{t=0}^{\infty}$. A strategy-allocation profile $(\sigma^g, \sigma^h)$ and an initial $b_0$ induce an outcome allocation and an outcome fiscal policy.

**Definition 4** A sequence of functions $\{x_t, \delta_t, g_t, c_t, b_{t+1}\}_{t=0}^{\infty}$ is an outcome of a strategy-allocation profile, $\sigma = (\sigma^g, \sigma^h) = \{x^g_t, g^g_t, \delta^h_t, c^h_t, l^h_t, b^h_{t+1}\}_{t=0}^{\infty}$, and initial $b_0 \in \mathbb{R}_+^N$ if $h_0 = b_0$ and for each $\theta^t$,

1. $x_t(\theta^t) = x^g_t(h_t(\sigma^g, \theta^{t-1}), \delta^h_t(h_t(\sigma^g, \theta^{t-1}), \theta_t))$,
2. $g_t(\theta^t) = g^g_t(h_t(\sigma^g, \theta^{t-1}), \delta^h_t(h_t(\sigma^g, \theta^{t-1}), \theta_t))$,
3. $\delta_t(\theta^t) = \delta^h_t(h_t(\sigma^g, \theta^{t-1}), \theta_t)$,
4. $c_t(\theta^t) = c^h_t(h_{t+1}(\sigma^g, \theta^t))$.

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Figure 1: Time Line

Definition 4 A sequence of functions $\{x_t, \delta_t, g_t, c_t, b_{t+1}\}_{t=0}^{\infty}$ is an outcome of a strategy-allocation profile, $\sigma = (\sigma^g, \sigma^h) = \{x^g_t, g^g_t, \delta^h_t, c^h_t, l^h_t, b^h_{t+1}\}_{t=0}^{\infty}$, and initial $b_0 \in \mathbb{R}_+^N$ if $h_0 = b_0$ and for each $\theta^t$,

1. $x_t(\theta^t) = x^g_t(h_t(\sigma^g, \theta^{t-1}), \delta^h_t(h_t(\sigma^g, \theta^{t-1}), \theta_t))$,
2. $g_t(\theta^t) = g^g_t(h_t(\sigma^g, \theta^{t-1}), \delta^h_t(h_t(\sigma^g, \theta^{t-1}), \theta_t))$,
3. $\delta_t(\theta^t) = \delta^h_t(h_t(\sigma^g, \theta^{t-1}), \theta_t)$,
4. $c_t(\theta^t) = c^h_t(h_{t+1}(\sigma^g, \theta^t))$.

---

7In doing so we follow other contributions to the literature, e.g. Athey, Atkeson and Kehoe (2003). Fudenberg and Tirole (1991) show that for many environments, this assumption does not alter the set of equilibrium payoffs. The exclusion of past private agent actions from public histories is justified by arguments in Chari and Kehoe (1990).
5. \( l_t(\theta^t) = b_t^g(h_{t+1}(\sigma^g, \theta^t)) \),
6. \( b_{t+1}(\theta^t) = b_{t+1}^g(h_{t+1}(\sigma^g, \theta^t)) \).

Call the allocation \( \{c_t, l_t, g_t\}_{t=0}^{\infty} \) obtained in this way the **outcome allocation** of \( \sigma \). Call \( \{\delta_t\}_{t=0}^{\infty} \) the **outcome fiscal policy** of \( \sigma \) and \( \{\delta_t\}_{t=0}^{\infty} \) the **outcome reporting policy** of \( \sigma \). Finally, call \( \{c_t, l_t, b_{t+1}\}_{t=0}^{\infty} \) the **outcome household plan** of \( \sigma \).

We will refer throughout to a profile of functions \( \{\delta_t\}_{t=0}^{\infty}, \delta_t : \Theta^{t+1} \rightarrow \Theta \) as a reporting policy. Finally, let \( V_t^g(\sigma^g, \sigma^h|h_t) \) denote the payoff to a government from a pair \((\sigma^g, \sigma^h)\) after a history \( h_t \).

### 3.2 Sustainable Equilibria and Allocations

A sustainable equilibrium of the policy game is a strategy-allocation profile \((\sigma^g, \sigma^h)\) that is consistent with household optimality and market clearing, and a sequential rationality requirement on \( \sigma^g \). The latter necessitates that, after each history, the government is better off adhering to \( \sigma^g \) than deviating to some feasible alternative strategy. Consequently, before giving a formal statement of an equilibrium, it is necessary to define the set of deviations available to a government. Within a period there are two sorts of feasible deviation. At the beginning of the period, the government can implement a fiscal menu different from that prescribed by its strategy. Later, after receiving the shock, it can deviate by altering its report. It is always free to give any report it wishes. However, it may only deviate to budget feasible fiscal menus.

**Definition 5** A fiscal menu \((\bar{x}, \bar{g})\) is **budget feasible** at \( h_t \) under \( \sigma \) if for each \( \delta \in \Theta \),

\[
\bar{g}(\delta)l_t^h(h_t, \bar{x}, \bar{g}, \delta) + \sum_{\delta' \in \Theta} \tilde{q}(\delta, \delta')b_{t+1}^g(h_t, \bar{x}, \bar{g}, \delta, \delta') \geq \bar{g}(\delta) + (1 - \bar{s}(\delta))b_t^g(h_t, \delta) + \tilde{S}(\delta).
\]

**Definition 6** Given a strategy-allocation profile \( \sigma = (\sigma^g, \sigma^h) \), \( \bar{\sigma}^g \) is a **feasible deviation** for the government if for all \( h_t \), the fiscal menu induced by \( \bar{\sigma}^g \), \( (x_t^g(h_t), g_t^g(h_t)) \), is budget feasible at \( h_t \) under \( \sigma \).

Let \( x(\sigma^g)|h_{t+1} \) denote the outcome fiscal policy induced by a government strategy \( \sigma^g \) after \( h_{t+1} \). Similarly, let \( \{e^{h^\infty}, b^\infty\}(\sigma)|h_t \) denote the outcome household plan induced by a strategy-allocation profile \( \sigma \). The formal statement of a sustainable equilibrium now follows.

**Definition 7** A **sustainable equilibrium** at \( \lambda_0 \) is a triple \((h_0, \sigma^g, \sigma^h)\) with \( h_0 \in \mathbb{R}_+^N \), \( \sigma^g = \{x_t^g, g_t^g, \delta_t^g\}_{t=0}^{\infty} \) and \( \sigma^h = \{c_t^h, l_t^h, b_{t+1}^h\}_{t=0}^{\infty} \) that satisfies the following conditions.

1. **Competitive equilibrium.**

   (a) After all histories \( h_{t+1} = (h_t, \tilde{s}, \tilde{S}, \tilde{r}, \tilde{q}, \tilde{g}, \delta) \), given the household’s current claim holdings \( b_t^g(h_t) \), the current policy \( \{s_t, \tilde{S}, \tilde{r}, \tilde{q}, \tilde{g}, \delta\} \) and the continuation outcome fiscal policy \( x(\sigma^g)|h_{t+1} \), the outcome plan \( \{e^{h^\infty}, b^\infty\}(\sigma)|h_t \) maximizes the household’s utility subject to its budget constraints and debt holding limits.

   (b) The functions \( \{b_{t+1}^g\}_{t=0}^{\infty} \) satisfy, for all \( t \geq 0 \), \( h_{t+1}, \delta_{t+1}, b_{t+1}^g(h_{t+1}, \delta_{t+1}) \geq 0 \).
(c) At each $h_t$, $(x_t^g(h_t), g_t^g(h_t))$ is budget feasible.
(d) $\lambda_0 = E^{\sigma^g} \left[ (c_0^g)^{-\rho} (1 - s_0^g)b_0 \right].$

2. Government optimality. For all $h_t$ and all feasible deviations $\hat{\sigma}^g$ given $(\sigma^g, \sigma^h)$:

$$V_t^g(\sigma^g, \sigma^h| h_t) \geq V_t^g(\hat{\sigma}^g, \sigma^h| h_t).$$

Notice that (1a) in the Definition 7 requires that, after each $h_{t+1}$, households behave optimally given the strategy of the government. Without loss of generality, we assume that the government sets claim prices so that asset markets clear and households do not wish to choose negative quantities of claims (that the government is unable to supply). (1a) and (1c) then imply that the outcome allocation induced by $\sigma$ is competitive.

We are interested in the set of allocations that can be supported by sustainable equilibria. Therefore, we identify a collection of direct constraints on allocations that are necessary and sufficient for sustainability. To obtain this characterization, we begin by defining an autarkic equilibrium. For this definition let $\delta^*$ be the truthful report function: $\forall \theta, \delta^*(\theta) = \theta$.

**Definition 8** The autarkic menu $(x^{aut}, g^{aut})$ consists of the functions $s = 1$, $S = 0$, $q = 0$, $T^{aut} = 1 - \frac{(c^{aut})^\rho}{(1 - (T^{aut})^\gamma)}$ and $g^{aut} = I^{aut} - c^{aut}$, where $c^{aut}(\theta)$ and $I^{aut}(\theta)$ solve:

$$\max_{\{(c,l):\theta = c^{\hat{\lambda}}, \rho\}} W(c, l, l - c; \theta).$$

An autarkic equilibrium is a strategy-allocation profile $(\sigma^{g,aut}, \sigma^{h,aut})$ and an initial portfolio $b_0$ that satisfies:

1. for all $h_t$, $\sigma_t^{g,aut}(h_t) = (x^{aut}, g^{aut}, \hat{\delta}^*)$;
2. for all $(h_t, x, \theta)$, $\sigma_t^{h,aut}(h_t, x, \theta) = \{c(x, \theta), l(x, \theta), b'(x, \theta)\}$, where $b'(x, \theta) = 0$ and

$$(c(x, \theta), l(x, \theta)) = \arg \max_{\{(c,l):\theta = c^{\hat{\lambda}}\}} W(c, l, l - c; \theta).$$

Define $V^{aut}$ to be the payoff from repeatedly implementing the autarkic menu and being truthful: $V^{aut} = \frac{1}{1 - \rho} E[W(c^{aut}, I^{aut}, g^{aut}, \theta)]$. The following proposition establishes that the autarkic equilibrium is the worst sustainable equilibrium and, hence, $V^{aut}$ is the worst equilibrium payoff.\footnote{As in Chari and Kehoe (1993a), this result relies on the government’s inability to lend. When that ability is reinstated, the worst equilibrium payoff is less severe and less easy to solve for (see Chari and Kehoe, 1993b).} Subsequently, we use this result to obtain a necessary condition for an allocation to be the outcome of a sustainable equilibrium.

**Proposition 4** An autarkic equilibrium $(b_0, \sigma^{g,aut}, \sigma^{h,aut})$ is a sustainable equilibrium. It gives the lowest payoff amongst sustainable equilibria.

**Proof:** See Appendix 1. ■

Proposition 5 provides necessary and sufficient conditions for the sustainability of allocations. To state the proposition, let $\delta^i(\theta^i)$ denote the sequence of reports given during a shock history $\theta^i$ under the reporting policy
\( \delta^\infty = \{ \delta_t \}_{t=0}^\infty \). Define \( \delta^{\infty}_s = \{ \delta_t^s \}_{t=0}^\infty \) to be the truthful reporting policy, with \( \delta^s(\theta^t) = \theta^t \). Also, define the payoff from a (report-contingent) allocation \( \epsilon^\infty = \{ c_t, l_t, g_t \}_{t=0}^\infty \) and a reporting policy \( \delta^\infty \) to be:

\[
V(\epsilon^\infty, \delta^\infty) = \sum_{t=0}^\infty \sum_{\theta^t \in \Theta^s} \beta^t W(c_t(\delta^t(\theta^t)), l_t(\delta^t(\theta^t)), g_t(\delta^t(\theta^t)); \theta^t) P^t(\theta^t).
\]

Let \( V(\epsilon^\infty, \delta^\infty|\theta^t) \) denote the continuation payoff from the pair \( (\epsilon^\infty, \delta^\infty) \) after shock history \( \theta^t \).

**Proposition 5** \( \epsilon^\infty = \{ c_t, l_t, g_t \}_{t=0}^\infty \in \Xi \) is the outcome allocation of a sustainable equilibrium at \( \lambda_0 \) if and only if it satisfies the following conditions:

**A:** (Competitive allocation) \( \forall t, \theta^t : l_t(\theta^t) - c_t(\theta^t) - g_t(\theta^t) \geq 0, \) \( \lambda_0 \leq \sum_{t=0}^\infty \sum_{\theta^t \in \Theta^s} \beta^t J(c_t(\theta^t), l_t(\theta^t)), \) \( \lambda^s = \sup_{\lambda \in A^s} \lambda. \) Let \( A^s = \sup_{\lambda \in A^s} \lambda. \) Also, let \( W \) be the set of initial payoffs and debt values supported by some SICCA:

\[ W = \{ (v, \lambda) : \exists \epsilon^\infty \in \mathcal{E}^IC(\lambda) \text{ and } v = V^S(\epsilon^\infty) \}. \]

For \( \lambda_0 \in A^s \), let

\[ V^S(\lambda_0) = \sup_{\epsilon^\infty \in \mathcal{E}^IC(\lambda_0)} V^S(\epsilon^\infty). \]

The graph of \( V^S \) may be interpreted as the upper surface of \( W \) when \( W \) is drawn in \( (\lambda, v) \)-space. Finally, define \( \epsilon^\infty \) to be an optimal SICCA at \( \lambda_0 \in A^s \) if \( V^S(\epsilon^\infty) = V^S(\lambda_0) \). Throughout the remainder of the paper we focus on optimal SICCA’s.

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9We do not impose the requirement of interiority on SICCA’s (i.e. the requirement that \( \forall t c_t > 0 \) and \( l_t \in (0, T) \)). SICCA’s continue to satisfy the first order conditions of households. However, these are neither necessary nor sufficient for household optimality, when the household’s budget set has an empty interior. Thus, \( \mathcal{E}^IC(\lambda_0) \) may contain some “boundary” allocations that are not competitive; it may also exclude some boundary allocations that are competitive. However, our subsequent focus is on optimal SICCA’s. It is easy to check that the optimal SICCA’s that we solve for are interior competitive allocations.
4 The Recursive Structure of Optimal SICCA’s

We show that the continuations of optimal SICCA’s are themselves optimal and that they are recursive in the debt value $\lambda$. We also show that the pair $(V^S, \Lambda^S)$ solves a functional equation and provide an iterative scheme for computing it.\footnote{This approach is related to, but distinct from, that of Abreu, Pearce and Stacchetti (APS) (1990). APS methods have been applied to macro-policy games, primarily of a complete information nature, by Chang (1998), Fernández-Villaverde and Tsyvinski (2002), Phelan and Stacchetti (2001) and Sleet (2001).}

The result that an optimal allocation is recursive in $\lambda$ is not immediate in policy games with private information. The difficulty is that the provision of good incentives ex ante may require the adoption of a bad continuation SICCA ex post. More precisely, suppose that $\epsilon^\infty$ is an optimal SICCA at $\lambda_0$, $\{\lambda_t\}_{t=0}^\infty$ is a consistent sequence of debt values for $\epsilon^\infty$ and $\theta^t$ a sequence of high shock reports. Potentially, it may be desirable for the continuation allocation $\epsilon^\infty|\theta^t$ to satisfy $V^g(\epsilon^\infty|\theta^t) < V^S(\lambda_t(\theta^t))$. In this way, the government could be discouraged from lying and reporting the high shock sequence when the true realization of shocks was low. The fact that this is not the case implies that the sequence of continuation payoffs and debt values associated with an optimal SICCA remain on the upper surface of the set $W$. This in turn allows optimal SICCA’s to be reconstructed from the policy functions that solve a dynamic programming problem. In solving for an optimal SICCA we also exploit the fact that the worst equilibrium payoff available at each $\lambda$ is $V^{aut}$ and that this can be easily and independently computed.

We first introduce some additional concepts and notation. Let $\Lambda^R = [0, \bar{\Lambda}] = \{\lambda \geq 0 : V^R(\lambda) \geq V^{aut}\}$. Notice that since $V^R \geq V^S$, $\Lambda^S \subseteq \Lambda^R$. Next fix a candidate set of value functions and debt value upper bounds:

\[
Q = \{(V, \bar{\Lambda}) : \Lambda^S \subseteq [0, \bar{\Lambda}] \subseteq \Lambda^R, V : [0, \bar{\Lambda}] \to \mathbb{R} \text{ and is 1) continuous and concave},
\]

\[
2) V \geq V^S \text{ on } \Lambda^S, V \leq V^R \text{ on } [0, \bar{\Lambda}],
\]

\[
3) V(\bar{\Lambda}) = V^{aut}. \}
\]

Endow $Q$ with the partial ordering $\succeq$, where $(V_1, \bar{\Lambda}_1) \succeq (V_2, \bar{\Lambda}_2)$ if $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$ and $V_1(\lambda) \geq V_2(\lambda)$, $\lambda \in [0, \bar{\Lambda}_2]$. Let $\{(V_n, \bar{\Lambda}_n)\}_{n=1}^\infty$ be a sequence in $Q$. For each $\lambda$, define $\{n^\lambda_k\}_{k=1}^\infty$ to be the subsequence consisting of those $n$ such that $\lambda \in [0, \bar{\Lambda}_n]$. We say that the sequence is decreasing if $(V_n, \bar{\Lambda}_n) \succeq (V_{n+1}, \bar{\Lambda}_{n+1})$ for all $n \geq 1$. We say that the sequence converges to $(V_\infty, \bar{\Lambda}_\infty)$ if $\lim_{n \to \infty} V_n = V_\infty$ and, $\forall \lambda \in [0, \bar{\Lambda}_\infty]$, $\lim_{n \to \infty} V_{n^\lambda_k}(\lambda) = V_\infty(\lambda)$.

Let $\Gamma_{V, \bar{\Lambda}}(\lambda)$ equal all tuples $\{c, l, g, \lambda', v'\}$ that satisfy the following constraints:

\[
\lambda \leq \sum_{\theta \in \Theta} [J(c(\theta), l(\theta)) + \beta \lambda'(\theta)] P(\theta),
\]

\[
\forall \theta, \ l(\theta) - c(\theta) - g(\theta) \geq 0,
\]

\[
\forall \theta, \ J(c(\theta), l(\theta)) + \beta \lambda'(\theta) \geq 0,
\]

\[
\forall (\theta, \hat{\theta}), \ W(c(\theta), l(\theta), g(\theta); \theta) + \beta v'(\theta) \geq W(c(\hat{\theta}), l(\hat{\theta}), g(\hat{\theta}); \theta) + \beta v'(\hat{\theta}),
\]
∀θ, \( v'(\theta) \in [V^{aut}, V(\lambda'(\theta))] \), \hspace{1cm} (22)
\[ \lambda'(\theta) \in [0, \lambda]. \] \hspace{1cm} (23)

These constraints have the following interpretations. Inequality (18) is a restatement of the recursive implementability constraint (see (4)), while (19) is the resource constraint. These conditions are consistent with \( \{c, l, g\} \) forming part of a competitive allocation. Inequalities (20) and (21) are recursive versions of, respectively, the no lending constraint (13) and the incentive compatibility constraint (14). Finally, (22) and (23) restrict the continuation payoff and debt value. Notice that if \((V, \lambda) \succeq (V^S, \lambda^S)\), then \(W\) is contained within the set of payoffs and debt values described by (22) and (23).

We now formally define the SICCA operator \( T \) (Q). For \((V, \lambda) \in Q\), and \( \lambda \in \{\lambda \geq 0 : \Gamma_{V, X}(\lambda) \neq \emptyset\}\), let

\[ V'(V, \lambda)(\lambda) = \sup_{\{c, l, g, \lambda', v'\} \in \Gamma_{V, \lambda}(\lambda)} \sum_{\theta \in \Theta} [W(c(\theta), l(\theta), g(\theta); \theta) + \beta v'(\theta)] P(\theta), \] \hspace{1cm} (24)

and define \( T(V, \lambda) = (T_V(V, \lambda), T_{\lambda}(V, \lambda)) \), where

\[ T_{\lambda}(V, \lambda) = \sup \left\{ \lambda \geq 0 : \Gamma_{V, X}(\lambda) \neq \emptyset \text{ and } V'(V, \lambda)(\lambda) \geq V^{aut} \right\}, \]
\[ T_V(V, \lambda)(\lambda) = V'(V, \lambda)(\lambda), \text{ for } \lambda \in [0, T_{\lambda}(V, \lambda)]. \]

With this definition in place, we now state the main proposition of this section.

**Proposition 6** \((V^S, \lambda^S)\) satisfies the following conditions.

1. \( \Lambda^S = [0, \lambda^S] \) and \((V^S, \lambda^S) \in Q\).
2. \((V^S, \lambda^S) = T(V^S, \lambda^S)\).
3. Let \((V_n, \lambda_n) = T^n(V^R, \lambda^S)\). The sequence \(\{(V_n, \lambda_n)\}_{n=0}^\infty\) is decreasing and converges to \((V^S, \lambda^S)\).

**Proof:** See Appendix 1.

The first part of Proposition 6 establishes that \( \Lambda^S \) is an interval of the form \([0, \lambda^S]\). Thus, there is an endogenous upper bound on debt values. Sleet (2003) shows in a model with incentive compatibility constraints, but no sustainability constraint, that the upper bound on debt values is the natural one, \( \lambda^{aut} \), obtained in Lemma 1. It is easy to show that \( \lambda^S < \lambda^{aut} \); introduction of the limited commitment friction tightens the upper debt value limit. This is an analogue for our environment of the endogenous solvency constraints that Alvarez and Jermann (2000) find in their model of households who are unable to commit to making debt repayments.

The second and third parts of Proposition 6 show that \((V^S, \lambda^S)\) is a fixed point of \( T \) and provide an iterative scheme for computing this pair. We put this iteration to work in later sections. Notice that the second part of the proposition implies that

\[ V^S(\lambda) = \sup_{\{c, l, g, \lambda', v'\} \in \Gamma_{V, \lambda}(\lambda)} \sum_{\theta \in \Theta} [W(c(\theta), l(\theta), g(\theta); \theta) + \beta v'(\theta)] P(\theta). \] \hspace{1cm} (25)
We show in the proof of this proposition that (25) has a solution satisfying:

\[ \forall \theta, (v'(\theta), \lambda'(\theta)) = (V^S(\lambda'(\theta)), \lambda'(\theta)) \in W. \] (26)

This result is central to our construction. In economic terms, it implies that there exist optimal SICCA’s whose continuation allocations are optimal SICCA’s.

Combining (25) and (26) implies that \( V^S \) satisfies the functional equation (27). We call this the optimal SICCA problem.

\[ V^S(\lambda) = \sup_{(c,l,g,\lambda') \in \Gamma^S(\lambda)} \sum_{\theta} [W(c(\theta), l(\theta), g(\theta); \theta) + \beta V^S(\lambda'(\theta))] P(\theta), \] (27)

where \( \Gamma^S(\lambda) \) consists of all \((c,l,g,\lambda')\) satisfying for all \( \theta \), \( c(\theta) \geq 0 \), \( g(\theta) \geq 0 \), \( l(\theta) \in [0,T] \) and,

\[ \lambda \leq \sum_{\theta \in \Theta} \left[ J(c(\theta), l(\theta)) + \beta \lambda'(\theta) \right] P(\theta), \] (28)

\[ \forall \theta : J(c(\theta), l(\theta)) + \beta \lambda'(\theta) \geq 0, \] (29)

\[ \forall (\theta, \hat{\theta}) : W(c(\theta), l(\theta), g(\theta); \theta) + \beta V^S(\lambda'(\theta)) \geq W(c(\hat{\theta}), l(\hat{\theta}), g(\hat{\theta}); \theta) + \beta V^S(\lambda'(\hat{\theta})), \] (30)

\[ \forall \theta : \lambda'(\theta) \in [0, \lambda^S] \equiv \Lambda^S, \] (31)

\[ \forall \theta : l(\theta) - c(\theta) - g(\theta) \geq 0. \] (32)

By Proposition 6, \((V^S, \lambda^S) \in Q\). Hence, \( V^S \) is continuous and since the graph of \( \Gamma^S \) is a compact set, there exist continuous policy functions, \( c^S, l^S, g^S \) and \( \lambda^S \), that describe a solution to this problem. These functions are of the form \( c^S : \Lambda^S \times \Theta \to \mathbb{R}_+, l^S : \Lambda^S \times \Theta \to [0,T], g^S : \Lambda^S \times \Theta \to [0,T] \) and \( \lambda^S : \Lambda^S \times \Theta \to \Lambda^S \). They can be applied recursively to reconstruct an optimal SICCA.

The following preliminary lemma collects various results about the optimal SICCA problem. It establishes that “resource burning” is not used to induce truth-telling and that the optimal policy functions are unique.

**Lemma 2**

1. \( V^S \) is strictly decreasing, strictly concave and almost everywhere differentiable.

2. The resource constraint binds in each \( \theta \) state.

3. If \( N = 2 \), the “downwards” incentive compatibility constraint between states \((\hat{\theta}_2, \hat{\theta}_1)\) does not bind.

4. The optimal policy functions \( \{c^S, l^S, g^S, \lambda^S\} \) are uniquely defined.

**Proof:** See Sleet and Yeltekin (2004). \( \blacksquare \)

## 5 Properties of Optimal SICCA’s

In this section we first show that optimal SICCA are generally dynamic. We then characterize the evolution of both the excess burden of taxation and the tax rates implied by an optimal SICCA.
5.1 History dependence of optimal SICCA’s

Recall that the Ramsey allocation is static and that the Ramsey policy function $\lambda^R$ satisfies $\lambda^R(\lambda, \theta) = \lambda$ for all $(\lambda, \theta) \in \Lambda^R \times \Theta$. In addition, Athey, Atkeson and Kehoe (2003) show, in a related model of monetary policy under private government information, that optimal policy takes the form of a repeated static contract. In contrast to both of these results, we show that optimal fiscal policy under private information and limited commitment is typically dynamic. To simplify the analysis, we set $N = 2$. By Lemma 2, the downwards incentive constraint does not bind and can be dropped. We do this in the remainder of this section and retain only the upwards incentive constraint between states $(\hat{\theta}_1, \hat{\theta}_2)$.

To state the following results, we let $\mu^S(\lambda), \xi^S(\lambda, \theta), \varphi^S(\lambda, \hat{\theta}_1, \hat{\theta}_2), \Delta^S(\lambda, \theta)$ and $\zeta^S(\lambda, \theta)$ denote the Lagrange multipliers on, respectively, the implementability (28), no lending (29), upwards incentive compatibility (30), and lower and upper $\lambda$ boundary (31) constraints in the optimal SICCA problem.

**Lemma 3** Suppose that $\lambda \in (0, \bar{\lambda}^S)$ is a point of differentiability of $V^S$ and is such that the upwards incentive compatibility constraint binds. Then either

1. $\lambda^S(\lambda, \hat{\theta}_2) > \lambda^S(\lambda, \hat{\theta}_1)$, or
2. $\lambda^S(\lambda, \hat{\theta}_i) \neq \lambda$ for some $i = 1, 2$.

**Proof:** If $\lambda^S(\lambda, \hat{\theta}_i)$ is not a point of differentiability of $V^S$ then $\lambda^S(\lambda, \hat{\theta}_i) \neq \lambda$ and Condition 2 holds. Suppose then that for each $i$, $V^S$ is differentiable at $\lambda^S(\lambda, \hat{\theta}_i)$. The first order conditions for $\lambda^S(\lambda, \theta)$ are:

$$
\frac{\partial V^S}{\partial \lambda}(\lambda^S(\lambda, \theta))(P(\theta) + \Delta \varphi^S(\lambda, \theta)) + \mu^S(\lambda)P(\theta) + \xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta) = 0,
$$

where $\Delta \zeta^S(\lambda, \theta) = \xi^S(\lambda, \theta) - \zeta^S(\lambda, \theta)$ and $\Delta \varphi^S(\lambda, \hat{\theta}_1) = \varphi^S(\lambda, \hat{\theta}_1, \hat{\theta}_2) = -\Delta \varphi^S(\lambda, \hat{\theta}_2)$. If (29) and (31) do not bind, then for each $\theta$, $\xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta) = 0$. Hence,

$$
\frac{\partial V^S}{\partial \lambda}(\lambda^S(\lambda, \hat{\theta}_1)) \left(1 + \frac{\varphi^S(\lambda, \hat{\theta}_1, \hat{\theta}_2)}{P(\theta_1)}\right) = \frac{\partial V^S}{\partial \lambda}(\lambda^S(\lambda, \hat{\theta}_2)) \left(1 - \frac{\varphi^S(\lambda, \hat{\theta}_1, \hat{\theta}_2)}{P(\theta_2)}\right),
$$

and Condition 1 follows from the strict concavity of $V^S$.

Assume then that for some $\theta$, $\xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta) \neq 0$. (33) also implies:

$$
E\left[\frac{\partial V^S}{\partial \lambda}(\lambda^S(\lambda, \theta))\right] + \mu^S(\lambda) + E\left[\xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta)\right] = 0.
$$

From the envelope theorem, $\mu^S(\lambda) = -\frac{\partial V^S}{\partial \lambda}(\lambda)$. Hence, if $E[\xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta)] \neq 0$, then $E[\frac{\partial V^S}{\partial \lambda}(\lambda^S(\lambda, \theta))] \neq \frac{\partial V^S}{\partial \lambda}(\lambda)$ and Condition 2 follows from the strict concavity of $V^S$.

This leaves the case $E[\xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta)] = 0$ and $\xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta) > 0 > \xi^S(\lambda, \theta) + \Delta \zeta^S(\lambda, \theta')$, $\theta \neq \theta'$. This, in turn requires that $\zeta^S(\lambda, \theta) > 0$ and $\lambda^S(\lambda, \theta) = \bar{\lambda}^S$ for at least one $\theta$. Since $\lambda \neq \bar{\lambda}^S$, Condition 2 holds. $

\text{---}

11 See Sleet (2001) for a related result.
Proposition 7 Suppose that $\lambda \in (0, \lambda^*)$ is a point of differentiability of $V^S$ and is such that the upwards incentive compatibility constraint binds, then the optimal SICCA with initial $\lambda_0 = \lambda$ is not static.

Proof: Since $V^S$ is strictly decreasing, an optimal SICCA at $\tilde{\lambda}$ is not feasible at $\lambda' > \tilde{\lambda}$. It then follows from Lemma 3 that the continuation optimal SICCA after one (or both) realizations of $\theta$ must differ from the initial optimal SICCA.

Intuitively, the government is tempted to claim its taste shock is high when it is really low, but not vice versa. Away from the boundary constraints (29) and (31), equation (34) and the strict concavity of $V^S$ imply that the government’s debt value rises after a high taste shock report and falls otherwise. The logic is straightforward. One way the government can be discouraged from exaggerating its taste shock is via an increase in its future debt value. As we discuss below, such an increase is costly because it implies higher future tax rates. Consequently, the need to provide incentives for truth-telling imparts intertemporal persistence to fiscal policy that is absent in the Ramsey economy.

5.2 Incentive constraints and the evolution of the excess burden

We now elaborate on the role of each friction in determining the dynamic evolution of the economy. Initially, we focus on the private information friction. To that end, consider the optimal SICCA problem and assume that the incentive compatibility constraint, but not the no lending and boundary constraints on $\lambda_0$ bind. For simplicity, assume that $V^S$ is differentiable. Under these assumptions, the first order condition for $\lambda^S(\lambda, \theta)$ and the envelope condition from the optimal SICCA problem imply the intertemporal smoothing equation:

$$E_{\theta} \left[ \frac{1}{\mu^S(\lambda^S(\lambda, \theta))} \right] = \frac{1}{\mu^S(\lambda)}. \quad (36)$$

Using the fact that the multipliers on the implementability constraints at periods $t$ and $t+1$ are given by $\mu_t(\theta^{t-1}) = \mu^S(\lambda_t(\theta^{t-1}))$ and $\mu_{t+1}(\theta^t) = \mu^S(\lambda^S(\lambda_t(\theta^t), \theta_t))$, we obtain:

$$E_{\theta^{t-1}} \left[ \frac{1}{\mu_{t+1}} \right] = \frac{1}{\mu_t(\theta^{t-1})}. \quad (37)$$

Since, $\mu_t \geq 0$, for all $t$, it follows from Jensen’s inequality that $E[\mu_{t+1}] \geq \mu_t$. If the incentive constraint (30) binds and the allocation is dynamic, then this inequality is strict. Consequently, the incentive constraint imparts an upward drift to the $\{\mu_t\}$ sequence and, thus, to the excess burden of taxation. The strict concavity of $V^S$ also implies that this constraint imparts an upward drift to the corresponding sequence of debt values $\{\lambda_t\}$. However, any such drift will eventually be arrested by the upper bound on $\lambda$ that stems from the sustainability constraints. Numerical calculations in the subsequent section indicate that (29), (30) and (31) interact to ensure that the debt value $\lambda$ evolves according to an ergodic Markov process.

5.3 Tax rate functions

Recall the Ramsey formula for labor tax rates:
\[ \tau^R(\lambda_0, \theta) = \frac{\mu^R(\lambda_0) \left( 1 - \frac{(1-\rho)}{\phi(l)^*(\lambda_0, \theta)} \right)}{\mu^R(\lambda_0) + \frac{1}{\phi(l)^*(\lambda_0, \theta)}}, \]

where \( \phi(l) = 1 + \gamma l/(T - l) \) and that after each \( \theta^t \), the Ramsey tax rate is given by \( \tau^R(\lambda_0, \theta_t) \). As in the Ramsey case, the tax rates associated with the optimal SICCA can be obtained from the household’s first order conditions and the optimal SICCA policy functions \( c^S \) and \( l^S \). In this case, after each \( \theta^t \), the tax rate \( \tau^S(\theta^t) \) is given by a function \( \tau^S(\lambda_t(\theta^{t-1}), \theta_t) \). Manipulation of the first order conditions from the optimal SICCA problem yields the following expression for \( \tau^S(\lambda, \theta) \).

**Lemma 4** The optimal SICCA tax rate function, \( \tau^S \), is given by:

\[ \tau^S(\lambda, \theta) = \frac{\left[ \mu^S(\lambda) + \xi^S(\lambda, \theta) \right] \left( 1 - \frac{(1-\rho)}{\phi(l)^*(\lambda, \theta)} \right)}{\mu^S(\lambda) + \frac{1}{\phi(l)^*(\lambda, \theta)} \left[ 1 + \Delta \phi^S(\lambda, \theta) \right]}. \]  

(38)

Equation (38) has implications for both the volatility and the persistence of optimal SICCA tax rates. The former are most easily seen in the special case \( \rho = 1 \) and \( \gamma = 0 \). Then the tax rate is given by:

\[ \tau^S(\lambda, \theta) = \frac{\left[ \mu^S(\lambda) + \xi^S(\lambda, \theta) \right]}{\mu^S(\lambda) + \frac{1}{\phi(l)^*(\lambda, \theta)} \left[ 1 + \Delta \phi^S(\lambda, \theta) \right]}. \]  

(39)

Variations in the multipliers \( \xi^S \) and \( \Delta \phi^S \) cause the SICCA tax rate to respond to contemporaneous shocks. In contrast, in the Ramsey economy under these preferences, the government is fully hedged and there is no variation in tax rates. In this case, when the government has little debt, it hedges against high taste shock states by buying claims contingent on such states. In the optimal SICCA problem, the no lending constraint prevents this; in situations of low debt and high spending needs, the government must raise taxes relative to their level when spending needs are low. This effect is captured in (39) by the multiplier \( \xi^S \), which increases to a positive number when the no lending constraint binds. Other things being equal, this implies an increase in the tax rate in those states. The incentive constraints also disrupt the government’s ability to hedge fiscal shocks. This effect is captured by \( \Delta \phi^S \) which is positive when the shock is \( \hat{\theta}_1 \) and negative when the shock is \( \hat{\theta}_2 \), implying a higher tax rate when the taste shock is high.

Equation (38) also implies that the SICCA tax rate depends on the current debt value \( \lambda \). Insofar as the incentive compatibility and no lending constraints lead \( \lambda \) to depend on past shock values, the tax rate will be similarly history dependent. Moreover, high taste shocks tend to raise the current tax rate and, by raising the continuation debt value, future tax rates as well. Thus, tax rates exhibit persistence. This is in contrast to the static nature of the Ramsey tax rate.

Sharper results can be obtained by making stronger assumptions on preferences. Suppose that \( \rho = 1 \), that \( \lambda \) and \( \theta \) are such that the boundary conditions (29) and (31) do not bind, and that \( V^S \) is differentiable, then the following intertemporal tax smoothing condition can be obtained:

\[ \left( \frac{\tau^S(\lambda, \theta)}{1 - \tau^S(\lambda, \theta)} \right) = \kappa(\lambda, \theta) \left[ \sum_{\theta'} \hat{p}(\theta') \left( \frac{\tau^S(\lambda^S(\lambda, \theta), \theta')}{1 - \tau^S(\lambda^S(\lambda, \theta), \theta')} \right) \right]. \]  

(40)
where \( \kappa(\lambda, \theta) = \frac{(T-t)S(\lambda, \theta)^{-\gamma}}{E[(T-t)S(\lambda, \theta)^{-\gamma}]} \) and \( \hat{p}(\theta') = P(\theta') + \Delta \varphi^S(\lambda^S, \theta) \) can be interpreted as an “incentive-adjusted probability”. It follows that, absent binding boundary conditions and after an adjustment for expected growth in the marginal utility of labor supply, \( \tau_t/ (1 - \tau_t) \) satisfies a martingale-like condition with respect to the “incentive-adjusted” probability distribution \( \{\hat{p}\} \). When \( \rho = 1 \) and \( \gamma = 0 \), a starker result can be obtained. In this case, again under the assumptions that (29) and (31) do not bind, and that \( V^S \) is differentiable, the first order conditions from the SICCA problem yield:\[ E \left[ \frac{1}{\tau_{t+1}} \right] = \frac{1}{\tau_t}. \] Applying Jensen’s inequality, \( E \left[ \tau_{t+1} \right] \geq \tau_t \), with the inequality strict if the incentive constraints bind. Hence, as for the debt value and the excess burden, the incentive constraints impart an upward drift in taxes over time.

In summary, incentive compatibility considerations tend to imply less inter-state and more intertemporal smoothing of taxes. This is achieved through adjustments in the government’s future debt value in response to fluctuations in taste shocks. These issues are now explored further in some numerical examples.

6 An illustrative numerical example

The set of parameters for the model is \( \mathcal{P} = \{\Theta, P, \beta, \rho, \gamma, \eta\} \). In the example, \( \rho, \gamma \) and \( \eta \) are set to 1, and \( \beta \) is set to 0.98. \( \Theta \) is set to \( \{0.2, 0.3, 0.4\} \) and the shocks are assumed to be i.i.d. with a uniform distribution. This parameterization is selected to illustrate a number of interesting aspects of optimal fiscal policy under the frictions described above. It is representative of many other numerical experiments that we have undertaken.\[ ^{14} \]

Below we illustrate the policy functions from the optimal SICCA problem. We provide intuition for the implied fiscal policy and compare it to optimal fiscal policy from two benchmark models: the complete markets Ramsey model and a model in which the government is exogenously restricted to trading non-contingent debt.

6.1 Debt values

Figure 2 shows the debt value policy functions \( \lambda^R \) and \( \lambda^S \) from the Ramsey and the optimal SICCA problems. These functions depend on the current debt value \( \lambda \) and the \( \theta \) shock. The domain for the graphs is \( \Lambda^S \), which is computed along with \( V^S \), using the procedure described in Section 4. The endogenous upper debt value \( \lambda^S \) is computed to be 0.26. This corresponds to an upper bound on outstanding claims, \( b^S(\overline{\lambda}^S, \theta) \), of between 31% and 49% of output, depending upon the current shock realization.

12 Note, that \( \sum_{\theta'} \hat{p}(\theta') = 1 \).
13 As noted earlier, results can be easily extended to this case.
14 A complete calibration exercise should incorporate capital accumulation and persistent taste shocks. As well as being empirically more reasonable, the latter would obviate the need for such large current taste shocks. Since such shocks are short-lived in the formulation below, they need to be large to have quantitatively significant effects. A full calibration exercise would also require a stand to be taken on what proportion of fiscal, or other, shocks are privately observed by the government.
The computed policy functions from the Ramsey case confirm Proposition 2: for all $\lambda$ and $\theta$, $\lambda^R(\lambda, \theta) = \lambda$. It follows that the Ramsey allocation is static. In contrast, Figure 2 shows that the optimal SICCA policy functions for different $\theta$’s coincide neither with the 45 degree line nor with one another. Thus the optimal SICCA is dynamic. The figure also indicates that $\lambda^S(\lambda, \theta)$ is increasing in both of its arguments. Consequently, taste shocks have persistent effects on debt values and, hence, future allocations. A high $\theta$ shock leads to an increase in the debt value chosen in the current period and, through the dependence of $\lambda^S$ on $\lambda$, it increases the continuation debt value chosen in the subsequent period as well. As discussed previously, such intertemporal persistence in allocations is part of an arrangement for the efficient provision of incentives.

Repeated sequences of high or low shock reports cause the economy to drift towards the debt value boundaries. These boundaries stem from the sustainability and no lending constraints. The latter directly restricts the government’s ability to hedge fiscal shocks. Both exacerbate the incentive problem by limiting the government’s ability to vary the debt value $\lambda^S(\lambda, \theta)$. As we describe below, this leads to more abrupt adjustments of fiscal policy in the neighborhood of these boundaries. Figure 2 also indicates that $\lambda^S$ and $P$ induce a Markov process for debt values that is monotone, that satisfies a mixing condition and that, consequently, has an invariant measure.

### 6.2 Tax rates

Figure 3, Panel A shows the optimal tax rate functions $\tau^R$ and $\tau^S$ for the Ramsey and SICCA economies. Both functions are increasing in $\theta$, revealing an absence of complete tax smoothing even in the Ramsey case. Both are also increasing in $\lambda$, especially for the SICCA case. Combining the policy functions $\lambda^S$ and $\tau^S$, we observe that $\theta$ shocks have persistent effects on tax rates in the SICCA economy. Higher taste shock reports lead to an immediate increase in taxes. They also increase the continuation debt value, hence, future tax rates.

Over much of $\Lambda^S$, the variation in tax rates across $\theta$ states is of a similar magnitude in both economies. However, at the $\lambda$ boundaries, tax rates become much more volatile in the SICCA case; they are cut sharply in
the low $\theta$ state when $\lambda$ is close to zero, and raised sharply in the high $\theta$ state when $\lambda$ is close to $\lambda^S$. As noted, persistent sequences of high or low reports move the SICCA economy towards the debt value boundaries, and, hence, to regions with greater tax rate volatility. These characteristics of the function $\tau^S$ are picked up by simulations of the SICCA economy (Figure 3, Panel B) which exhibit occasionally sharp upward and downward hikes in tax rates as the boundaries are approached.

![Figure 3: Tax Rates and Tax Rate Simulation](image)

Panel A: Tax rate functions  
Panel B: Tax Simulation

### 6.3 Debt management and hedging

In this section, we compare debt management policies across three different asset market arrangements: Ramsey, the SICCA economy and the exogenously non-contingent debt market model of AMSS. In the latter, the government is constrained to sell only risk-free debt subject to exogenous debt limits. It faces a sequence of implementability constraints of the form: $\forall t, \theta^t$,

$$b_t(\theta^{t-1})c_t^{-\rho}(\theta^t) = E_{\theta^t} \left[ \sum_{s=0}^{\infty} \beta^s J(c_{t+s}, l_{t+s}) \right].$$

(42)

Note that here $b_t$ is measurable with respect to $\theta^{t-1}$, not $\theta^t$. Equivalently, $\forall t, \theta^{t-1}$, $i \in \{1, \cdots, N-1\}$,

$$c_t^p(\theta^{t-1}, \hat{\theta}_i)E_{\theta^{t-1}, \hat{\theta}_i} \left[ \sum_{s=0}^{\infty} \beta^s J(c_{t+s}, l_{t+s}) \right] = c_t^p(\theta^{t-1}, \hat{\theta}_{i+1})E_{\theta^{t-1}, \hat{\theta}_{i+1}} \left[ \sum_{s=0}^{\infty} \beta^s J(c_{t+s}, l_{t+s}) \right].$$

(43)

These “risk-free debt” constraints replace the incentive-compatibility constraints of earlier sections. Exogenously given lower and upper bounds on debt repayments of 0 and $\overline{b} > 0$ impose the following additional set of constraints on allocations, $\forall t, \theta^{t-1}$:

$$b_t(\theta^{t-1}) = c_t^p(\theta^{t-1}, \hat{\theta}_1)E_{\theta^{t-1}, \hat{\theta}_1} \left[ \sum_{s=0}^{\infty} \beta^s J(c_{t+s}, l_{t+s}) \right] \in [0, \overline{b}].$$

(44)
These constraints collectively imply an endogenous set of debt values, $\Lambda^N(b)$, such that for each $\lambda \in \Lambda^N(b)$ there exists a competitive allocation debt value $\lambda$ and non-contingent debt that respects the debt bounds $[0, b]$. For purposes of comparability, we compute $\Lambda^N(b)$ and the optimal non-contingent debt allocation at each $\lambda \in \Lambda^N(b)$. We select $b$ so that $\Lambda^N(b)$ is approximately equal to $\Lambda^S$, and focus attention on the contrasting implications of the incentive-compatibility and the risk-free debt conditions.

To compare debt management policies across the three economies, we use the policy functions from each to compute the variance of debt repayments and the revenues from debt sales across $\theta$ states conditional on the current $\lambda$. We also compute the covariance of these variables with government spending, again conditional on $\lambda$, and give a breakdown of the sources of finance used to fund the additional spending undertaken after a high taste shock. For completeness, we provide a similar set of statistics for tax rates. Collectively, these results detail the extent to which the government hedges shocks across the different environments.

We begin with Figure 4 which shows conditional variances as functions of the current $\lambda$. The figure indicates that debt repayments are most volatile in the Ramsey economy and least volatile in the non-contingent debt economy (where they exhibit no volatility by definition), and that the reverse ordering is true for tax rates, except at very high $\lambda$ levels. The volatility of debt repayments and tax rates in the SICCA economy is intermediate between the other two economies. However, tax rate volatility is low and much closer to that in the Ramsey case over much of the $\lambda$ range; only at the boundaries does it rise to levels close to those in the non-contingent model. The volatility of revenues from debt sales is much greater in the non-contingent than in the other cases. We discuss the explanation for this below.

Figure 5 elaborates on the previous figure by showing the $\lambda$-conditional covariances of these variables with government spending. The Ramsey economy exhibits significant negative covariance between debt repayments and government spending, confirming that the government uses its portfolio of claims to hedge shocks. This is reduced in the SICCA economy, indicating a much more limited ability to hedge. Debt sale revenues also covary negatively with spending in the Ramsey economy, but positively in the non-contingent economy. This contrast was noticed by Marcet and Scott (2001), who also observed a positive covariance between these variables in the data. Debt sale revenues are determined by both the price and the quantity of claims sold. In the Ramsey case, the same portfolio of claims is sold in each $\theta$ state, and the volatility of debt sale revenues is driven entirely by price movements. Since these are negatively correlated with the taste shock, so too are debt revenues. In the non-contingent case, this price effect is more than offset by variations in the numbers of claims sold. Since the government cannot hedge fiscal shocks by varying debt returns in this case, it relies on new debt sales and higher tax rates for additional revenues in the high shock states. More claims are sold in high shock states, leading to greater revenues in those states and a positive correlation of revenues with spending. The SICCA economy is intermediate, with a smaller variation in the quantity of claims sold across $\theta$ states. For most of the $\lambda$ range, the quantity effect dominates the price effect, leading to a covariance that is positive, but smaller than in the non-contingent case. Close to $\Lambda^S$, however, the price effect dominates and the covariance becomes negative. The nearly offsetting price and quantity effects also diminish the variance of debt sale revenues relative to the non-contingent case, and, at high $\lambda$’s, relative to the Ramsey case as well.

The final set of figures in this section gives a proportional breakdown of the extra finance needed to fund spending in the high $\theta_3$-shock state relative to the low $\theta_1$-shock state, across $\lambda$ values, for each of the three
economies. Between 60 and 80% of the extra revenue needed to fund the higher spending in the $\theta_3$-shock state comes from variations in the debt return in the Ramsey model. This falls to between 0 and 50% in the SICCA model and then to 0% in the non-contingent debt model. In the Ramsey economy, new debt sales make a negative contribution to the funding of higher spending, for the reasons described in the last paragraph, but they make a contribution of close to 50% across most $\lambda$-values in the non-contingent economy. In the SICCA economy, they make a small, but positive contribution.

Table 1 reports sample autocorrelations and covariances with government spending for tax rates and debt repayments. These numbers are obtained from long simulations of each economy, with each covariance normalized in the variance of government spending. In the Ramsey economy, both variables inherit the stochastic
properties of the underlying shocks; they exhibit zero serial correlation. In the other economies, they have positive autocorrelations. In particular, tax rates in the SICCA economy exhibit greater persistence than in the Ramsey economy, but less than in the non-contingent economy. The normalized covariances imply that taxes covary with government spending positively in all three economies and to an intermediate degree in the SICCA economy. Finally debt returns covary negatively with government spending in all economies, even in the non-contingent debt economy.\(^{15}\)

<table>
<thead>
<tr>
<th>Frictions</th>
<th>Ramsey</th>
<th>Non-contingent</th>
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</thead>
<tbody>
<tr>
<td>(\rho(\tau))</td>
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</tr>
<tr>
<td>(\rho(b))</td>
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<td>0.267</td>
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<tr>
<td>(\text{Cov}(b,g))</td>
<td>-0.453</td>
<td>-0.694</td>
</tr>
</tbody>
</table>

### 6.4 Welfare Comparisons

Lastly, we provide a welfare comparison between the SICCA economy and the benchmark Ramsey model. We calculate, for each initial \(\lambda_0\), the reduction in household consumption needed in all states and periods for the household to be indifferent between the Ramsey allocation \(e^R(\lambda_0)\) and the optimal SICCA at \(\lambda_0\). For our parameterization, this reduction averages 0.053% of the household’s Ramsey consumption across \(\lambda_0\) values, rising to a peak of 0.07% at the \(\lambda\) boundaries. This small loss is consistent with the welfare findings of AMSS, and indicates the capacity of tax smoothing across periods to substitute for tax smoothing across states.

### 7 Conclusions

The empirical properties of tax rate and debt processes contrast with the implications of the Ramsey model and are suggestive of incomplete government insurance. This paper develops techniques for analyzing optimal fiscal policy under two frictions, private government information and limited government commitment. We find that these frictions imply endogenous debt value limits, a reduced smoothing of taxes across states, and enhanced persistence in tax rates and debt levels. The government retains some ability to hedge against fiscal shocks, so that the ensuing allocations and fiscal policies in our model lie somewhere between those implied by the Ramsey model and a model with exogenously non-contingent debt. This brings our model into line with evidence reported by Marcet and Scott (2001).

### References


\(^{15}\)The latter result reflects the fact that at higher debt levels there is less government spending and larger sales of debt.


8 Appendix 1: Proofs

Proof of Lemma 4: Since, under \( x^{aut} \), \( s_t(h^g_t) = 1 \) for all \( t \), \( h^g_t \), households anticipate a full default by the government after each history. Thus, after each \( h_t \), it is weakly optimal for them to purchase no debt and their continuation problem reduces to a sequence of problems of the form (10). When the government implements the autarkic menu \( x^{aut} \), households’ optimal choices are given by \( c^{aut} \) and \( l^{aut} \). The government’s strategy implies choices that are budget feasible for it and resource feasible for the economy. Since households, anticipating future default, purchase no debt at a positive price, the government can not raise revenue from debt sales. It looses nothing from setting debt prices to zero. Its problem, after each \( h_t \), also reduces to a sequence of static problems. The government maximizes its payoff by defaulting on any outstanding debt, setting the transfer \( S_t \) to zero. Given that the government must balance its budget, \( \tau^{aut}(\theta) \) gives the optimal tax rate. The choices described by \( (x^{aut}(\theta), g^{aut}(\theta)) \) are optimal for the government given a current shock realization of \( \theta \). Consequently, the government has no incentive to lie and \( \sigma^{g,aut} \) satisfies the sequence of optimality conditions (8). It follows that \( (h_0, \sigma^{g,aut}, \sigma^{h,aut}) \) is a sustainable equilibrium.
For any $b_0$, the strategy $\sigma^{g,aut}$ is always feasible for the government. If the government implements $(x^{aut}, g^{aut})$ and reports $\theta$, the household will face a set of zero bond prices and, essentially, a static choice problem. Confronted with the tax rate $\tau^{aut}(\theta)$ and with $s(\theta) = 1$ and $S(\theta) = 0$, it will be optimal for households to make the consumption and labor choices $c^{aut}(\theta)$ and $l^{aut}(\theta)$. Thus, after any $h_t$, by repeatedly implementing the menu $x^{aut}$ and being truthful, the government can guarantee itself the autarkic payoff. Thus, this payoff acts as a lower bound on the set of sustainable equilibrium payoffs. □

**Proof of Proposition 5:** (Only if) Fix a sustainable equilibrium $\sigma = (\sigma^g, \sigma^h)$ at $\lambda_0$ with outcome allocation $e^\infty = \{c_t, l_t, g_t\}_{t=0}^\infty \in \Xi$. By the first condition of Definition 7, $e^\infty$ is a competitive allocation and, hence, satisfies Condition A of the proposition. By Condition 1(b) of Definition 7, $e^\infty$ is part of a competitive equilibrium that entails no government lending. Therefore, for each $t$, $\theta^t$, $c^\sigma_t(\theta^t) [(1 - s_t(\theta^t)) b_t(\theta^t) + S_t(\theta^t)] \geq 0$, where $s_t(\theta^t), b_t(\theta^t), S_t(\theta^t)$ are variables associated with the equilibrium. The household’s first order conditions and budget constraint then imply (13).

Denote the reporting functions associated with $\sigma^g$ by $\{\delta_t^g\}_{t=0}^\infty$. Note that the definition of a sustainable equilibrium does not require them to be truthful, i.e. $\delta_t^g(h_t(\sigma^g, \theta^{t-1}), \theta^t)$ need not equal $\theta^t$. Let $\{\delta_t^g\}_{t=0}^\infty$ denote an arbitrary reporting policy. Suppose the government sought to obtain the allocation $e^\infty(\delta) = \{c_t(\delta^t()), l_t(\delta^t()), g_t(\delta^t())\}_{t=0}^\infty$. It could do this in the policy game by implementing the menu functions implied by $\sigma^g$, and altering the reporting functions along the outcome path to: $\tilde{\delta}^g_t(h_t(\sigma^g, \delta^{t-1}(\theta^{t-1})), \theta^t)$ $= \delta^g_t(h_t(\sigma^g, \delta^{t-1}(\theta^{t-1})), \delta_t(\theta^t))$. Equivalently, the government can obtain the allocation $e^\infty(\delta)$ by making the reports that it would have made if the shock histories $\{\delta_t(\theta^t)\}$ had actually occurred. Since this is a feasible deviation in the policy game and the government chooses not to make such reports, it must be the case that the allocation $e^\infty$ gives a higher utility than $e^\infty(\delta)$, and, hence, satisfies the incentive compatibility condition (14). After any $h_t$, the government can always feasibly deviate to the policies associated with the autarkic equilibrium. Thus, the continuation of the outcome allocation must deliver a payoff above the autarkic one and (15) is satisfied.

(If) Suppose an interior allocation $e^\infty = \{c_t, l_t, g_t\}_{t=0}^\infty$ satisfies the conditions of the proposition, then a sustainable equilibrium at $\lambda_0$ can be constructed that supports it as an outcome allocation. This is done as follows. Set, for all $t$, $\theta^t$, $\tau_t(\theta^t) = 1 - \frac{c_t^g(\theta^t)}{(c_t^g(\theta^t))}$, and $s_t(\theta^t) = 0$. For $t \geq 1$, set $s_t(\theta^t) = 0$, and $b_t(\theta^t)$ so that $c_t^g(\theta^t) b_t(\theta^t) = E_{\theta^t} \sum_{s=0}^\infty \beta^s J(c_{t+s}(\theta^{t+s}), l_{t+s}(\theta^{t+s})).$ For $t = 0$, choose $s_0(\theta^0) \in \mathbb{R}_+$ and $b_0(\theta^0) \in \mathbb{R}_+$ to satisfy $c_0^g(\theta^0) b_0(\theta^0) + S_0(\theta^0) = E_{\theta^0} \sum_{s=0}^\infty \beta^s J(c_s(\theta^s), l_{t+s}(\theta^s))$, and $\lambda_0 = E [c_0^g(\theta^0)(1 - s_0) b_0(\theta^0)].$ Set $q_t$ so that the household’s Euler equations hold at the allocation. Set $x^\infty = \{x_t\}_{t=0}^\infty$, with $x_t(\theta^t) = \{s_t(\theta^t), \tau_t(\theta^t), S_t(\theta^t), q_t(\theta^t)\}.$

We construct $\sigma^g$ and $\sigma^h$ history-by-history as follows. Let $H^x_t = \{h_t | h_t = \{b_0, x_0(s_0^{-1}), g_0(s_0^{-1}), \theta^0\}_{s=0}^{t-1}\}$. Interpret $H^x_t$ as the set of $t$-period public histories that are consistent with the government implementing $(x^\infty, g^\infty)$. Note that the occurrence of $h_t \in H^x_t$ does not imply that the government has been truthful. If $h_t \in H^x_t$, set $\sigma^g_t(h_t)$ to $\{x_t(\theta^{-1}), g_t(\theta^{-1}), \delta^t\}$, and $\sigma^{g-1}_t(h_t)$ to $\{c_t(\theta^{-1}), l_t(\theta^{-1}), b_t(\theta^{-1})\}$. If $h_t \not\in H^x_t$, set $\sigma^g_t(h_t)$ to $\{x^{aut}, g^{aut}, \delta^t\}$ and $\sigma^{g-1}_t(h_t)$ to the autarkic choices that solve (10).

By construction, $e^\infty$ is the outcome allocation of $\sigma$. Condition A of the proposition and the proof of Proposition 1 then establishes that $e^\infty$, $\{b_t\}_{t=0}^\infty$ and $x^\infty$ form a competitive equilibrium with initial debt value.
λ_0. Thus, for all t, h_t ∈ H^T_t, Conditions (1a) and (1c) of Definition 7 are satisfied. After all histories h_t ∈ H^T_t, the definition of the autarkic equilibrium again implies that Conditions (1a) and (1c) of Definition 7 are satisfied. Condition (1d) of the definition is satisfied by construction. There is no borrowing in the autarkic equilibrium. The construction of the {h_t}^∞_{t=0} and (13) then implies that (1b) in the definition is satisfied. It remains to check Condition 2 of the definition. Since, after a history h_t(σ^g, θ^t−1), any deviation from (x_t(θ^t−1), g_t(θ^t−1)) leads households to make the autarkic choices, when it deviates from (x_t(θ^t−1), g_t(θ^t−1)), the government can do no better than to choose the autarkic menu in the current period, and implement this in all future periods. This will deliver a payoff of V^aut to the government. By (15), this is below the payoff that the government obtains from continuing to implement the menus associated with the allocation and being truthful. Thus, the government will never deviate from the sequence (x^∞, g^∞) and altering its reporting strategy. Any allocation obtained by such a deviation is of the form: 
\{c'^t, l'^t, g'^t\}^∞_{t=0}, where, for each t, \theta^t, c'^t(\theta^t) = c^t(\theta_{t+1}(σ^g, δ^t(\theta^t))) = c_t(δ^t(\theta^t)), l'^t(\theta^t) = l^t(\theta_{t+1}(σ^g, δ^t(\theta^t))) = l_t(δ^t(\theta^t)), g'^t(\theta^t) = g^t(\theta_t(σ^g, δ^t−1(\theta−1)), δ_t(\theta^t)) = g_t(δ^t(\theta^t)) and {δ^t}^∞_{t=0} is a reporting policy. By (14) such a deviation is not profitable for the government and Condition 2 of the definition is verified. \sigma is a sustainable equilibrium. 

We now turn to the task of characterizing a SICCA.

**Lemma A1** Suppose that e^∞ = \{c_t, l_t, g_t\}^∞_{t=0} ∈ E^{IC}(λ_0). Let e^∞_∞|θ^t = \{c^t(\theta^t, \cdot), l^t(\theta^t, \cdot), g^t(\theta^t, \cdot)\}^∞_{t=0} denotes the continuation allocation implied by e^∞ after shock history θ^t. Let \{λ_t\}^∞_{t=0} be a sequence of debt value functions consistent with e^∞. Then, for all t, \theta^t, e^∞|θ^t ∈ E^{IC}(λ_{t+1}(θ^t)).

**Proof:** Since e^∞ is a SICCA and \{λ_t\}^∞_{t=0} is consistent with e^∞, then for all θ^t, λ_t(θ^t) ≤ E^θ^t[∑^∞_{r=t+1}β^r J(c_{t+r}, l_{t+r})]. Also e^∞ satisfies the resource constraints after all histories. Hence, for all t, θ^t, e^∞|θ^t satisfies the competitive allocation requirements in Proposition 5. Additionally, e^∞ satisfies the no lending and sustainability constraints (13) and (15) after each history. Furthermore, since P^θ(θ^t) > 0, e^∞ is incentive compatible only if each e^∞|θ^t is. Hence, for all t, θ^t, e^∞|θ^t ∈ E^{IC}(λ_{t+1}(θ^t)).

**Lemma A2** A pair (v, λ) is supported by a SICCA if and only if there exist functions \{c, l, g, v', λ'\}, with c : \Theta → \mathbb{R}_+, l : \Theta → \mathbb{R}_+, λ' : \Theta → \mathbb{R}_+, v' : \Theta → \mathbb{R}, such that:

\[ v = \sum_{θ ∈ \Theta} [W(c(θ), l(θ), g(θ); θ) + βv'(θ)] P(θ), \] (45)

\[ λ ≤ \sum_{θ ∈ \Theta} [J(c(θ), l(θ)) + βλ'(θ)] P(θ), \] (46)

\[ ∀θ, l(θ) − c(θ) − g(θ) ≥ 0, \] (47)

\[ ∀θ, J(c(θ), l(θ)) + βλ'(θ) ≥ 0, \] (48)

\[ ∀(θ, ˆθ), W(c(θ), l(θ), g(θ); θ) + βv'(θ) ≥ W(c( ˆθ), l( ˆθ), g( ˆθ); θ) + βv'( ˆθ), \] (49)

\[ \sum_{θ ∈ \Theta} [W(c(θ), l(θ), g(θ); θ) + βv'(θ)] P(θ) ≥ V^{aut}, \] (50)

\[ ∀θ, (v'(θ), λ'(θ)) ∈ \mathcal{W}. \] (51)
PROOF: (ONLY IF) Suppose \((v, \lambda) \in \mathcal{W}\), then there exists a SICCA, \(e^\infty = \{c_t, l_t, g_t\}_{t=0}^\infty \in \mathcal{E}^{IC}(\lambda)\), with \(v = V^g(e^\infty)\). Set \(v'(\theta) = V^g(e^\infty | \theta)\) and \(\lambda'(\theta) = \lambda_1(\theta)\), where \(\lambda_1\) is a consistent debt value, \(c = c_0\) and \(l = l_0\). Then, \(v = V^g(e^\infty) = E[W(c_0(\theta), l_0(\theta), g_0(\theta); \theta) + \beta V^g(e^\infty | \theta)] = E[W(c(\theta), l(\theta), g(\theta); \theta) + \beta v'(\theta)]\), which confirms (45). Since \(\lambda_1\) is consistent, \(\lambda \leq \sum_{\theta \in \Theta} [J(c_0(\theta), l_0(\theta)) + \beta \lambda_1(\theta)] P(\theta)\), which confirms (46). (48), (49) and (50) follow from the fact that \(e^\infty\) satisfies (13), (14) and (15) respectively. By Lemma A1, \(e^\infty | \theta \in \mathcal{E}^{IC}(\lambda_1(\theta))\) and, so for each \(\theta\), \((V^g(e^\infty | \theta), \lambda_1(\theta)) \in \mathcal{W}\). This confirms (51). (47) follows from resource feasibility of \(e^\infty\) in period 0.

(1F) Suppose that for \((v, \lambda)\), there exists a four-tuple \(\{c, l, g, v', \lambda'\}\) satisfying the conditions in the lemma. Then, by (51), for each \(\theta\), there exists a \(e^\infty_1(\theta) = \{c_t, l_t, g_t\}_{t=0}^\infty \in \mathcal{E}^{IC}(\lambda'(\theta))\), with \(v'(\theta) = V^g(e^\infty_1(\theta))\). Define \(e^\infty = \{c, l, g, e^\infty_1\}\). It is straightforward to check that this satisfies all the properties of a SICCA and has initial payoff and debt value \((v, \lambda)\).

Let \(\Gamma_W(\lambda)\) denote the set of \(\{c, l, g, v', \lambda'\}\) tuples satisfying conditions (46) to (51). It follows from Lemma A2 that

\[
V^S(\lambda) = \sup_{\{c, l, g, v', \lambda'\} \in \Gamma_W(\lambda)} \sum_{\theta \in \Theta} [W(c(\theta), l(\theta), g(\theta); \theta) + \beta v'(\theta)] P(\theta). 
\]

To prove Proposition 6, we first give a series of preliminary lemmas. These require a change of variables.\(^{16}\) Given \(c : \Theta \to \mathbb{R}_+, l : \Theta \to [0, T]\), and \(g : \Theta \to \mathbb{R}_+\), let \(u(\theta) = c^{1-\rho}(\theta), y(\theta) = (T - l(\theta))^{1-\gamma}, r(\theta) = g(\theta)^{1-\eta}\) and \(\mathcal{V} = \{(T - l)^{1-\gamma} : l \in [0, T]\}\). We recast problem (52) in terms of these utility variables:

\[
\sup_{\{u, y, r, v', \lambda'\} \in \Gamma_W(\lambda)} \sum_{\theta \in \Theta} [F(u, y, r, \theta, \theta) + \beta v'(\theta)] P(\theta) 
\]

where \(F(u, y, r, \theta', \theta) = \frac{u(\theta')}{1-\rho} + \frac{y(\theta')}{1-\gamma} + \frac{r(\theta')}{1-\eta}\) and \(\Gamma_W(\lambda)\) equals all \(\{u, y, r, v', \lambda'\}\) satisfying:

\[
\lambda \leq \sum_{\theta \in \Theta} \left[ u(\theta) + \left( y(\theta) - Ty(\theta)^{1-\gamma} \right) + \beta \lambda'(\theta) \right] P(\theta), 
\]

\[
\forall \theta, \ T - y \frac{1}{1-\gamma}(\theta) - u \frac{1}{1-\rho}(\theta) - r \frac{1}{1-\eta}(\theta) \geq 0, 
\]

\[
\forall \theta, \ u(\theta) + \left( y(\theta) - Ty(\theta)^{1-\gamma} \right) + \beta \lambda'(\theta) \geq 0, 
\]

\[
\forall (\theta, \theta'), \ F(u, y, r, \theta, \theta') + \beta v'(\theta) \geq F(u, y, r, \theta, \theta) + \beta v'(\theta), 
\]

\[
\sum_{\theta \in \Theta} [F(u, y, r, \theta, \theta) + \beta v'(\theta)] P(\theta) \geq V^{aut},
\]

\[
\forall \theta, \ (v'(\theta), \lambda'(\theta)) \in \mathcal{W}, \ u(\theta) \in \mathbb{R}_+, \ y(\theta) \in \mathcal{V}, \ r(\theta) \in \mathbb{R}_+. 
\]

\(^{16}\)This change of variables is valid for \(\rho \in (0, 1)\) and \(\gamma \in (0, 1)\) and \(\eta \neq 1\). For \(\rho, \gamma\) or \(\eta\) equal to 1, analogous results hold with \(u(\theta) = \log c(\theta), y(\theta) = \log(T - l(\theta))\) and \(r(\theta) = \log(g(\theta))\). Related results also hold for the case \(\gamma = 0\).
The operator $T$ can be re-expressed using utility variables. For $(V, \bar{\lambda}) \in \mathcal{Q}$, let

$$V'(V, \bar{\lambda})(\lambda) = \sup_{\{u,y,r,\lambda',\theta\} \in \Gamma_{V,\bar{\lambda}}(\lambda)} \sum_{\theta \in \Theta} [F(u, y, r, \theta, \theta) + \beta v'(\theta)] P(\theta),$$

where $\Gamma_{V,\bar{\lambda}}(\lambda)$ is defined to be all $\{u,y,r,v',\lambda'\}$ that satisfy (54), (55), (56), (57), (60) and

$$\forall \theta, v'(\theta) \leq V'(\lambda(\theta))$$

$$\lambda'(\theta) \in [0, \bar{\lambda}]$$

(62)

(63)

Also, let $\Lambda'(V, \bar{\lambda}) = \{\lambda \in \Omega(V, \bar{\lambda}) : V'(V, \bar{\lambda})(\lambda) \geq V^{\text{aut}}\}$. Then, $T_V(V, \bar{\lambda}) = \sup \Lambda'(V, \bar{\lambda})$, while for $\lambda \in \Lambda'(V, \bar{\lambda})$, $T_V(V, \bar{\lambda})(\lambda) = V'(V, \bar{\lambda})(\lambda)$.

**Lemma A3** $T : \mathcal{Q} \rightarrow \mathcal{Q}$.

**Proof:** Fix $(V, \bar{\lambda})$ in $\mathcal{Q}$. The convexity and compactness of the graph of $\Gamma_{V,\bar{\lambda}}$ implies that $\Omega(V, \bar{\lambda})$ is convex and compact. The continuity of $V'(V, \bar{\lambda})$ on $\Omega(V, \bar{\lambda})$ follows from the Theorem of the Maximum; the concavity of $V'(V, \bar{\lambda})$ on $\Omega(V, \bar{\lambda})$ follows from the concavity of the objective and the convexity of the graph of $\Gamma_{V,\bar{\lambda}}$.

It follows that $\Lambda'(V, \bar{\lambda})$ is an interval of the form $[\underline{\lambda}', \overline{\lambda}']$ and $T_V(V, \bar{\lambda})$ is concave and continuous. Since, the constraint set for problem (61) at a $\lambda \in \Lambda^S$ relaxes that for the SICCA problem (53) $(V(\lambda) \geq V^S(\lambda))$ and $\Lambda^S \subseteq [0, \bar{\lambda}]$, so $\Gamma_{V,\bar{\lambda}}(\lambda) \subseteq \Gamma_{V,\bar{\lambda}}(\lambda)$), and tightens that governing an arbitrary competitive allocation, $V^S(\lambda) \leq T_V(V, \bar{\lambda})(\lambda) \leq V^R(\lambda)$. Also, $\Lambda^S \subseteq \Lambda'(V, \bar{\lambda}) \subseteq \Lambda^R$. The latter implies that $\lambda' = 0$. It remains to check that $T_V(V, \bar{\lambda})(\overline{\lambda}') = V^{\text{aut}}$. Suppose that $T_V(V, \bar{\lambda})(\overline{\lambda}') > V^{\text{aut}}$. At each $\lambda$ from 0 to $(V^R)^{-1}(V^{\text{aut}})$, there exists an optimal “static” SICCA that solves:

$$\sup(1 - \beta)^{-1} \sum_{\theta \in \Theta} [F(u, y, r, \theta, \theta)] P(\theta)$$

subject to (55), (60) and

$$\lambda \leq (1 - \beta)^{-1} \sum_{\theta \in \Theta} \left[ u(\theta) + \left( y(\theta) - T y(\theta) \right) \hat{\omega} \right] P(\theta)$$

$$\forall \theta, u(\theta) + \left( y(\theta) - T y(\theta) \right) \hat{\omega} \geq 0$$

$$\forall(\theta, \hat{\omega}), F(u, y, r, \theta, \omega) \geq F(u, y, r, \theta, \hat{\omega})$$

(64)

(65)

(66)

(67)

At any $\lambda \in [0, \overline{\lambda}']$, the payoff from this static allocation is less than $T_V(V, \lambda)(\lambda)$. Let $\hat{\lambda} = \lambda + \epsilon$, for $\epsilon > 0$ and small. Then it is possible to take a convex combination of the solution to the optimal static SICCA at $\hat{\lambda}$ and the solution at $\overline{\lambda}'$. Label the corresponding convex combination of $\hat{\lambda}$ and $\overline{\lambda}'$, $\lambda_{\psi}$. If the weight attached to the solution at $\overline{\lambda}'$ is large enough, the convex combination of solutions will be feasible at $\lambda_{\psi}$. It will be incentive compatible, deliver a payoff above autarky, and satisfy the implementability constraint at $\lambda_{\psi}$. But since $\lambda_{\psi} > \overline{\lambda}'$, the maximality of $\overline{\lambda}'$ is contradicted. 

**Lemma A4** For $(V, \bar{\lambda}), (V', \overline{\lambda}) \in \mathcal{Q}$ with $(V, \bar{\lambda}) \geq (V', \overline{\lambda})$ then $T(V, \bar{\lambda}) \geq T(V', \overline{\lambda})$.

**Proof:** Follows directly from the fact that the constraint sets for the optimizations defined by $T(V', \overline{\lambda})(\lambda)$, $\lambda \in \Lambda(V', \overline{\lambda})$ are smaller than for the optimizations defined by $T(V, \bar{\lambda})(\lambda)$. 

32
Lemma A5 Each term of the sequence \( \{T^n(V^R, \lambda^n)\} \) lies in \( Q \) and converges to a point \((V_\infty, \lambda_\infty) \in Q\). \((V_\infty, \lambda_\infty) \) is a fixed point of \( T \).

Proof: It is easily verified that \((V^R, \lambda^R) \) is a fixed point of an operator that does not incorporate the no lending, sustainability and incentive compatibility constraints. Thus, \( T(V^R, \lambda^R) \geq (V^R, \lambda^R) \) and the sequence \( \{T^n(V^R, \lambda^n)\} \) is a decreasing one. Let \((V_n, \lambda_n) = T^n(V^R, \lambda^R) \). Since it is decreasing and bounded below by 0, the sequence \( \{\lambda_n\}_{n=1}^\infty \) has a limit, \( \lambda_\infty \). Since \( \{\lambda_n\}_{n=1}^\infty \) is decreasing, for each \( \lambda \in [0, \lambda_\infty] \), \( \{V_n(\lambda)\}_{n=1}^\infty \) is a well defined sequence. It is decreasing and bounded below by \( V^\text{aut} \), and, hence, it converges to a limit \( V_\infty(\lambda) \). Since \( V_\infty \) is the pointwise limit of the sequence \( \{V_n\} \) and since each \( V_n \) satisfies \( V^S \leq V_n \) on \( \Lambda^S \), \( V_n \leq V^R \) on \( [0, \lambda_\infty] \) and is continuous and concave on \([0, \lambda_\infty]\), \( V_\infty \) also satisfies these properties. Also, \( \lim_{n \to \infty} V_n(\lambda_\infty) \geq V^\text{aut} \), so \( V_\infty(\lambda_\infty) \geq V^\text{aut} \). If \( V_\infty(\lambda_\infty) > V^\text{aut} \) then for all \( n \), \( V_n(\lambda_\infty) > V^\text{aut} \), but then by continuity of the \( V_n \) functions and the fact that \( \lim_{n \to \infty} \lambda_n = \lambda_\infty \), there exists an \( N \) such that for \( n > N \), \( V_n(\lambda_\infty) > V^\text{aut} \), a contradiction. Thus, \( \{(V_n, \lambda_n)\}_{n=1}^\infty \) converges to \((V_\infty, \lambda_\infty) \in Q \). We now establish that \((V_\infty, \lambda_\infty) = (T_\infty(V_\infty, \lambda_\infty), T_\infty(V_\infty, \lambda_\infty)) \).

Part 1: \((V_\infty, \lambda_\infty) \geq (T_\infty(V_\infty, \lambda_\infty), T_\infty(V_\infty, \lambda_\infty)) \). If \( \lambda \notin \Lambda_\infty \), then \( \lambda > \lambda_\infty \) and there exists some \( N \) such that \( \lambda > \lambda_n, n > N \). By the monotonicity of \( T_\infty(V_\infty, \lambda_\infty) \leq T_\infty(V_n, \lambda_n) = \lambda_{n+1} \) for all \( n \). Hence, \( \lambda > T_\infty(V_\infty, \lambda_\infty) \) and \( T_\infty(V_\infty, \lambda_\infty) \leq \lambda_\infty \). If \( \lambda \in [0, T_\infty(V_\infty, \lambda_\infty)] \), \( T_\infty(V_\infty, \lambda_\infty)(\lambda) = \sup\{u, y, r, v, \theta \} \in V_\infty(\lambda, \lambda_\infty) \sum_{\theta \in \Theta} [F(u, y, r, \theta, \theta) + \beta v'(\theta)] P(\theta) \leq \sup_{u, y, r, v, \lambda, \lambda_\infty} \sum_{\theta \in \Theta} [F(u, y, r, \theta, \theta) + \beta v'(\theta)] P(\theta) = T_\infty(V_n, \lambda_n)(\lambda) = V_{n+1}(\lambda) \). Taking the limit as \( n \) goes to infinity, we have, \( T_\infty(V_\infty, \lambda_\infty)(\lambda) \leq V_\infty(\lambda) \).

Part 2: \((V_\infty, \lambda_\infty) \leq (T_\infty(V_\infty, \lambda_\infty), T_\infty(V_\infty, \lambda_\infty)) \). Let \( \lambda \in \Lambda_\infty \). Denote a solution to the problem \( T_\infty(V_\infty, \Lambda_\infty)(\lambda) \) by \( \{u_n, y_n, r_n, v_n', \lambda_n'\} \). Each \( \{u_n, y_n, r_n, v_n', \lambda_n'\} \in \Gamma'_{V_\infty, \lambda_\infty}(\lambda) \). Since \( \Gamma'_{V_\infty, \lambda_\infty}(\lambda) \) is a compact set, \( \{u_n, y_n, r_n, v_n', \lambda_n'\}_{n=1}^\infty \) has a convergent subsequence \( \{u_{n_k}, y_{n_k}, r_{n_k}, v_{n_k}', \lambda_{n_k}'\}_{k=1}^\infty \). Denote the limit \( \{u_\infty, y_\infty, r_\infty, v_\infty, \lambda_\infty\} \). Since \( \lambda_{n_k}' \leq \lambda_{n_k} \) each \( k \), \( \lambda_\infty = \lim_{k \to \infty} \lambda_{n_k} \leq \lambda_{n_k} \) and \( V_\infty(\lambda_\infty(\theta)) \) is well defined for each \( \theta \). Additionally, \( V_\infty(\lambda_\infty(\theta)) = \lim_{n \to \infty} V_n(\lambda_\infty(\theta)) = \lim_{n \to \infty} \lim_{k \to \infty} V_n(\lambda_{n_k}(\theta)) \geq \lim_{n \to \infty} \lim_{k \to \infty} V_n(\lambda_{n_k}(\theta)) = \lim_{n \to \infty} \lim_{k \to \infty} V_n(\lambda_{n_k}(\theta)) = V_\infty(\lambda_\infty(\theta)) \). The second inequality comes from the continuity of \( V_n \), and the first inequality uses the fact that \( k \), and, hence, \( n_k \), and that the sequence of value functions is a decreasing one. The second inequality uses \( V_{n_k}(\lambda_{n_k}(\theta)) \geq v_{n_k}(\theta) \). The other constraints that comprise \( \Gamma'_{V_\infty, \lambda_\infty}(\lambda) \) define closed sets. Hence, \( \{u_\infty, y_\infty, r_\infty, v_\infty, \lambda_\infty\} \in \Gamma'_{V_\infty, \lambda_\infty}(\lambda) \) and \( T_\infty(V_\infty, \lambda_\infty)(\lambda) \geq \sum_{\theta \in \Theta} [F(u_\infty, y_\infty, r_\infty, \theta, \theta) + \beta v_\infty'(\theta)] P(\theta) = \lim_{n \to \infty} \sum_{\theta \in \Theta} [F(u_{n_k}, y_{n_k}, r_{n_k}, \theta, \theta) + \beta v_{n_k}'(\theta)] P(\theta) = \lim_{n \to \infty} V_{n_k}(\lambda_{n_k}(\theta)) = V_\infty(\lambda(\theta)) \). Moreover, if \( \lambda \in \Lambda'(V_\infty, \lambda_\infty) \subseteq \Lambda'(V_n, \lambda_n) \), then \( V_n(\lambda) = \lim_{n \to \infty} V_n(\lambda) \geq V^\text{aut} \). Hence, for all \( \lambda \in \Lambda'(V_\infty, \lambda_\infty) \), \( T_\infty(V_\infty, \lambda_\infty)(\lambda) = V_\infty(\lambda) \geq V^\text{aut} \) and \( T_\infty(V_\infty, \lambda_\infty)(\lambda) \geq \lambda_\infty \).

Lemma A6 For \((V, \lambda) \in Q \) and each \( \lambda \in T_\infty(V, \lambda) \), there exists a solution to problem (61) that satisfies, for all \( \theta, v'(\theta) = V(\lambda'(\theta)) \).

Proof: Let \( \mu, \xi(\theta), \chi(\theta), \tau(\theta), \zeta(\theta) \) be Lagrange multipliers on, respectively, the implementability constraint (54), the constraint on debt repayment values in state \( \theta \) (56), constraint (62), and the upper and lower boundary constraints on \( \lambda'(\theta) \) (63). Let \( \partial V(\lambda) \) denote the set of sub-differentials of \( V \) at \( \lambda \).

The first order condition for \( \lambda'(\theta) \) is 0 \( \mu P(\theta) + \xi(\theta) + \chi(\theta) \partial V(\lambda'(\theta)) + \zeta(\theta) - \tau(\theta) \). Suppose there is a
solution such that \( \mu P(\theta) + \xi(\theta) > 0 \) and \( v'(\theta) < V(\lambda(\theta)) \). Then \( \chi(\theta) = 0, \varpi(\theta) > 0 \) and \( \lambda'(\theta) = \overline{\lambda} \). But at \( \lambda'(\theta) = \overline{\lambda} \), \( V^{aut} \leq v'(\theta) < V(\lambda'(\theta)) = V^{aut} \), a contradiction. Suppose then that there is a solution with \( \mu P(\theta) + \xi(\theta) = 0 \) and \( v'(\theta) < V(\lambda'(\theta)) \). Consider raising \( \lambda'(\theta) \) slightly. This relaxes the constraints (54) and (56) in state \( \theta \), and if \( v'(\theta) \) is still feasible it does not reduce the government’s payoff. Raise \( \lambda'(\theta) \) until either i) \( v'(\theta) = V(\lambda'(\theta)) \), or ii) \( \lambda'(\theta) = \overline{\lambda} \). In the first case, the required condition holds immediately. In the second, it follows from \( V^{aut} \leq v'(\theta) \leq V(\overline{\lambda}) = V^{aut} \). ■

Lemma A7 Each \((V_\infty(\lambda), \lambda) \in \Lambda_\infty \equiv [0, \overline{\lambda}_\infty]\) is supported by a SICCA.

Proof: Let \( u_\infty : \Lambda_\infty \times \Theta \to \mathbb{R}_+ \), \( y_\infty : \Lambda_\infty \times \Theta \to \mathbb{R}_+ \), \( r_\infty : \Lambda_\infty \times \Theta \to \Lambda_\infty \), \( v_\infty : \Lambda_\infty \times \Theta \to [V^{aut}, V^R(0)] \) denote policy functions from the problem \( T_V(V_\infty, \overline{\lambda}_\infty) \) that satisfy \( v_\infty'(\theta) = V_\infty(\lambda_\infty(\theta)) \). Define the resource policy functions: \( c_\infty = u_\infty^{1-\rho}, l_\infty = T - y_\infty^{1-\gamma} \) and \( g_\infty = r_\infty^{1-\eta} \). Given an initial \( \lambda_0 \in \Lambda_\infty \), construct an allocation recursively by iterating on the equations: \( l_t(\theta^t) = l_\infty(\lambda(\theta^t), \theta_t), c_t(\theta^t) = c_\infty(\lambda_t(\theta^t), \theta_t), g_t(\theta^t) = g_\infty(\lambda_t(\theta^t), \theta_t) \) and \( \lambda_{t+1}(\theta^t) = \lambda_\infty(\lambda_t(\theta^t), \theta_t) \). After each \( \theta^t \), the functions \( c_t, l_t, g_t \) satisfy the resource constraint and the recursive implementability constraint (46). Since \( \lambda_{t+1}(\theta^t) \in \Lambda_\infty \) for all \( t, \theta^t \), \( \lim_{t \to \infty} \beta^t E_\theta[\lambda_{t+1}(\theta^t)] = 0 \) and the allocation \( \{c_t, l_t, g_t\}_{t=0}^{\infty} \) is competitive. Similarly, the constraint (46), the boundedness of the \( \lambda_\infty \) function and the recursive no lending condition (48) imply that, for all \( t, \theta^t \), \( E_\theta[\sum_{s=t}^{\infty} \beta^{s-t} J(c_s, \theta_s)] \geq 0 \) so that the allocation satisfies the no lending conditions (13).

Repeatedly using \( T_V(V_\infty, \overline{\lambda}_\infty) \) and the constructed allocation yields \( V_\infty(\lambda_0) = E \left[ \sum_{t=0}^{\infty} \beta^t W(c_t, l_t, g_t; \theta_t) \right] + E \left[ \beta^{t+1} V_\infty(\lambda_{T+1}) \right] \). Taking the limit as \( T \) goes to infinity and using \( V^R(0) \geq V_\infty(\lambda) \geq V^{aut} \) implies that \( V_\infty(\lambda_0) \) gives the payoff from the allocation. Similarly, \( V_\infty(\lambda_t(\theta^t)) \) gives the continuation payoff from the allocation after the history \( \theta^t \). Since, for all \( \theta^t \), \( \lambda_t(\theta^t) \in \Lambda_\infty \), then for all \( \theta^t \), \( V_\infty(\lambda_t(\theta^t)) \geq V^{aut} \), and the allocation satisfies the sustainability constraints. It also satisfies a sequence of recursive incentive compatibility constraints. \( \forall t, \theta^t, \theta^t : W(c_t(\theta^t), l_t(\theta^t), g_t(\theta^t); \theta_t) + \beta V_\infty(\lambda_{t+1}(\theta^t)) \geq W(c_t(\theta_{t-1}), l_t(\theta_{t-1}), g_t(\theta_{t-1}, \theta_t); \theta_t) + \beta V_\infty(\lambda_{t+1}(\theta_{t-1}, \theta_t)) \).

By an argument of Atkeson and Lucas (1992), these recursive incentive compatibility constraints and the boundedness of \( V_\infty \) imply that the allocation is incentive compatible. Thus, the allocation is a SICCA. ■

Proof of Proposition 6:

Part 2: (Sketch). The first part of the proof is a standard dynamic programming argument. It proceeds by showing that if \( V^S(\lambda) < T_V(V^S, \overline{\lambda}^S)(\lambda) \), \( \lambda \in \Lambda^S \), then there exists a SICCA consistent with \( \lambda \) that delivers a payoff in excess of \( V^S(\lambda) \), a contradiction. Conversely, if \( V^S(\lambda) > T_V(V^S, \overline{\lambda}^S)(\lambda) \), then there exists a feasible choice for problem (61) that delivers a payoff in excess of \( T_V(V^S, \overline{\lambda}^S)(\lambda) \), again a contradiction. So \( V^S(\lambda) = T_V(V^S, \overline{\lambda}^S)(\lambda) \), \( \lambda \in \Lambda^S \). It then follows from the definition of \( T_X \) that \( \overline{\lambda} = T_X(V^S, \overline{\lambda}^S) \). ■

Part 3: Using the monotonicity of \( T \) and the result from Part 2, \( (V^S, \overline{\lambda}^S) = T(V^S, \overline{\lambda}^S) \leq T(V^R, \overline{\lambda}^R) \leq (V^R, \overline{\lambda}^R) \). Hence, iterating on these conditions: \((V^S, \overline{\lambda}^S) \preceq (V_\infty, \overline{\lambda}_\infty) = \lim_{n \to \infty} T^n(V^R, \overline{\lambda}^R) \). But since each \((V_\infty, \lambda), \lambda \in \Lambda_\infty \) is supported by a SICCA, \((V^S, \overline{\lambda}^S) \succeq (V_\infty, \overline{\lambda}_\infty) \). Hence, combining inequalities, \((V^S, \overline{\lambda}^S) = (V_\infty, \overline{\lambda}_\infty) \). ■

Part 1: Follows from the fact that \((V^S, \overline{\lambda}^S) = (V_\infty, \overline{\lambda}_\infty) \in Q \). ■