On Recent Expositions of Horizontal and Vertical Equity

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This paper constructs an analytical framework for the analysis of equity subsuming both the traditional public finance concepts of horizontal and vertical equity, that deal with the ideas of "equal treatment of equals" and progressivity, and a relatively new notion of horizontal equity, which requires that the before tax rank of a taxpayer in the distribution of income or welfare be maintained after tax in order for this second form of horizontal equity to be achieved. The purpose of developing this framework is to relate traditional and more recent equity concepts to each other. The concept of a Lipschitz tax system is introduced to facilitate the analysis.

1. Introduction

Summary measures of income and other distributions have long interested economists and statisticians. For example, Musgrave and Thin (1948) examined a variety of formulas for calculating the degree of progression of a personal income tax system. Much earlier,

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Mill (1921) sought to ascertain whether one could produce a progressive income tax regime if one knew consumers' marginal utilities; Samuelson (1947) made this approach more precise.

In a number of related papers, Atkinson (1970), Blackorby and Donaldson (1978), Sen (1973), Kondor (1975), Rosen (1978), Fields and Fei (1978), and King (1983) have pointed out that index numbers of the income distribution should be consistent with a social welfare function. Atkinson (1970), for example, develops on the basis of certain characteristics of, or postulates concerning an underlying social welfare function, a particular index of vertical income inequality while Fields and Fei (1978) examine a number of commonly used index measures (coefficient of variation, Gini coefficient, Atkinson's index, and Theil's index) to see if they are consistent with three axioms that they recommend for the development of inequality comparisons. Lambert (1989) provides a nice survey of the social welfare function approach.

Feldstein (1976), Atkinson (1980), Plotnick (1981), Jenkins (1988), and Kaplow (1989) have rekindled interest in horizontal equity, and in an important paper, King (1983) unified consideration of the vertical and horizontal characteristics of tax systems using a social welfare function approach suggested by the earlier papers.

The goals of this paper are to develop a framework sufficiently general and precise to permit description of this newer literature in a consistent manner, to develop and make precise some older concepts of vertical and horizontal equity found in the public finance literature, and to demonstrate clearly some relationships between the two literatures. Some semantic problems that have caused confusion are resolved in the process. Further, one is able to compare the relative strengths of principles of equity used by various authors. In terms of organization, Section 2 provides the foundations of the framework. Section 3 describes how the newer literature fits into the framework. Section 4 describes how the older concepts of public finance fit into the framework. Section 5 provides a look at a comparison of the two literatures. Section 6 provides a characterization of an older concept. Section 7 provides some examples for illustrative purposes, while Section 8 contains our conclusions.

2. A Framework for Analyzing Concepts of Vertical and Horizontal Equity

In order to be able to discuss the interrelationships of various notions of equity, it is first necessary to formalize the structure of the framework.

The following are its primitives. Taxpayer units (or consumers) are represented by a measure space $(A,\ \mathcal{A},\ \nu)$ where A is the set of consumers, $\mathcal{A}$ is a σ-algebra (subsets of consumers), and ν is a measure representing the relative size of consumers. An example of this representation is $([0,\ 1],\ B,\ m)$, where B is the collection of measurable subsets of $[0,\ 1]$ and m is Lebesgue measure on $[0,\ 1]$. Of course, if the number of consumers is finite, then ν is purely atomic.

To each taxpaying unit is associated a set of n characteristics that are independent of any tax system imposed.[1] For example, these might include family size, endowment, race, and location of residence. Let D_i, a subset of a Euclidean space, be the domain of the i^{th} characteristic. Let $D \equiv \prod_{i=1}^{n} D_i$. An economy is then defined to be a measurable map $s\colon\ A \rightarrow D$ that assigns n characteristics to each consumer, just as in the Hildenbrand (1974) model of an economy. Let S be the set of all such measurable maps.

In order to talk about equity as it is known in the public finance literature, it is necessary to introduce some concept of income. In a partial equilibrium setting, once an income concept has

[1]This may include a person's name to allow perfect discrimination.

been chosen, income can be introduced exogenously as, say, the first characteristic of each consumer, since it is assumed that the introduction of a tax system will not change prices. On the other hand, if this framework is embedded in a larger general equilibrium model of an economy, then prices and consequently income can change as a result of changing tax systems. One logical way to define *ex-ante* income is to compute equilibrium prices in this larger general equilibrium model without taxes and multiply the prices so obtained by the endowment of a consumer to obtain his income. This procedure is undesirable from several viewpoints. First, in the setting under discussion, equilibrium prices might not be unique, so income might not be well-defined. Second, computation could be cumbersome and complex. Third, the general equilibrium impact of a tax system is unpredictable. If one introduces a tax system that, for example, transfers wealth from the rich to the poor, and embeds this in a general equilibrium model, the new equilibrium (with the tax system) could make the poor worse off than in the original equilibrium. In essence, this derives from the transfer problem of international trade theory transplanted to the new context of the present model. Thus, in a general equilibrium system the computation of a measure of ability to pay is difficult. Certainly the evaluation of a tax system is likely to be just as difficult.

In any case, most concepts of equity that are familiar to the authors require some notion of ability to pay. Such a measure can be derived in either a partial or a general equilibrium setting. In any setting, it is possible to compute the income of a taxpaying unit $a \in A$, and we define[2] $s_1(a) \in D_1$ to be this income, where $D_1 = \Re$.

A tax function in this model is a measurable map $f : D \to \Re$,

[2]Subscripts denote vector components.

where the image of f is interpreted as post-tax income.[3] Let F be the set of all such measurable maps. Once again, there is a distinction in the interpretation of f for partial and general equilibrium models. If the system is embedded in a partial equilibrium model where taxpayers are assumed not to react to the imposition of a tax system, the f can simply be applied to D. If the system is embedded in a larger general equilibrium model, then the image of f must take into account reactions to the tax system. For example, a taxpayer may alter his income sources in reaction to the imposition of a tax. To take this into account, one simply computes a new equilibrium with the tax system imposed. In either case, it is possible to assign an after tax income to each vector of characteristics, whether the model is partial or general equilibrium in nature.[4]

An *equity concept* is a complete preorder over $F \times S$, a ranking of tax system-attribute distribution pairs. Examples of equity concepts include measures of vertical and horizontal equity, which will be discussed further below. The preorder is over both the space of tax systems and distributions of characteristics, since it is impossible to separate completely the two objects. For example, if there is a very inequitable feature of a tax system that applies to no one, then this tax system should be ranked the same as one without the feature. This should not be true if the inequitable feature applies to some taxpayers. Thus, attributes of taxpayers come into play in the evaluation of tax systems.

At this point, it should be noted that Arrow's Theorem applies to an equity concept if it is derived from individual preferences over $F \times S$. In our view, equity concepts are not derived from individual

[3]The examples and discussion in this paper can easily be rephrased in terms of tax liability or effective tax rates rather than post-tax income. One must choose a criterion for equity.

[4]Random taxes are not dealt with here for two reasons. First, real tax systems do not generally have random components. Second, this extension of the theory, like many other possible extensions, would complicate the arguments without altering the conclusions.

preferences. Instead, they either represent the preference ordering of a policy maker or a particular statistic describing certain aspects of the tax system.

To make the exposition easier, we assume that to each equity concept, $\geq$, there corresponds a top equivalence class, i.e., $\{(f,s) \in F \times S \mid (f,s) \geq (f',s'), \forall (f',s') \in F \times S\}$. These are the tax system-attribute distribution pairs that are most equitable under the equity concept. Of course, such a class need not always exist (unless the image of an index measure is bounded and closed), but such a class can always be created through an appropriate compactification.

A weaker idea than that of an equity concept is the notion of an *equity principle.* The latter is defined to be any restriction placed on the top equivalence class of an equity concept. Of course, there can be many equity concepts that satisfy a given equity principle. Many of the notions of equity found in the literature are in fact equity principles and not equity concepts.

One postulate, A1, used by all of the measures of equity under consideration is that the preorder depends only on the graph of f:

A1: The preorder $\geq$ depends only on $Gr(f,s) \equiv \{(x,y) \in D \times \Re \mid y = f(x) \text{ a.s. } (\nu \circ s^{-1})\}$.

That is, for $(f,s), (f',s') \in F \times S$, $Gr(f,s) = Gr(f',s') \Rightarrow (f,s) \sim (f',s')$. In particular, the ranking does not depend on tax provisions that apply to nobody.

3. The Social Welfare Function Approach

As mentioned above, recent research on equity has centered around work related to the social welfare function approach to inequality measure design. This literature focuses on the design of measures that capture either the vertical or the horizontal equity of a tax system. In order to proceed further, it is necessary to present the definitions of the equity principles associated with this literature.

For a tax system to be horizontally equitable, it is required that tax units ranked by income or well-offness (utility level) not be *re*ordered by this tax system.[5] This rank preservation condition is occasionally taken to be a logical implication of the definition,[6] rather than a part of the definition, but the bulk of the literature[7] takes it as the definition. This approach requires no explicit determination of which taxpayers are equals and which are not. As pointed out in King (1983, p.108), the literature is closely tied to the income mobility literature.[8]

The following equity principle is suggested by this literature:

> **A2:** A sufficient condition for $(f, s) \in F \times S$ to be in the top equivalence class of an equity concept is that if $d,\ d' \in D,\ d_1 > d_1'$ iff $f(d_1) > f(d_1')$ a.s. $(\nu \circ s^{-1})$.
>
> That is, a tax system is equitable if there are no rank reversals due to the tax system.

In this literature, a measure of horizontal equity (or inequity) is defined to be an equity concept satisfying **A1** and **A2**. Note that the condition of no rank reversal is generally taken to be only *sufficient* for a tax system to be equitable; this will be verified by an example later. It is thus possible to have peculiar tax systems (that induce rank reversals) in the top equivalence class.

The concept of vertical equity used in this literature is essentially that of income equality. A tax system is said to be vertically equitable if and only if it makes the post-tax distribution of income more equitable than the pre-tax distribution of income. To compare the vertical equity of two tax systems, it is necessary to compare the post-tax distributions of income generated by the two systems.

[5]See Feldstein (1976, p.87), Atkinson (1980, p.17), and Plotnick (1981, p.283).

[6]See Feldstein (1976).

[7]See King (1983, p.102).

[8]See also Shorrocks (1980).

Mathematically, this amounts to a comparison of the images of the tax systems, a stronger restriction than **A1**, which is:

> **A3:** The preorder $\geq$ depends only on $Im(f,s) \equiv \nu \circ s^{-1} \circ f^{-1}$.
>
> That is, for (f,s), $(f',s') \in F \times S$, $Im(f,s) = Im(f',s') \Rightarrow (f,s) \sim (f',s')$.

Thus, in this literature, a measure of horizontal equity (or inequity) is an equity concept satisfying **A1** and **A3**. For example, Atkinson (1970) develops a specific measure of vertical inequity by inverting a social welfare function to obtain the income equivalent of the social loss due to an unequal distribution of income.

One problem with this classification scheme for equity concepts is obvious. There are many equity concepts that satisfy **A1** but do not satisfy **A2** or **A3**. Thus, such a scheme is not exhaustive.

4. Older Concepts of Vertical and Horizontal Equity

The traditional ideas about equity seem to divide equity measures into three categories rather than the two noted above while at the same time using a similar nomenclature; indeed, we believe that this has been the source of some confusion. To distinguish these three categories from the previous two categories, we use the nomenclature income inequality, VE, and HE for categorizing equity concepts according to the older system.

The use of three categories for equity derives from Musgrave (1959) among others. By creating a distinction between the distributive and allocative functions of government, a distinction is also made between income redistribution (a distributive idea) and determination of the method of taxation to pay for public goods (an allocative idea). The latter includes as a partial solution the use of taxes based on ability to pay, which in turn includes as considerations vertical and horizontal equity (VE and HE). It is in this sense that we shall develop three categories of equity concepts.

Income inequality has been of interest to economists for quite some time. The term income inequality refers to exactly the same idea as vertical equity of the newer literature. Again, it is an equity concept satisfying **A1** and **A3**. Recent contributions include those of Bourguignon (1979) and Shorrocks (1980).

The traditional concept of horizontal equity (HE) is stated, for example, by Musgrave: "Perhaps the most widely accepted principle of equity in taxation is that people in equal positions should be treated equally."[9] Once again, this is only an equity principle, not an equity concept. A tax system that treats equals in the same manner is to be placed in the top equivalence class.

We may formalize this principle in the following manner. First, in order to ensure that the preorder is consistent with the traditional definition of equity, it is necessary to say who are equals and who are not equals. Sets of equals are groups of people for whom there is no moral justification for unequal treatment. This is distinguished from groups of individuals that are treated equally by the government or tax system; formally, the latter partition is dependent on the tax system and given by the inverse image of points under f.

Let $B_j \subseteq D$ for $j \in J$ be such that $\{B_j\}_{j \in J}$ partitions D. The collection of sets $\{B_j\}_{j \in J}$ are an exogenously determined collection of cells of equals. In this section we use the term "equals" in the sense of taxpayers with given characteristics in a set B_j. To examine whether a tax system treats equals in the same manner, it is only necessary to examine those taxpayers in any given cell without reference to those in other cells. Let 1_k be the indicator function

[9] Musgrave (1959, p.160).

for the set $K \subseteq D$. Let $\prod_1$ be the projection map from $\Re^n$ onto its first component, and let *supp* denote the support of a measure.

A4: A sufficient condition for $(f, s) \in F \times S$ to be in the top equivalence class of an equity concept is that for each j, $sup\{| a-b || a, b \in supp\,(\nu \circ s^{-1} \circ 1_{B_j} \circ f^{-1})\} \leq sup\{| a-b || a, b \in \prod_1 \circ [supp(\nu \circ s^{-1} \circ 1_{Bj})]\}$.

This means that the set of after tax incomes spanned by the consumers in B_j is not larger than the set of pre-tax incomes spanned by the consumers in B_j. Those who start as equals end as equals. It should be emphasized that this condition is only sufficient, not necessary. An equity concept satisfying **A1** and **A4** is defined to be a concept of HE. Examples of such measures can be found in Wertz (1975) and Berliant and Strauss (1983). Also, Pechman and Okner (1974) study empirically variations in effective tax rates by income class; this is basically an example of a measure of HE as well. The contrast between horizontal equity and HE will be given in the next section. However, it is important to note that measures of HE compare the post-tax circumstances of only those who are equals, while measures of horizontal equity may employ the post-tax circumstances of those who are not equals.

Our development of the traditional concept of vertical equity (VE) here is complementary to the concepts of horizontal equity (HE) and distributional equity presented above in Section 4. It is often referenced as the axiom that people with greater ability to pay should pay more in taxes.[10] We shall expand on this.

A5: **A1** is satisfied, $\exists\ (f, s) \in F \times S$ such that **A4** is not satisfied, and $\exists\ (f', s') \in F \times S$ such that **A3** is not satisfied.

A measure of VE is an equity concept satisfying **A5**. In other words, measures of VE do not depend solely on the post-tax income

[10] Musgrave and Musgrave (1976, p.242).

distribution (they depend on some pre-tax variables), and they do not depend solely on the relative post-tax positions of equals. Thus, they involve pre- and post-tax positions as well as comparisons of taxpayers who are not equals.

In summary, the equity concepts of distributional equity, HE and VE exhaust all possible equity concepts satisfying **A1**. Next, the classification systems of sections 3 and 4 are compared.

5. A Comparison of the Two Approaches

To begin, note that the concepts of vertical equity and distributional equity are the same, so the difference in classification schemes arises from a difference between horizontal equity on the one hand and HE and VE on the other hand.

First, it is shown that horizontal equity and HE are logically divorced from one another, so that one should not be used as justification for the other, and acceptance of one should be independent of acceptance of the other. The fact is demonstrated using two counter-examples.[11]

For the first counter-example, taxpaying units are evenly divided between two narrow pre-tax income brackets, one high and one low, where the brackets have the same width and the same internal distribution within each bracket, and plenty of space between them containing no taxpaying units (see Figure 1). Further, suppose the tax/transfer system maintains the overall distribution of these units, but is such that the corresponding units in each band switch places. Certainly, given that these units *within* each band are considered to be equals, this tax system conforms to the classical notion of equity, that of equals being treated equally (HE). However, this tax system also plays havoc with the rank ordering

[11] The counter-examples and discussion in this section closely follow Berliant and Strauss (1985).

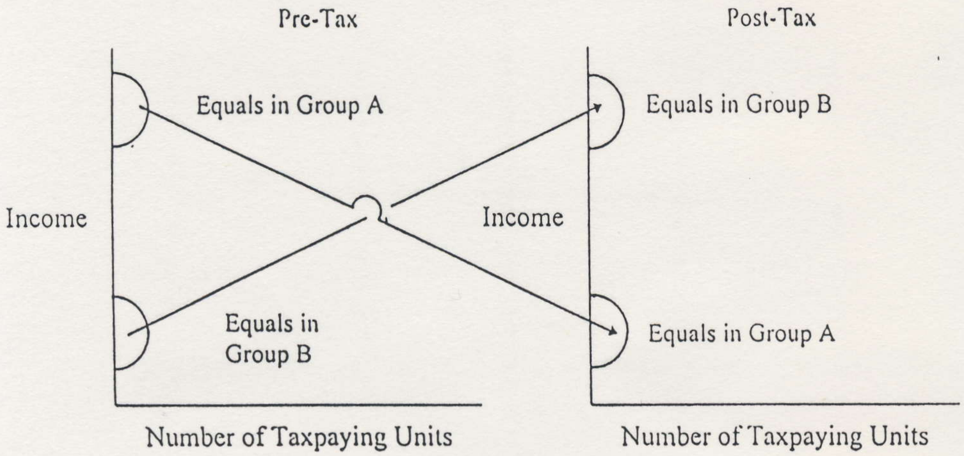

Figure 1

of the units. Thus, changes in the rank ordering do not imply that there are horizontal inequities (HE) present in the tax system. The tax system is HE but not horizontally equitable.

Two obvious objections may be raised to the structure of this example. First, the term "equals" is never defined; but this is not needed since the bands can be made as narrow as necessary (even degenerate). Second, no real income distribution looks like this one. However, it is equally obvious that this example may be embedded in a larger distribution while maintaining its purpose and conclusion.

The second counter-example postulates a pre-tax regime with one narrow income bracket in which the entire population is concentrated (see Figure 2). Suppose the tax/transfer system spreads the distribution proportionally over a much wider range (i.e., its support becomes larger).[12] Certainly the rank ordering of all individuals does not change under this tax scheme. Also, if the pre-tax income band is narrow enough to allow all taxpaying units to be considered to be equals, then the tax system is not horizontally

[12] Such an operation can be made precise through the use of a convolution; see Rudin (1974, p.155).

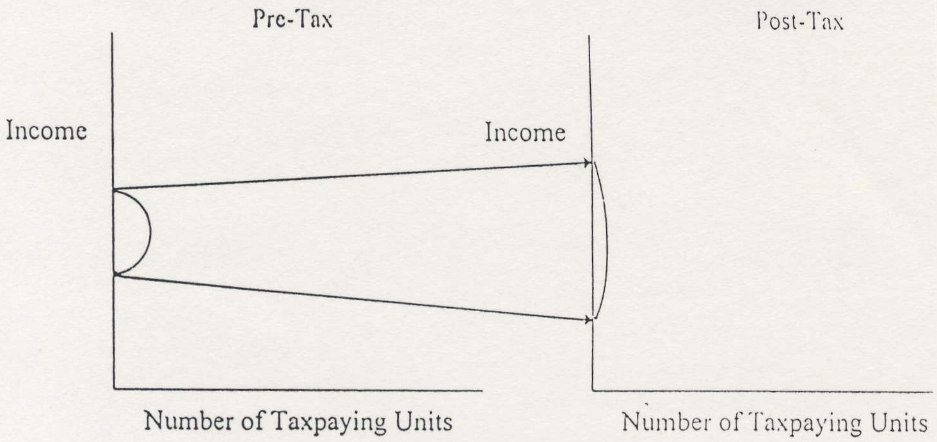

Figure 2

equitable in the classical sense (HE); some taxpayers receive windfalls while others experience huge losses through the imposition of the tax system. Thus, tax systems characterized by horizontal inequities (HE) do not necessarily change the rank order of taxpaying units. The tax system is horizontally equitable but not HE.

It is sometimes claimed, for example in Plotnick (1982, p.375), that a generalization of "equal treatment of equals," namely "similar treatment of similars," is at the heart of the rank reversals measures. However, note that "similar treatment of equals" is a weakening of the "equal treatment of equals" concept, while "equal treatment of similars" is a strengthening, so that "similar treatment of similars" is neither stronger nor weaker than "equal treatment of equals."

If the reader objects to the assumption that all of those units in the pre-tax income bands above are equals, and would prefer the "similar treatment of similars" concept, the examples may be reinterpreted. If those taxpaying units within each pre-tax income band are regarded to be in similar circumstances, the first counterexample shows that changes in the rank ordering do not imply that similars are not treated similarly. In addition, the second example shows that when similars are not treated similarly, the rank ordering might not change.

These counter-examples serve to illustrate that the traditional idea of HE, along with the independent notion of "similars treated similarly," are divorced from the ideas of rank order preservation and horizontal equity. Thus the burden lies with those who wish to postulate changes in the rank ordering as part of the definition of a horizontally inequitable tax system. These examples can, of course, be made mathematically precise so as to fit into the framework developed above, but it seems repetitive to do this. It suffices to say that **A2** and **A4** are unrelated unless further assumptions are made.

Under strong assumptions, a weak relationship does exist.

Proposition 1: If the B_j are points in D rather than sets and the hypothesis of **A4** is not satisfied for some tax system-attribute distribution pair, there exists a ranking system so that the hypothesis of **A2** is not satisfied.

Proof: The proof of this proposition is trivial.

How are rank-reversal measures related to VE? Suppose the effects of a tax system on the taxpaying units in some narrow band of income (such that these units are considered equals) are examined. The changes in the rank ordering of these individuals depends on the positions (pre-post-tax) of taxpayers outside this band, including some who have incomes very different from those in the band if the rank reversals are severe (as in the first example). This implies that rank reversal measures and VE are not independent.

Another approach to horizontal equity has been suggested. A tax system-distribution pair (f, s) is said to be weakly horizontally equitable if for $a, a' \in A, s_1(a) = s_1(a') \Rightarrow f(s(a)) = f(s(a'))$ *a.s.*. **A tax system distribution pair (f, s) is said to be strongly horizontally equitable if for $a, a' \in A, s_1(a) = s_1(a') \iff f(s(a)) =$**

$f(s(a'))$ *a.s.*. Weak horizontal equity means that a tax system-distribution pair is perfectly equitable when it treats taxpayers with the same income in the same manner. Strong horizontal equity means that a tax system-distribution pair is perfectly equitable when it is weakly horizontally equitable and when taxpayers with different pre-tax incomes necessarily end up with different post-tax incomes.

What these two *principles* have in common is that they specify restrictions on the ideal (perfectly equitable) class of tax-system-distribution pairs, but not how to rank inequitable tax system-distribution pairs. Mathematically, only the top equivalence class of an equity concept is restricted. That is, for weak horizontal equity, if $(f, s) \in F \times S$ and $a, a' \in A, [s_1(a) = s_1(a') \Rightarrow f(s(a)) = f(s(a'))]$ *a.s.* (ν) means that (f, s) is in the top equivalence class. For strong horizontal equity, if $(f, s) \in F \times S$ and $a, a' \in A, [s_1(a) = s_1(a') \iff f(s(a)) = f(s(a'))]$ *a.s.* (ν) means that (f, s) is in the top equivalence class.

In order to turn these two principles into equity concepts, the other equivalence classes and an ordering over them must be given. In terms of classifying these notions, weak horizontal equity is exactly **A4** with $B_j = j \times \prod_{i=2}^{n} D_i (j \in \Re)$. Thus, weak horizontal equity falls under the auspices of HE in Section 4.

On the other hand, the principle of strong horizontal equity is inconsistent with **A4**, so it falls in the category of VE. It is also obvious that an equity concept satisfying strong horizontal equity does not necessarily satisfy **A2** or **A3**, and hence cannot be classified in the scheme of Section 3.

The principle of weak horizontal equity implies that of strong horizontal equity. When these two principles are employed by equity concepts, little can be said about the relative sizes of the top equivalence classes of the measures from just the principles. If a principle is a necessary condition, it is generally possible to make a statement about the relative size of top equivalence classes of equity concepts satisfying it. Thus, it is important to state whether a principle is *necessary* or *sufficient* (or both) in order to precisely determine the meaning of a restriction on an equity concept. For example, the restriction **A4**, a sufficient condition, is a test of coarseness of equivalence classes. Thus, it is not surprising that equity concepts satisfying strong horizontal equity can fail the test, as can equity concepts satisfying the strong restriction of **A2**. Certainly those equity concepts taking no rank reversal and strong horizontal equity as necessary conditions cannot satisfy **A4**.

In summary, the strength of an equity principle is reflected in the fineness or coarseness of the top equivalence class (of tax system-attribute distribution pairs) generated by an equity concept. If the principle is a sufficient condition, such as **A2** or **A4**, then the equivalence class is relatively coarse. If the principle is a necessary condition, then the equivalence class is relatively fine. This discussion can be formalized, in part, as follows:

> **Proposition 2:** If $\geq$ and $\geq'$ are equity concepts such that the top equivalence class of $\geq$ is contained in the top equivalence class of $\geq'$, and if $\geq$ satisfies a given sufficient equity principle, so does $\geq'$. If $\geq'$ satisfies a given necessary equity principle, so does $\geq$.

6. Smooth Tax Systems

Many of the assumptions discussed have topological interpretations. For example, the principle **A2** and that of strong horizontal equity are related to whether a tax system is one-to-one over the relevant domain, and hence **A2** is closely related to degree theory. The principle **A4**, on the other hand, is related to the continuity properties of the tax map.

Theorem 3: Let $f \in F$ satisfy the following Lipschitz-type condition on $D : | f(x) - f(y) | \leq | x_1 - y_1 |$. Then the hypothesis of **A4** is satisfied for any $s \in S$, so that any equity concept satisfying the principle **A4** places (f, s) in the top equivalence class. That is, (f, s) is HE for any $s \in S$.

Proof: First fix j. Pick $a, b \in supp(\nu \circ s^{-1} \circ 1_{B_j} \circ f^{-1})$. To this pair there corresponds at least one pair $x, y \in supp(\nu \circ s^{-1} \circ 1_{Bj}) \subseteq D$ with $f(x) = a, f(y) = b$. By the condition of the theorem, $| a - b | \leq | x_1 - y_1 |$.

Thus, for each $a, b \in supp(\nu \circ s^{-1} \circ 1_{Bj} \circ f^{-1}) \ \exists \ x, y \in supp(\nu \circ s^{-1} \circ 1_{Bj})$ with $| a - b | \leq | x_1 - y_1 |$. Hence the supremum over such a and b has to be less than or equal to the supremum over such x and y.

Q.E.D.

The condition cited in the theorem is strong, indicating that a small deviation in the income characteristic means (at most) a small change in tax liability, and that if two people are identical in income, they have identical after-tax income. Of course, this condition is necessary if we wish to have f be HE for any $\{B_j\}$ and any $s \in S$.

Theorem 4: If (f, s) is HE for any $s \in S$ and any collection $\{B_j\}$, then $| f(x) - f(y) | \leq | x_1 - y_1 |$ for all $x, y \in D$.

Proof: Fix $x, y \in D$. Pick $B_j = [x_1, y_1] \times \Re^{n-1}$ (assuming $x_1 \leq y_1$) and pick s so that $x, y \in supp(\nu \circ s^{-1} \circ 1_{Bj})$.

Then $| f(x) - f(y) | \leq sup\{| a - b | \mid a, b \in supp(\nu \circ s^{-1} \circ 1_{Bj} \circ f^{-1})\} \leq sup\{| v - w | \mid v, w \in \prod_1 \circ [supp\ (\nu \circ s^{-1} \circ 1_{Bj})]\} = | x_1 - y_1 |$.

Q.E.D.

7. Examples

In this section, examples of measures of vertical and horizontal equity are presented so as to clarify the aspects of vertical and horizontal equity that are represented. A full axiomatization of the measures cannot be provided due to space limitations; however, references to such work will be given where possible.

King (1983) presents an overall index of inequality that can be multiplicatively decomposed into horizontal and vertical parts in the sense of Section 3. These equity concepts are defined using an ordering generated by a social welfare function. In particular, the following measure, among others, is derived:

$$I_H = 1 - \left[\sum_i (y_i e^{-\eta s_i})^k / \sum_i y_i^k\right]^{1/k}$$

where y_i is the *ex post* income of consumer i, s_i is a scaled order statistic, and η and k are parameters $(\eta \geq 0, k \neq 0)$. Of course, no inequality is achieved when $I_H = 0$. Setting this equality to obtain the top equivalence class, and letting $x_i = e^{-\eta s_i}$, we obtain

$$\sum_i y_i^k (x_i^k - 1) = 0.$$

Since x_i^k can be any positive number, we see that no rank reversal (*i.e.*, $s_i = 0 \ \forall \ i$) is sufficient but *not necessary* for a tax system to be placed in the top equivalence class. Tax systems with rank reversals can be placed in the top equivalence class. Thus **A2** is satisfied and I_H is a measure of horizontal equity. In the second classification scheme, this would be a measure of VE.

Atkinson (1980) and Plotnick (1981) used a measure described in Plotnick (1982, p.389) as:

$$(1/N^2 Y) \sum r_i (y_i^P - y_i^f)$$

where there are M units reranked of N total units, Y is mean income, r_i is the initial rank of individual i, y_i^f is the final income of individual i and y_i^P is the rank-preserving final income of individual i. No rank reversals imply that the value of the inequity index is zero, but there are other types of tax systems that generate an inequity value of zero. Hence **A2** holds as a sufficient condition, and this is a measure of horizontal equity or VE.

To clarify the distinctions between principles, it would be best to consider a tax reform proposal. It has been proposed that the child benefit (a flat benefit) be limited to couples with income below a certain level. Suppose that for simplicity of argument, we take pre-tax income to be that generated under current law. If only 2 parent, 2 child families are considered, **A4** is satisfied so any HE measure will show that the proposal does satisfy equal treatment of equals, except around the cutoff. The principles of no rank reversal and strong horizontal equity are violated around the cutoff, and measures of VE might show a change. Income distribution or vertical equity measures will generally show an improvement.

Now consider a proposal to limit the child benefit to couples with income *above* a given cutoff. Both the no-rank-reversal and strong horizontal equity principles are satisfied. The HE measures will show a small loss in horizontal equity around the cutoff. Measures of VE will show a loss in progressivity, while income distribution or vertical equity measures will show a deterioration in the distribution.

Besides illustrating what the different principles measure, the above suggests that the principle of no rank reversal (**A2**) and strong horizontal equity are related. In fact, it is easy to construct an example where they are unrelated. More importantly, equity principles are generally conditions *sufficient* (not necessary) to place a tax system-attribute distribution pair in the top equivalence class of an equity concept. Few measures yield economically meaningful necessary conditions. Thus, it is possible to have measures that satisfy several principles (sufficient conditions), such as strong horizontal equity and **A2**, or one but not others.

8. Conclusions

We have sought in this paper to develop a framework which permits the precise description of traditional and new notions of horizontal and vertical equity, and in turn provides a comparison of these ideas as they relate to operational index numbers of horizontal and vertical equity. Based on this framework, we find that some of the newer notions of horizontal equity, that involve the maintenance of relative ranks of taxpayers' before and after tax positions for the newer notions of horizontal equity to be present, are more properly viewed as vertical equity (VE) measures. Further, we find that the traditional notion of horizontal equity, summarized by "equal treatment of equals," is quite distinct from the newer notions of horizontal equity which require maintenance of relative rank positions. Indeed, the former is neither a necessary nor a sufficient condition of the latter.[13] Berliant and Strauss (1995) provide an axiomatic characterization of index numbers of HE and VE, consistent with the older notions found in the public finance literature.

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[13]This logical result is consistent with empirical work reported in Berliant and Strauss (1985), which reports correlations among various empirical index measures of horizontal and vertical equity, and finds generally a lack of correlation between some of the newer vertical measures such as Atkinson's I, and measures of progressivity developed by the authors.

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