Einstein boundary conditions for the 3+1 Einstein equations

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In the 3+1 framework of the Einstein equations for the case of a vanishing shift vector and arbitrary lapse, we calculate explicitly the four boundary equations arising from the vanishing of the projection of the Einstein tensor along the normal to the boundary surface of the initial-boundary value problem. Such conditions take the form of evolution equations along (as opposed to across) the boundary for certain components of the extrinsic curvature and for certain space derivatives of the three-metric. We argue that, in general, such boundary conditions do not follow necessarily from the evolution equations and the initial data, but need to be imposed on the boundary values of the fundamental variables. Using the Einstein-Christoffel formulation, which is strongly hyperbolic, we show how three of the boundary equations up to linear combinations should be used to prescribe the values of some incoming characteristic fields. Additionally, we show that the fourth one imposes conditions on some outgoing fields.

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The advent of sensitive gravitational wave detectors such as the Laser Interferometer Gravitational Waves Observatory (LIGO) [1] and the Laser Interferometer Space Antenna (LISA) [2] motivates the need for accurate templates of gravitational wave emission from powerful sources such as the collisions of black holes with other black holes or with neutron stars [3]. Even though the early and late stages of such merging events can be modeled analytically by perturbative methods [4,5], the actual merger phase is highly nonlinear and can be expected to be modeled accurately only by numerical simulations. Most numerical simulations of dynamical black-hole spacetimes break down well before the relevant information can be extracted, for reasons that are yet to be pinned down. Numerical analysis on the side, there are many fundamental issues in the Einstein equations that may be relevant to the stability of any numerical implementation of the binary black hole merger, among them; the intrinsic ill posedness of the related Cauchy problem, constraint violations, poor physical choices of binary black hole data and poor choices of boundary conditions. The solution to long term numerical instabilities of binary black hole mergers is likely to involve an orchestrated control of all these factors, in addition to the other elements of general numerical analysis that apply to time-dependent problems with boundaries. In this work, we point out that even before thinking about a numerical integration of the Einstein equations in a region with timelike boundaries, the possible implications of the Einstein equations on the boundaries need to be taken into consideration, as a matter of principle. To date, most numerical simulations with boundaries have largely disregarded this issue, as have most mathematical studies of the mixed initial-boundary value problem of the Einstein equations, with exceptions explicitly noted in the following.

Consider the 3+1 formulation of the vacuum Einstein equations with vanishing shift vector, in which the metric has the form

\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij} dx^i dx^j \]  

where \( \alpha \) is the lapse function and \( \gamma_{ij} \) is the Riemannian metric of the spatial three-slices of fixed value of \( t \). The Einstein equations \( G_{ab}\ =\ 0 \) take the Arnowitt-Deser-Misner (ADM) form [6]

\[ \gamma_{ij} = -2\alpha K_{ij}, \]  

\[ K_{ij} = \alpha(R_{ij} - 2K_{ij} K^l_l + K K_{ij}) - D_i D_j \alpha, \]  

with the constraints

\[ R - K_{ij} K^{ij} + K^2 = 0, \]  

\[ D_i K^i_j - D_j K = 0. \]

Here an overdot denotes a partial derivative with respect to the time coordinate \( \partial / \partial t \), indices are raised with the inverse metric \( \gamma^{ij} \), \( D_i \) is the covariant three-derivative consistent with \( \gamma_{ij} \), \( R_{ij} \) is the Ricci curvature tensor of \( \gamma_{ij} \), and \( R \) its Ricci scalar. Because there are no time derivatives of the lapse in any of the equations, the lapse function is completely free. Because a time derivative of the constraints turns out to be a linear combination of the constraints themselves and their space derivatives by virtue of the evolution equations, they will vanish at subsequent times in the domain of dependence of the initial slice if vanishing initially (how stable the propagation of the zero values of the constraints is to small perturbations does not concern us at this time [7]).

Strictly speaking, the initial-value problem consists of finding a spatially periodic or square integrable solution to the Einstein equations in the half spacetime \( t \geq 0 \). In practice,
one looks for a spatially non-periodic solution in $t \gg 0$, $x^i \in \mathcal{V}$, where $\mathcal{V}$ is a box in $\mathbb{R}^3$. There arises the problem of specifying data on the timelike boundaries of the region of interest. The problem is rooted in the fact that in order to specify such data one needs to know the values of the solution on the boundary for all time, which, of course, one does not. In an initial value problem without constraints, the boundary values may or not be constrained by the choice of initial data. It seems to us that if the problem, in addition, has constraints, the boundary values will know it, and it should be natural to expect consequences to the boundary values from the presence of the constraints. The point has been raised in a precise sense by Stewart [8] and further pursued in a similar direction by Calabrese et al. [9] and by Szilágyi and Winicour [10].

The existence of at least some such consequences is intuitively predictable from the point of view that the 3+1 equations derive from a 4D covariant formulation, which is more fundamental, that is, $G_{ab} = 0$. The 3+1 split is simply a choice of linearly independent combinations of the ten components of the Einstein tensor. The choice is precisely that which removes any time derivatives of second order from four of the equations, and is accomplished by projecting the Einstein tensor along the direction normal to the slicing $G_{ab} n^b = 0$.

One can easily convince oneself (or else refer to [11]) that the projection of the Einstein tensor along the direction normal to a surface given by a fixed value of any of the coordinates $x^i$ removes all second derivatives with respect to that coordinate $x^i$. Let us say we look at our boundary at a fixed value of $x$ and write down the projection of the Einstein tensor of the metric (1) along the normal vector $e^b = (0, \gamma^{ii})$, which may be transformed to unit length by dividing by $\sqrt{\gamma^{ii}}$, but that is clearly not necessary. The four components of the projection of the Einstein tensor along $e^b$ are

\[
G_{i b} e^b = -\frac{1}{2} \gamma^{i i} \left( \ln \gamma_{i j} - \gamma^{i j} \gamma_{j k} \right) - K D^i \alpha + K_i x D^i \alpha
+ \alpha \left( \gamma^{i k} \Gamma_{k j}^i + \gamma^{i j} \Gamma_{j k}^i \right) = 0,
\]

(6a)

\[
G_{s b} e^b = -\frac{K_s}{\alpha} + K K_s + R_s - \frac{1}{\alpha} D^s D_\alpha \alpha = 0,
\]

(6b)

\[
G_{z b} e^b = -\frac{K_z}{\alpha} + K K_z + R_z - \frac{1}{\alpha} D^z D_\alpha \alpha = 0,
\]

(6c)

\[
G_{x b} e^b = \frac{K - K_x}{\alpha} - \frac{1}{2} \left( R + K^{i j} K_{i j} + K^2 \right) + K K_x + R_x
+ \frac{1}{\alpha} \left( D^i D_\alpha \alpha - D^z D_\alpha \alpha \right) = 0.
\]

(6d)

Here the time derivative of the components of the extrinsic curvature is applied after raising an index, that is, $K^i_j = (\gamma^{i k} K_{k j})_{, i}$. The reader can verify that $R^i_i$ and $R^i_z$ do not involve second derivatives with respect to $x$ of any of the variables and that neither does the combination $R^i_s - \frac{1}{2} R$. In the framework of the ADM equations (2), even though, to our knowledge, there is no clear notion of which kind of boundary conditions would be appropriate for a unique solution, one could think of Eq. (6a) as an evolution equation for a particular combination of the first space-derivatives of the metric, of Eqs. (6b) and (6c) as evolution equations for $K^i_s$ and $K^i_z$ respectively, and of Eq. (6d) as an evolution equation for the combination $K^i_s + K^i_z$. Or one could freely combine them linearly to obtain evolution equations for combinations of these fundamental variables. In any case, these are evolution equations along the boundary itself which can be viewed as a sort of mixed Neumann-Dirichlet boundary conditions [12].

The situation clarifies in the case that one considers a first-order reduction of the ADM equations. In that case, the space derivatives of the metric are adopted as additional variables $q_{ij k} = \gamma_{ij k, k}$. This has the effect of eliminating all appearances of $x$ derivatives of the fundamental variables from Eqs. (6). However, a straight reduction in this fashion is not well posed in the sense that it is not strongly hyperbolic. If one is to consider a first-order reduction, one is much better off adopting one that is strongly hyperbolic in the sense that there exists a complete set of null eigenvectors of the principal symbol of the resulting system of equations.

In a generic strongly hyperbolic system of partial differential equations (PDE’s), the null eigenvectors constitute characteristic fields that propagate with the characteristic speeds of the system, and the notion of consistent boundary conditions can be defined unambiguously, as follows [13]. Characteristic fields that are outgoing at the boundary are determined by the initial values and cannot be prescribed freely at the boundary. On the other hand, characteristic fields that are incoming at the boundary must be prescribed in order for there to be a unique solution, and, furthermore, can be prescribed arbitrarily (we do not concern ourselves with stability for the moment).

In the case that there exist constraints, it seems to us that this notion of consistent boundary conditions may still be used but must be adapted to the particularities of a constrained system. To wit, if the incoming characteristic fields are to lead to a solution that satisfies the constraints inside the region of interest, then they must “know” about the constraints and, in general, may not be prescribed as freely as in the case where there are no constraints. In the case of the Einstein equations, we have shown that there are four boundary equations implied by the ten Einstein equations themselves, which are completely equivalent to linear combinations of the evolution equations and constraints that have no second derivatives across the boundary. Moreover, in a sense, the boundary equations are to the boundary surface what the constraints are to the initial slice. To see this consider the Bianchi identities $\nabla_a G^{ab} = 0$, which take the following form in a generic coordinate system $(t, x^i)$ adapted to spacelike slices:

\[
\partial_k G^{ab} = -\partial_k G^{ib} - \partial_k G^{ib} - \partial_k G^{ac} - \partial_k G^{ib} - 4 \Gamma^{ac} \gamma^{ib} - 4 \Gamma^{ac} G^{ac}
\]

(7)
where $\Gamma^{c}_{ab}$ are the Christoffel symbols of the four-metric $g_{ab}$. The right-hand side obviously contains no derivatives with respect to $x$ higher than second order. Therefore, $\partial_{t}G^{tb}$ must be of second order in $\partial_{t}$ at most, which implies that $G^{ab}$ involves, at most, first-order derivatives with respect to $x$. Of course, the reasoning runs as well for the $y$ derivatives, the $z$ derivatives, and the $t$ derivatives, so that $G^{zt}$ has no second $y$ derivatives, $G^{zt}$ has no second $z$ derivatives and $G^{tb}$ has no second time derivatives. In fact, traditionally this reasoning has been implicitly used to argue that the constraints $G^{tb}=0$ are the only conditions needed at the initial slice to specify a solution to the Einstein equations via the initial-value problem [14].

We conclude that $G_{ab}e^{b}=g_{ab}G^{cx}=0$ contain as much knowledge about the solution of the Einstein equations in the interior as the boundary values may be expected to "know."

For this reason, we think that imposing the boundary conditions (6) on the incoming characteristic fields is necessary in order to evolve a solution to the evolution equations from constrained initial data that will satisfy the four constraints at a later time slice as well.

To illustrate the point, we now consider the Einstein-Christoffel formulation of the 3+1 equations [15]. By defining first-order variables as the following 18 linearly independent combinations of the space derivatives of the metric:

$$\gamma_{ij} = -2\alpha K_{ij},$$

$$K_{ij} + \alpha \gamma^{kl} \partial_{kl} f_{ij} = \alpha \{ \gamma^{ij}(K_{ik}K_{lj} - 2K_{ij}K_{kl}) + \gamma^{kl} \gamma^{mn}(4f_{kmn}f_{lj} - f_{km}f_{ln} + 8f_{kij}f_{mn} + 4f_{kmn}f_{lj}) - 8f_{kli}f_{mn} + 20f_{k}f_{ln} - 13f_{ik}f_{mn} - \delta_{ij} \delta_{kl} \ln Q - \delta_{ij} \ln Q \delta_{kl} \ln Q + 2\gamma^{kl} \gamma^{mn} \gamma^{ij} \gamma^{mn} \} \times (f_{mn} \partial_{l} \ln Q - f_{kl} \partial_{m} \ln Q) + \gamma^{kl}(2f_{ijk} - f_{ijk}) \partial_{l} \ln Q + 4f_{kij} \partial_{l} \ln Q \ln Q - 3f_{ijkl} \partial_{l} \ln Q \}$$

$$f_{ij} + \alpha \partial_{k} K_{ij} + \alpha \gamma^{mn} \{ 4K_{km}f_{j} - f_{mn}K_{j} - 2f_{mn}(K_{ik} + 2f_{jmn} - 3f_{kmn}) + 2\gamma^{mn} \gamma^{ij} \gamma [f_{km} \gamma_{ij} - 2f_{kl} \gamma_{mj} + 13f_{ij}f_{km} + 4f_{i}f_{j} + 8f_{k} - 20f_{m}m + f_{mn}] \times (8f_{m}m - 20f_{m}m) = 0, \}$$

$$C_{ij} = \gamma^{kl} \gamma^{mn} (K_{ik}f_{jm} - f_{jm}f_{ik} - K_{km}f_{ij} - K_{mj}f_{ik} + K_{km} - K_{mj}f_{ij}) = 0, \}$$

$$C_{kij} = \partial_{k} \gamma_{ij} - 2f_{kij} + 4\gamma^{mn}(f_{ln} \gamma_{ij} - \gamma_{kl}f_{ln} \gamma_{ij}) = 0 \}$$

with the constraints

$$\gamma^{ij}(K_{ik}K_{lj} - 2K_{ij}K_{kl}) + \gamma^{kl} \gamma^{mn}(4f_{kmn}f_{lj} - f_{km}f_{ln} + 8f_{kij}f_{mn} + 4f_{kmn}f_{lj}) - 8f_{kli}f_{mn} + 20f_{k}f_{ln} - 13f_{ik}f_{mn} - \delta_{ij} \delta_{kl} \ln Q - \delta_{ij} \ln Q \delta_{kl} \ln Q + 2\gamma^{kl} \gamma^{mn} \gamma^{ij} \gamma^{mn} \}

\times (f_{mn} \partial_{l} \ln Q - f_{kl} \partial_{m} \ln Q) + \gamma^{kl}(2f_{ijk} - f_{ijk}) \partial_{l} \ln Q + 4f_{kij} \partial_{l} \ln Q \ln Q - 3f_{ijkl} \partial_{l} \ln Q \}$$

$R^{\alpha}_{\beta}, \ R^{\beta}_{\alpha} \ (1/2)R$ expressed in terms of $f_{ij}$, the point being that there is a way to express their occurrence so that no $x$ derivative of any $f_{ij}$ appears. As they stand, they remain as evolution equations for the components $K_{ij}^{\alpha}, K_{ij}^{\beta}$ and $K_{ij}^{\xi} + K_{ij}^{\eta}$ respectively. However, Eq. (6a) takes the form

$$G_{ik}e^{b} = f_{m}^j - f_{m}^j + \alpha(2K_{j}(4f_{m}^j - 3f_{m}^j) + K_{j}f_{m}^j + f_{m}^j) - K_{ij}f_{m}^j \partial_{l} \ln Q \} \}$$

$$= 0.$$
are another six characteristic fields that are incoming at the boundary. The remaining characteristic fields are the six components of \( f^{\mu \nu} \), the six components of \( f^{\mu \nu} \) and the six components of \( \gamma_{ij} \), all 18 of which travel upward along the boundary (they have vanishing characteristic speeds).

Inverting to find the fundamental fields in terms of the characteristic fields we have

\[
K_i^j = \frac{+U_i^j - -U_i^j}{2},
\]

\[
f^{\mu \nu} = \frac{+U_i^j - -U_i^j}{2\sqrt{\gamma^{kl}}}. \tag{15}
\]

One can immediately see that the boundary conditions (6b) and (6c) turn into evolution equations for \( -U_i^j \) and \( -U_i^j \) assuming that the outgoing fields \( +U_i^j \) and \( +U_i^j \) are known, respectively. In fact, these equations yield the values of \( -U_i^j \) and \( -U_i^j \) linearly in terms of the outgoing fields and in terms of sources known from the previous time step. So the transverse components \( a=y,z \) of the projection \( G_{ab}e^b=0 \) are quite welcome and are actually necessary and consistent with the initial-boundary value problem of the Einstein equations.

Turning now to the longitudinal component \( a=x \) of \( G_{ab}e^b=0 \), which is Eq. (6d), we see that, because \( K-K_i^j = K_i^j + K_i^j \), the equation takes the form

\[
( +U_i^j + -U_i^j ) + ( -U_i^j + -U_i^j ) + \cdots = 0. \tag{16}
\]

On the other hand, because

\[
j^m_m - j^m_m = j^x_y + j^z_z - j^z_x - j^z_z \tag{17}
\]

then the time component \( a=t \) of \( G_{ab}e^b=0 \), which is Eq. (11), takes the form

\[
( +U_i^j + +U_i^j ) - ( -U_i^j + -U_i^j ) - j^x_x - j^z_z + \cdots = 0. \tag{18}
\]

Both equations thus involve the same incoming and outgoing characteristic fields: \( -U_i^j + -U_i^j \) and \( +U_i^j + +U_i^j \). This means that they determine both the outgoing and incoming fields \( +U_i^j + +U_i^j \) and \( -U_i^j + -U_i^j \). One way (by no means the only one) to make this point explicit is the following. Subtracting the two equations with appropriate factors one obtains an evolution equation for \( -U_i^j + -U_i^j \), which is welcome as a prescription of boundary values that are consistent with the Einstein equations:

\[
2\alpha G_{ab}e^b - \frac{2}{\sqrt{\gamma^{kl}}} C_{ab}e^b = +U_i^j + +U_i^j + \cdots = 0, \tag{19}
\]

which also involves time derivatives of some of the fields that travel along the boundary, but this is fine. In fact, we have shown already [11] that at least in the case of the Einstein-Christoffel formulation with spherical symmetry, this resulting boundary evolution equation is being used in numerical integration and is referred to as a constraint-preserving boundary condition [9].

So far, the projection of the Einstein equations in the direction normal to the boundary is seen as providing prescriptions for the boundary values that are necessary in the light of the initial-boundary value problem of this strongly hyperbolic formulation. Any solution to the ten Einstein equations must satisfy such boundary equations. Therefore, failure to impose such conditions on the boundary would result in a solution to the evolution equations with constrained initial data that would not solve all four of the constraints at a later time slice.

There remains the question of the fourth boundary equation implied by adding \( G_{ib}e^b=0 \) and \( G_{ib}e^b=0 \) with appropriate factors in order to eliminate the occurrence of the incoming fields. This equation is linearly independent of the other three; therefore it will not be satisfied as a consequence of imposing the other three. One can see by inspection that this is an evolution equation for the outgoing field \( +U_i^j + +U_i^j + +U_i^j + +U_i^j = 0 \),

\[
2\alpha G_{ib}e^b + \frac{2}{\sqrt{\gamma^{kl}}} G_{ib}e^b = +U_i^j + +U_i^j + +U_i^j + +U_i^j + \cdots = 0, \tag{20}
\]

which involves also the time derivatives of some of the fields that travel along the boundary, but does not involve the time derivative of any incoming characteristic fields. This equation is not “necessary” in the sense that the outgoing fields are already determined by propagation from their initial values. However, this equation must be consistent with the outgoing fields determined by the initial values, otherwise one would not be solving the full set of Einstein equations. Therefore, it is actually necessary for this equation to be satisfied. In fact, in principle, one may simply use this equation for those outgoing characteristic fields involved, with the confidence that this boundary prescription is consistent with the initial values, so long as the initial values satisfy the constraints with sufficient accuracy. The actual numerical implementation of this boundary condition is a subject for subsequent study and lies beyond our current scope.

In summary, in the case of the Einstein-Christoffel formulation, the four “Einstein boundary conditions” defined as \( G_{ab}e^b=0 \) up to linear combinations constrain the values of three of the six incoming characteristic fields. The other three incoming characteristic fields are free, since there are no other available conditions on the boundary. In fact, the 18 first-order constraints \( C_{ijk} \)—which are needed in order to pick an actual solution of the Einstein equations out of the larger set of solutions of the Einstein-Christoffel equations—are preserved trivially by the evolution. This can be seen easily by taking a time derivative of \( C_{ijk} \) and using the evolution equations to eliminate the occurrence of time derivatives of the fundamental variables in the right-hand side. The reader may verify that this leads to \( \dot{C}_{ijk} = F_{ijk}(C, C_i, C_{ijk}) \), that is, ordinary differential equations for the propagation of \( C_{ijk} \). This means that the 18 first-order constraints are trivi-
ally satisfied on the boundary by virtue of the initial data and the evolution equations, and, consequently, they may not prescribe conditions on the incoming fields. There thus remain three incoming fields that can be specified in any arbitrary fashion for a solution to the Einstein equations. They may, for instance, be “frozen” by setting their time derivatives to zero, as is done by some authors [16,17].

We have thus obtained a complete set of boundary conditions for the Einstein-Christoffel formulation from the projection of the Einstein tensor normally to the boundary and the fact that the 18 constraints that realize the first-order reduction are preserved by ordinary differential equations. The incoming fields that are prescribed by the three boundary equations carry, in a sense, the vanishing of $G_{ab}e^b$ into the interior, where the evolution equations hold. Since $G_{ab}e^b$ are linear combinations of the evolution equations and the constraints, it follows that the constraints will be satisfied in the interior. Thus the boundary conditions preserve the constraints.

The argument makes no use of the auxiliary system of propagation of the constraints implied by the evolution equations. Still, the resulting boundary conditions appear to be completely consistent with constraint propagation in the following sense. In the three-dimensional case that interests us here, if the evolution equations are satisfied, the constraint functions propagate according to a strongly hyperbolic system [17]. In this auxiliary system, three characteristic constraints are incoming, three are outgoing and the rest are “static.” In the context of this auxiliary system, the three incoming constraints must be set to zero along the boundary. One may thus infer that there will be three conditions on the boundary values of the fundamental fields that are necessary and sufficient to preserve the constraints. Since we have three Einstein boundary conditions and no other conditions on the boundary, the Einstein boundary conditions and the three conditions that arise from constraint propagation must be equivalent in some definite sense. We argue that the equivalence is through linear combinations of the evolution equations. In the spherically symmetric case we have shown that this is precisely the case [11]. Work is currently in progress to demonstrate explicitly the equivalence of the two sets of conditions in the three dimensional case and will appear elsewhere. Interestingly, since constraint propagation is at the basis of Calabrese et al.’s “constraint-preserving” boundary conditions [18], it would thus appear that projecting the Einstein equations normally to the boundary would shortcut the procedure leading to “constraint-preserving” boundary conditions, providing an exact nonlinear version of the conditions and a geometric basis for them at the same time.

We emphasize that there is a very subtle issue of consistency between the constrained initial data traveling toward the boundary, and the conditions $G_{ab}e^b=0$ on the boundary that apply to the outgoing fields. The issue turns critical in numerical simulations because of the inaccuracies involved in satisfying the initial constraints. Clearly, the outgoing fields propagating out from the initial data will carry and perhaps increase the effects of the initial errors, and there is a good chance that they will fail to satisfy Eq. (20), the odds of failure increasing with time. This issue may have large consequences for achieving long time numerical evolution. This observation applies especially to numerical simulations implementing black hole excision [19]. As far as we know, no consistency issues in strongly hyperbolic systems with constraints have been addressed in the reference literature on finite-difference methods such as [13].

Interestingly, the other constrained system of equations in physics which is oftentimes used as a model for the Einstein equations—that is, the Maxwell equations—does not have this issue. When the Maxwell equations $\partial^0 F_{ab} = J_b$ are written in terms of a vector potential $F_{ab} = \partial_0 A_{b\parallel}$, the other half of the system $\partial_0 F_{b\parallel} = 0$ is identically satisfied and the time component $A_0 = \phi$ of the vector potential may be considered arbitrary (just as the lapse function is arbitrary insofar as the 3+1 equations are concerned). It is a straightforward exercise to put the equations in “ADM form” by defining a variable $K_i = A_i$, and to show that the projection $e^b \partial^0 \partial_0 A_{b\parallel} = e^b J_b$ for $e^b = \delta^b_0$ turns out to be exactly one of the evolution equations. Exactly the same thing happens in the Lorenz gauge. The underlying reason is, of course, that there is only one constraint $C = n^b \partial^0 \partial_0 A_{b\parallel} - n^b J_b$ with $n^b = \delta^b_0$ which is preserved by the evolution by means of the trivial equation $C = 0$.

A separate issue not treated here is whether or not the “Einstein boundary conditions,” that is, $G_{ab}e^b = 0$ up to linear combinations, would lead to a well-posed initial-boundary value problem for the Einstein-Christoffel formulation in the sense that the boundary conditions preserve the well-posedness of the initial value problem. In this respect, Calabrese et al. [18] show how to obtain a subset of the constraint-preserving boundary conditions that lead to an overall well-posed initial-boundary value problem—including constraint propagation—for the linearized “generalized” Einstein-Christoffel system. The main difference between the Einstein-Christoffel system and its modified counterpart (aside from the fact that the variable $f_{\kappa\lambda\mu}$ is defined differently) is that the constraints of the modified system propagate in a symmetric hyperbolic manner, a difference that does not affect the well-posed character, so one may expect that a subset of well-posed boundary conditions can be found in the Einstein-Christoffel case as well. Furthermore, the “Einstein boundary conditions” clearly exist and can be written down for any formulation of the Einstein equations, including those that are symmetric hyperbolic and that have symmetric hyperbolic constraint propagation, by simply expressing Eqs. (6) in terms of the fundamental variables of the problem. There are preliminary indications that, in such cases, linear combinations of $G_{ab}e^b = 0$ exist which are well posed in the sense of [18].

But most of the numerical relativity effort is currently driven by formulations that are not strongly hyperbolic (see, for instance, [20–26]) in which case sensible boundary conditions are also needed, whether or not they are well posed. For those formulations, the constraint-preserving boundary conditions of Calabrese et al. may not even be defined, yet the projections $G_{ab}e^b = 0$ evaluated on the boundary are always available, are easy to write in exact non-linear form,
and provide physical boundary conditions that are clearly necessary in order to enforce the constraints, the question of well posedness becoming irrelevant.

There is the question of why numerical simulations should use these boundary equations at all if they can do just fine by simply “pushing the boundaries out”—even though doing so increases the computational requirements considerably. In this respect, we point out the existence of the following conjecture based on the fact, observed by many authors, that the run time of numerical simulations is noticeably shortened by growing constraint violations. The conjecture, yet to be proven rigorously, is that the run time might be extended by efficiently controlling the constraint violations. Since we have shown that the boundaries are intimately connected with constraint propagation, if the conjecture turns out to be true then the “Einstein boundary conditions” may have a role to play in extending the run time of numerical simulations.

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[1] LIGO is the Laser Interferometer Gravitational-Wave Observatory with twin 4-km interferometers based at Livingston, Louisiana, and Hartford, Washington, see www.ligo.caltech.edu
[2] LISA is the projected Laser Interferometer Space Antenna based in a heliocentric orbit, see lisa.gsfc.nasa.gov