Classical and quantum dynamics of the Faraday lines of force

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(Received 20 December 1993)

We study the vacuum Maxwell theory by expressing the electric field in terms of its Faraday lines of force. This representation allows us to capture the two physical degrees of freedom of the electric field by means of two scalar fields. The corresponding classical canonical theory is constructed in terms of four scalar fields, is fully gauge invariant, has an attractive kinematics, but a rather complicated dynamics. The corresponding quantum theory can be constructed in a well-defined functional representation, which we refer to as the Euler representation. This representation turns out to be related to the loop representation. The resulting quantization scheme is, perhaps, of relevance for non-Abelian theories and for gravity.

PACS number(s): 03.50.De, 03.70.+k

I. INTRODUCTION

Considerable effort [1,2] has been expended over many years to the problem of giving a reduced phase-space formulation, and eventual quantization, of physical theories [such as general relativity (GR) or Yang-Mills (YM) theory] that possess a gauge group. The problem has been to find the transformation from the "large" phase space, including the gauge degrees of freedom, to a smaller phase space for which all the variables are gauge invariant but yet contain all the physical information of the original large space. A formulation of this type was recently given for GR and YM theory in Ref. [3]. In this formulation, the basic variables of the reduced phase space were a generalization of Faraday's lines of force from electromagnetic theory. This formulation, though very geometrical, was nevertheless quite involved, presenting some difficult conceptual issues. It is the purpose of this article to clarify some of these issues by studying the same ideas but in a simpler system, namely, in vacuum Maxwell theory on Minkowski space.

In Sec. II we present the classical canonical Maxwell theory in terms of the new version of the reduced phase space. In Sec. III we show how the quantization can be carried out in terms of the new variables. An important aspect of the formalism that we develop is that the reduction to the physical degrees of freedom is entirely defined on physical space with no recourse to Fourier transforms or momentum space. In particular, the gauge-invariant degrees of freedom of the field are captured without using the (background) metric structure of spacetime. This feature may become very important in gravity.

II. CLASSICAL THEORY

We begin with the standard canonical form of Maxwell theory on a $3+1$ decomposition ($R\times E^3$) of Minkowski space. The Maxwell potential $A_a(x)$ and the electric field $E^a(x)$ are canonical conjugate pairs (x is in E^3, and $a=1,2,3$). They satisfy the first class Gauss-law constraint

$$\partial_a E^a(x)=0 \tag{2.1}$$

and evolve under the Hamiltonian

$$H=\int d^3x\,[E^a(x)E^a(x)+B^a(x)B^a(x)]\,, \tag{2.2}$$

where $B^a(x)$ is the magnetic field. Indices are raised and lowered by the Euclidean metric. A_0 is set to zero throughout the paper.

A common way to resolve the problems associated with the gauge transformations generated by the Gauss-law constraint is to decompose both the electric field and the connection into their transverse and longitudinal components, discard the longitudinal component and separate the two helicities of the transverse components by means of a transverse basis in momentum space. The resulting two components represent the physical degrees of freedom of the field. We follow here an alternative path, always remaining in coordinate space: we will describe the electric field in terms of its Faraday (or electric) lines of force [4] or the integral curves of E^a. The Faraday lines can be described (in general) by a congruence in the three-dimensional space E^3.

We denote curves in E^3 by Greek letters α, β, etc.; and real parameters labeling the curves by $\underline{u}$ and $\underline{v}$. For the purpose of this paper, we define a congruence $\mathcal{C}$ as a two parameter family $\alpha(\underline{u},\underline{v})$ of (unparametrized) nonintersecting curves that fill space. Thus a congruence is given by a map $\mathcal{C}$: $(\underline{u},\underline{v})\to\alpha(\underline{u},\underline{v})$ or, more explicitly, by $x^a=\alpha^a(s,\underline{u},\underline{v})$ with s an arbitrary parametrization of the individual curves. Note that, from this definition, a congruence carries more information than just a foliation of E^3 by one-dimensional curves. The extra information is contained in the *fixed* labeling of the curves by the two parameters $\underline{u}$ and $\underline{v}$. (The information in this labeling will be used to characterize the "density" of the Faraday lines, and thus fix the strength of the electric field.)

A congruence $\mathcal{C}$ is uniquely related to a pair of scalar fields as follows. Given a point x in E^3, there is a unique curve $\alpha(\underline{u},\underline{v})$ in the congruence $\mathcal{C}$ that goes through x, and therefore $\mathcal{C}$ determines two numbers $u=\underline{u}$ and $v=\underline{v}$ for every point x, i.e., two scalar fields, which we denote

as $u(x)$ and $v(x)$. Conversely the congruence is determined by any two scalar fields $u(x)$ and $v(x)$ as the set of the unparametrized curves $\alpha(\underline{u},\underline{v})$ defined as the intersection of the surfaces $u(x)=\underline{u}$ and the surfaces $v(x)=\underline{v}$. From now on, we will equivalently talk of the congruence, and of the pair of scalar fields $u(x)$ and $v(x)$.

A. Euler potentials and their conjugate momenta

We now consider the relation between congruences and electric fields. Let us fix a region R of the three-dimensional space. Given a congruence in R, determined by the two fields $u(x)$ and $v(x)$, we can construct the vector field

$$E^a(x)=\varepsilon^{abc}\partial_b u(x)\partial_c v(x) . \tag{2.3}$$

It is immediately seen that the vector field $E^a(x)$ constructed in this way is divergence-free. We interpret it as an electric field. Note that this electric field is everywhere tangent to the curves of the congruence—the curves of the congruence are therefore the integral curves of $E^a(x)$, or its Faraday lines of force.

Conversely, given an electric field $E^a(x)$, we can, locally, find two scalar fields $u(x)$, and $v(x)$, and therefore a congruence, such that (2.3) is satisfied. Note that in general this construction works only locally, because the integral lines of an arbitrary electric field may have wild global behavior. For instance, a line may get arbitrarily close to a point an arbitrary number of times so that the Faraday lines may fail to form a congruence. In order to avoid these global difficulties, and for simplicity, we restrict ourselves to a finite region of space. More precisely, we fix a fiducial two-surface Σ, and assume that we are only interested in a finite region R which lies in a neighborhood of Σ, and that all the Faraday lines in this region R have an "origin" on Σ and intersect Σ once and only once. We can then "coordinatize" the two-surface Σ by (u,v) parameters and label the lines of force intersecting the surface accordingly. Later, we will fix boundary conditions for the fields on this fiducial surface. Given this restriction, we can construct the $u(x)$ and $v(x)$ fields for any given electric field.

The two scalar fields $u(x)$ and $v(x)$ capture the two degrees of freedom of the *divergence-free* electric field. These fields were first introduced by Euler, and are sometimes referred to as *Euler potentials*. Following Ref. [3], we want to view $u(x)$ and $v(x)$ as the two gauge-invariant canonical coordinates of the electromagnetic dynamical system. Thus, we have to find their canonically conjugate momenta, which we denote as $p_u(x)$ and $p_v(x)$. These can be found by means of the "singular" canonical transformation [3,5] from the phase space given by the six canonical variables $[E^a(x), A_a(x)]$ (with some boundary conditions, which we will discuss below) to the set of four functions $[u(x), v(x), p_u(x), p_v(x)]$. These four functions are to be canonically conjugate to each other, gauge invariant, and are to span the *physical* phase space (namely the constraint surface modulo gauge transformations). Such a canonical transformation was introduced in Ref. [3] and is given by

$$E^a=\varepsilon^{abc}\partial_b u\partial_c v , \tag{2.3a}$$

$$\begin{aligned} p_u &= -B^a\partial_a v , \\ p_v &= B^a\partial_a u . \end{aligned} \tag{2.3b}$$

$p_u(x)$ and $p_v(x)$ are then two scalar densities conjugate to $u(x)$ and $v(x)$. The generating functional $S[A,u,v]$ of the canonical transformation (2.3) is given by the natural pairing between a congruence $\alpha(\underline{u},\underline{v})$ and a one-form A: namely,

$$\begin{aligned} S[A,u,v] &= \int d^3x\, \varepsilon^{abc}\partial_b u\partial_c v A_a \\ &= \int d\underline{u}\int d\underline{v}\int_{\alpha(\underline{u},\underline{v})} A , \end{aligned} \tag{2.4}$$

where the first integral is over the region R, and the line integral is therefore over the portion of the curve $\alpha(\underline{u},\underline{v})$ contained in R. This generating functional leads to (2.3) via

$$\begin{aligned} E^a &= \frac{\delta S}{\delta A_a} = \varepsilon^{abc}\partial_b u\partial_c v \\ p_u &= -\frac{\delta S}{\delta u} = -\varepsilon^{abc}\partial_b A_c\partial_a v = -B^a\partial_a v , \\ p_u &= -\frac{\delta S}{\delta v} = -\varepsilon^{abc}\partial_b A_c\partial_a u = B^a\partial_a u . \end{aligned} \tag{2.5}$$

Equation (2.3) is the *implicit* form of the canonical transformation between E and A and $[u,v,p_u,p_v]$. It is, of course, not possible to invert these equations for A unless a gauge is fixed. We can, on the other hand, try to solve for the magnetic field $B^a=\varepsilon^{abc}\partial_a A_b$ (which is gauge invariant) as $B^a=B^a(u,v,q_u,q_v)$. Note that two components of the magnetic field [along the gradients of $u(x)$ and $v(x)$] are directly determined by Eq. (2.5); the last component (along the curves) can be obtained by integration, from the fact that B^a is divergenceless.

In doing so, however, we encounter a difficulty, because the third component of B^a is only determined up to an integration constant. More precisely, one can easily see that if $B^a(x)$ solves Eqs. (2.5), so does

$$\begin{aligned} B'^a(x) &= B^a(x)+\rho(u(x),v(x))E^a(x) \\ &= B^a(x)+\rho(u(x),v(x))\varepsilon^{abc}\partial_b u(x)\partial_c v(x) \end{aligned} \tag{2.6}$$

for any arbitrary function of two variables $\rho(u,v)$. Thus, there is a "lower dimensional" degeneracy in the canonical transformations (2.5). This is related to the fact that in the derivation of (2.5) from $S[A,u,v]$ we perform an integration by parts, which gives boundary terms

$$\int\partial_b(\varepsilon^{abc}u\partial_c v A_a) \text{ and } \int\partial_c(\varepsilon^{abc}\partial_b uv A_a) .$$

The simplest way to eliminate these complications is to choose appropriate boundary conditions for the fields. It is particularly convenient to fix this "lower dimensional set of degrees of freedom" of the fields on the fiducial surface Σ itself. If the surface is given by the mapping $\Sigma:(u,v)\varepsilon R^2\to x^a(u,v)$, we thus fix "boundary" conditions on Σ, by assuming that A_a is normal to the surface, by which we mean

$$A_u \equiv A_a \partial x^a/\partial u = 0 \ , \quad A_v \equiv A_a \partial x^a/\partial v = 0 \ . \tag{2.7}$$

Note that the boundary term considered above is then vanishing on Σ. It is then a straightforward exercise to show that the magnetic field is tangent to Σ, namely $B^a \varepsilon_{abc}(\partial x^b/\partial u)(\partial x^c/\partial v)=0$. This condition is sufficient to invert Eq. (2.3b), because it uniquely fixed the arbitrariness displayed by Eq. (2.6). The magnetic field is then uniquely determined in terms of u, v, p_u, and p_v by three requirements: (i) Eqs. (2.3b); (ii) the divergenceless property $\partial_a B^a=0$; (iii) the boundary conditions (2.7) on the fiducial surface.

B. Explicit inversion of the canonical transformations

We can compute explicitly the magnetic field as follows. We introduce the notation

$$f(x)=\int_0^{s(x)} g J^{-1} ds \tag{2.8}$$

to indicate that the scalar function $f(x)$ is obtained by integrating the scalar density $g(x)$ along the (unique) curve of the congruence that passes through x, the integral going from the origin of the curve on the fiducial surface Σ to the point x itself. Hence s is an arbitrary parametrization of the curves and J is the Jacobian of the transformation $(s,u,v)\rightarrow x^a=x^a(s,u,v)$, i.e.,

$$J \equiv \varepsilon^{abc} \partial_a u \, \partial_b v \, \partial_c s \ . \tag{2.9}$$

It is then straightforward to check that all conditions (i), (ii), and (iii) are satisfied by

$$B^a = \varepsilon^{abc} \partial_b u \, \partial_c \Pi_u + \varepsilon^{abc} \partial_b v \, \partial_c \Pi_v \tag{2.10}$$

with

$$\Pi_u = \int_0^{s(x)} p_u J^{-1} ds \ , \quad \Pi_v = \int_0^{s(x)} p_v J^{-1} ds \ . \tag{2.11}$$

It is also easily seen that the connection is given by

$$A_a = u \partial_a \Pi_u + v \partial_a \Pi_v + \partial_a \lambda \tag{2.12}$$

where $\lambda(x)$ is an arbitrary scalar function. An important property of the relation between B^a and A_a and the new variables is that the dependence of B^a and A_a on p_u and p_v is nonlocal, but *linear homogeneous* (except for the gauge term $\partial_a \lambda$). This fact will play a major role in the next section.

There is an alternative, but equivalent, method of obtaining Π_u and Π_v, that avoids the use of the arbitrary parametrization s (and J). Given the two densities (three-forms) p_u and p_v, it is easily seen that one can find two one-forms ${\Pi^u}_a$ and ${\Pi^v}_a$ that satisfy $p_u = \varepsilon^{abc} \partial_a u \, \partial_b v \, {\Pi^u}_c$ and $p_v = \varepsilon^{abc} \partial_a u \, \partial_b v \, {\Pi^v}_c$. The scalars Π_u and Π_v are then given by the line integral of these one-forms along the loops $\Pi_u = \int_0^{s(x)} {\Pi^u}_a d\alpha^a$, $\Pi_v = \int_0^{s(x)} {\Pi^v}_a d\alpha^a$.

Notice that if we did not fix the boundary conditions on the fiducial surface, the magnetic field would have been determined only up to an additive term proportional to E^a, the proportionality factor being constant along the curves. In other words, in the absence of boundary conditions, the inversion from the reduced phase space to the original one is unique up to a lower dimensional ambiguity. More precisely, this ambiguity would be given by a single function of two variables. The same ambiguity appears if we try to invert (2.3a) and expressed the reduced variables as functions of E_a and A_a. In fact, Eq. (2.3a) determines u and v only up to arbitrary transformations $(u,v)\rightarrow(u',v')=(u'(u,v),\, v'(u,v))$ with a unit Jacobian. These too depend on a single function of two variables, and should be fixed as boundary conditions.

C. Equations of motion

By inserting $E[u,v]$ and $B[u,v,p_u,p_v]$ into the Hamiltonian, we obtain the Hamiltonian $H[u,v,p_u,p_v]$ written in terms of these (gauge-invariant) reduced phase-space variables. One can then calculate the equations of motion

$$\begin{aligned}
dp_u/dt &= 2\partial_{[a} v \, \partial_{b]}(\partial_a u \, \partial_b v) \\
&\quad - B^d \partial_d \int_0^{s(x)} \varepsilon^{abc} \partial_a B_b \partial_c v J^{-1} ds \ , \\
dp_v/dt &= -2\partial_{[a} u \, \partial_{b]}(\partial_a u \, \partial_b v) \\
&\quad + B^d \partial_d \int_0^{s(x)} \varepsilon^{abc} \partial_a B_b \partial_c u J^{-1} ds \ , \\
du/dt &= -\int_0^{s(x)} \varepsilon^{abc} \partial_a B_b \partial_c u J^{-1} ds \ , \\
dv/dt &= -\int_0^{s(x)} \varepsilon^{abc} \partial_a B_b \partial_c v J^{-1} ds \ ,
\end{aligned} \tag{2.13}$$

with B^a given by (2.10). Note that all the equations that we have written are invariant under a change in the parametrization of each curve: $\alpha^a(s)\rightarrow\alpha^a(s'(s))$.

The equations that we have derived are quite cumbersome, and highly nonlocal. They can be greatly simplified by working with a preferred choice of the parameter $s=s_p$ along the curves, fixed as follows. We define the quantity

$$S(x)=\int_0^{s(x)} B_a d\alpha^a \tag{2.14}$$

and choose the preferred parameter s_p along each curve by

$$S(\alpha^a(s))=s_p \ . \tag{2.15}$$

Note that $S(x)$ is completely determined in terms of u, v, p_u, p_v and hence its evolution (i.e., dS/dt) is also uniquely determined, but quite difficult to calculate. Given this choice of parametrization, we have immediately that the projection of the magnetic field on the tangent of the curve is one:

$$B_a(x) t^a(x) = 1 \tag{2.16}$$

where we have introduced the curve tangent

$$t^a(\alpha^a(u,v,s_p)) = d\alpha^a(u,v,s_p)/ds_p \tag{2.17}$$

or

$$t^a(x) = J^{-1}(x) \varepsilon^{abc} \partial_b u(x) \partial_c v(x) = J^{-1}(x) E^a(x) \ . \tag{2.18}$$

Thus, this parametrization is equivalently defined by

$$J(x) = B_a(x) E^a(x) \ . \tag{2.19}$$

Two disadvantages of this parametrization are first that it makes explicit use of the background metric, and, perhaps more seriously, s_p is not in general a monotonic function along the curve—it can even give rise [see Eq. (2.12)] to discontinuities in the tangent vector t^a and perhaps have t^a not even defined. The equations of motion, however, do take the much simpler local form

$$\begin{aligned} dp_u/dt &= \partial_{[a}v\partial_{b]}(\partial_a u\partial_b v) - B^d\partial_d(N^b B_b)\ , \\ dp_v/dt &= -\partial_{[a}u\partial_{b]}(\partial_a u\partial_b v) - B^d\partial_d(M^b B_b)\ , \\ du/dt &= -N^b B_b\ , \\ dv/dt &= M^b B_b\ , \end{aligned} \tag{2.20}$$

where $M^a=\partial\alpha^a/\partial u$ and $N^a=\partial\alpha^a/\partial v$ [3]. In this parametrization, the nonlocality is hidden in the fact that the parametrization $s_p(x)$ is uniquely, but nonlocally, determined by $u(x)$, $v(x)$, $p_u(x)$, $p_v(x)$.

From the evolution of these variables we can obtain the time evolution of the curves of the congruence themselves. The meaning that we attach to this idea is as follows: A particular curve (at one instant of time) can be followed into the future by considering the succession of curves that all intersect the fiducial surface at the same value of $(\underline{u},\underline{v})$. The successive curves $x^a=\alpha^a(t,s,\underline{u},\underline{v})$ will have a "connecting vector," connecting points with the same values of $(s_p,\underline{u},\underline{v})$. This can be obtained by the following argument: taking the time derivative of $(u,v,s_p)\equiv\omega^i=\omega^i(x^a,t)$ where $s_p=S(x)$ is our privileged parameter, and remembering that we are now treating the u,v,s_p as fixed, we have that

$$0=\omega^i{}_{,a}\,d\alpha^a/dt+d\omega^i/dt\ . \tag{2.21}$$

Using the set $\theta^a{}_i=(N^a,M^a,t^a)$ which is dual to $\omega^i{}_{,a}$ and (2.12), we obtain

$$d\alpha^a/dt=(N^aM_b-M^aN_b)B^b-(dS/dt)t^a\ , \tag{2.22}$$

the evolution equation for the electric lines of force. Note that in the connecting vector (2.22), any term proportional to the tangent vector is "nonphysical" in the sense that it can be changed arbitrarily by a change in the curve parametrization.

D. Positive-frequency version of the formalism

In order to discuss the quantization of the theory, it is useful to introduce an alternative version of the reduced phase-space construction and dynamics that we have presented, based on the introduction of the "positive- and negative-frequency" fields E^+ and A^-. For simplicity, let us fix the gauge by $\partial_a A^a=0$, $\partial_a A^{-a}=0$, so that $A^-{}_a$ is uniquely determined by $B^-{}_a$ (boundary conditions are fixed). The two complex fields $E^{+a}(x)$ and $A^-{}_a(x)$ are defined by

$$\begin{aligned} E^{+a}(x) &= E^a(x)-i\Delta^{1/2}A^a(x)\ , \\ A^{-a}(x) &= A^a(x)-i\Delta^{-1/2}E^a(x)\ , \end{aligned} \tag{2.23}$$

where the operator $\Delta^{1/2}$ is defined as

$$\begin{aligned} \Delta^{1/2}f(x) &\equiv (2\pi)^{-3}\int d^3k\int d^3y\,|k|e^{ik(x-y)}f(y)\ , \\ \Delta^{-1/2}f(x) &\equiv (2\pi)^{-3}\int d^3k\int d^3y\frac{1}{|k|}e^{ik(x-y)}f(y)\ . \end{aligned} \tag{2.24}$$

The transformation (2.23) from the real canonical variables E^a and A_a to the complex fields E^{+a} and $A^-{}_a$ is a canonical transformation; in particular, it is simple to check that

$$\{E^{+a}(x),A^-{}_b(y)\}=\delta^a{}_b\delta^3(x,y)\ . \tag{2.25}$$

Moreover, the two complex fields E^{+a} and $A^-{}_a$ provide a complex coordination of the phase space. They are not independent (the phase space has only six real dimensions per space point), but related, by construction, via the relation

$$[E^{+a}(x)]^*=i\Delta^{1/2}A^{-a}(x)\ , \tag{2.26}$$

which we will refer to as the reality conditions. The Hamiltonian can be written in terms of these fields as

$$H=i\int d^3x\,E^{+a}\Delta^{1/2}A^-{}_a\ . \tag{2.27}$$

Therefore the canonical theory can be entirely reexpressed in terms of the E^{+a} and $A^-{}_a$ fields. We will use this formulation below.

The fields E^{+a} and $A^-{}_a$ have of course an interpretation as the positive and negative components of the Maxwell field. To see this, it is sufficient to write explicitly the Hamiltonian equations

$$\partial_t A^-{}_a(x,t)=\frac{\delta H}{\delta E^{+a}(x)}=i\Delta^{1/2}A^{-a}(x,t)\ . \tag{2.28}$$

If we rewrite this equation in momentum space, we have

$$\partial_t A^-{}_a(k,t)=i|k|A^{-a}(k,t), \tag{2.29}$$

so that

$$A^-{}_a(k,t)=e^{i\omega t}A^-{}_a(k)\ , \tag{2.30}$$

where $\omega=|k|$; namely, $A^-{}_a$ is a negative-frequency field. Analogously we get

$$E^{+a}(k,t)=e^{-i\omega t}E^{+a}(k)\ ; \tag{2.31}$$

namely, E^{+a} is a positive-frequency field.

[Alternatively, we could write the four-dimensional Maxwell field $F_{\mu\nu}(x,t)$ as $F_{\mu\nu}(x,t)=F^+{}_{\mu\nu}(x,t)+F^-{}_{\mu\nu}(x,t)$, where $F^+{}_{\mu\nu}$ and $F^-{}_{\mu\nu}$ are the positive- and negative-frequency components of $F_{\mu\nu}$, defined by

$$\begin{aligned} F^+{}_{\mu\nu}(x,t) &= (2\pi)^{-3/2}\int d^3k\int d\omega\,\theta(\omega)e^{i(kx-\omega t)}f_{\mu\nu}(k,\omega)\ , \\ F^-{}_{\mu\nu}(x,t) &= (2\pi)^{-3/2}\int d^3k\int d\omega\,\theta(-\omega)e^{i(kx-\omega t)}f_{\mu\nu}(k,\omega)\ , \end{aligned} \tag{2.32}$$

where

$$f_{\mu\nu}(k,\omega)=(2\pi)^{-3/2}\int d^3x\int dt\, e^{-i(kx-\omega t)}F_{\mu\nu}(x,t)\ ,$$

and $\theta(x)$ is the step function, i.e., the characteristic function of R^+. Then it is easy to see that $E^{+a}(x)=F^{+0a}(x,0)$ and $B^-{}_a(x)=\varepsilon^{abc}\partial_b A^-{}_c=\varepsilon_{abc}F^{-bc}(x,0)$.]

We can now repeat the previously described construction by replacing E^a and A_a, with E^{+a} and $A^-{}_a$; i.e., we have

$$\begin{aligned} E^{+a}&=\varepsilon^{abc}\partial_b u'\partial_c v'\ ,\\ p'_u&=-B^{-a}\partial_a v'\ ,\\ p'_v&=B^{-a}\partial_a u'\ . \end{aligned} \tag{2.3'}$$

The inversion of these equations can be treated as before: B^{-a} and $A^-{}_a$ (up to gauge) are uniquely determined by (2.3'), by $\partial_a B^{-a}=0$, and by boundary conditions on the fiducial surface. The two resulting nonlocal functionals $B^{-a}[u',v',p'_u,p'_v]$ and $A^{-a}[u',v',p'_u,p'_v]$ are linear homogeneous (in some gauge) in p'_u and p'_v.

We have denoted with a prime the new variables defined by E^{+a} and $A^-{}_c$, to distinguish them from the ones defined by real fields. The four complex reduced phase-space variables $[u',v',p'_u,p'_v]$ are still canonically conjugate to each other and still coordinatize the physical phase space. Thus, the structure of the theory is the same as before, except for the existence of the nontrivial reality condition (2.25) and the new form of the Hamiltonian:

$$H=i\int d^3x\ \varepsilon^{abc}\partial_b u'\partial_c v'\Delta^{1/2}A^{-a}[u',v',p'_u,p'_v]\ . \tag{2.2'}$$

Note, however, that the quantities $u'(x),v'(x)$ do not admit a direct interpretation as Euler potentials of real Faraday lines, since they are complex. This does not affect the consistency of the above construction. Finally, note that a third version of our construction can be obtained by replacing the positive and/or negative fields with self-dual and/or anti-self-dual fields. This may be of interest for (the Ashtekar version of) GR.

In the following section, we base the quantization of the theory on the positive-frequency version of the reduction of the phase space, described here, namely on the primed $[u',v',p'_u,p'_v]$ variables (2.3'). This is not fully satisfactory for two reasons. First, the positive-frequency version of our construction does not have the immediate clean geometrical flavor of the real version of the construction. Second, the positive and/or negative frequencies of the fields are defined in terms of the background metric, and this weakens the interest of the original construction, which does not require a background metric, particularly in view of a generalization to GR. However, using the positive- and/or negative-frequency fields simplifies the construction drastically, and allows us to avoid dealing with infinities. Thus, we prefer to describe the much simpler quantization of the positive-frequency variables $[u',v',p'_u,p'_v]$ here, and leave the analysis of the viability of the quantization of the real $[u,v,p_u,p_v]$ variables (and of the ones based on the self-dual fields) to future investigations.

III. THE QUANTUM THEORY

The complications of the nonlinear dynamics of the theory described above may suggest that the quantum theory could become difficult too. Indeed, since the Hamiltonian is a complicated function of the reduced variables, the Schrödinger equation becomes quite cumbersome. However, we shall see that the action of the Hamiltonian operator is simple. We use the positive-frequency version of the reduction of the phase space, that was described at the end of the preceding section, namely on the primed $[u',v',p'_u,p'_v]$ variables. For simplicity of notation, however, we omit the use of the prime on the fields $[u',v',p'_u,p'_v]$, for the remainder of this section—all fields are of the primed type.

We quantize the theory by means of a Schrödinger functional representation where we represent states as functionals Ψ of the Euler potentials u and v: the operators u and v as multiplicative operators and the operators p_u, p_v as the functional derivatives

$$\begin{aligned} p_u(x)\Psi[u,v]&=-i\hbar[\delta/\delta u(x)]\Psi[u,v]\ ,\\ p_v(x)\Psi[u,v]&=-i\hbar[\delta/\delta v(x)]\Psi[u,v]\ . \end{aligned} \tag{3.1}$$

We may denote this representation of the quantum Maxwell field as the Euler representation. We choose the ordering of the Hamiltonian (2.2') with all the functional derivatives at the right.

A. *n*-photon states

Our task is to solve the time-independent and the time-dependent Schrödinger equation. The Hamiltonian depends on the derivative operators p_u, p_v via B_a (or A_a) which is a nonlocal function of p_u,p_v. At first sight, this may seem to indicate that the dynamics will be as difficult to treat as in the classical case. However, we will see that there is a simple short cut that can be taken, which allows us to compute the action of the operator B_a on certain states without difficulty.

Since B^{-a} is linear-homogeneous in p_u, p_v, it follows that the Hamiltonian is linear-homogeneous in the first order derivative operators $\delta/\delta u(x)$ and $\delta/\delta v(x)$. It is then straightforward to see that the zero energy state, namely the vacuum, is simply given by $\Psi_0[u,v]=1$.

Consider now the one-particle states. Since in the Bargman electric field representation the one-particle states can be represented as $\Psi_f[E^+]=\int d^3x\, f_a E^{+a}$, a natural ansatz, with $f=f_a dx^a$ a one-form, is

$$\Psi_f[u,v]=\int d^3x\ \varepsilon^{abc}f_a\partial_b u\partial_c v\ . \tag{3.2}$$

We restrict the possible choices of f by requiring that f satisfies the same boundary conditions, Eq. (2.6), that we required on A_a in the previous section; the reason for this will be clear in a moment. To compute the action of the Hamiltonian on this state we first note, from (3.2), that

$$p_u\Psi_f[u,v]=i\hbar F^a\partial_a v\ ,$$
$$p_v\Psi_f[u,v]=-i\hbar F^a\partial_a u\ , \tag{3.3}$$

where $F^a=\varepsilon^{abc}\partial_b f_a$. Next, we compute the action of the operator B^{-a}. From (3.3), and the fact that B^{-a} is *linear* in p_u, p_v we have

$$B^{-a}[u,v,p_u,p_v]\Psi_f[u,v] =B^a[u,v,-i\hbar F^a\partial_a v, i\hbar F^a\partial_a u]\ . \tag{3.4}$$

Now, $B^{-a}[u,v,p_u,p_v]$ is a highly nontrivial function of its arguments. However, consider the following: $B^{-a}[u,v,p_u,p_v]$ is defined as the unique divergence-free function such that $B^{-a}\partial_a v=-p_u$ and $B^{-a}\partial_a u=p_v$, satisfying our boundary conditions. Therefore $B^{-a}[u,v,-i\hbar F^a\partial_a v, i\hbar F^a\partial_a u]$ is the unique divergence-free function such that $B^{-a}\partial_a v=i\hbar F^a\partial_a v$ and $B^{-a}\partial_a u=i\hbar F^a\partial_a u$. This function is of course $i\hbar F^a$. Therefore, without any additional computation we can conclude

$$B^{-a}[u,v,p_u,p_v]\Psi_f[u,v]=i\hbar F^a \tag{3.5}$$

and therefore

$$A^{-}{}_a[u,v,p_u,p_v]\Psi_f[u,v]=i\hbar f_a\ . \tag{3.6}$$

Thus we have, from (2.2′),

$$H\Psi_f[u,v]=-\hbar\int\varepsilon^{abc}\partial_b u\partial_c v\Delta^{1/2}f_a\ .$$

It follows that $\Psi_f[u,v]$ is an eigenstate of the Hamiltonian with eigenvalue $\mathcal{E}$ if an only if f_a satisfies

$$-\hbar\Delta^{1/2}f_a=\mathcal{E}f_a\ . \tag{3.7}$$

Let us now consider the time-dependent Schrödinger equation. Consider an arbitrary one-particle state of the form (3.2), but let f_a be time dependent, namely, let it be a four-dimensional field $f_a(x,t)$. We immediately see that the time-dependent Schrödinger equation

$$-i\hbar\partial_t\Psi_f[u,v]=H\Psi_f[u,v] \tag{3.8}$$

gives

$$\partial_t f_a(x,t)=i(\Delta)^{1/2}f_a(x,t)\ . \tag{3.9}$$

Thus $f_a(x,t)$ evolves in time as a negative-frequency solution of the Maxwell equations. Equation (3.9) should be compared with Eqs. (2.28) and (2.29). The eigenstates of the quantum Maxwell Hamiltonian are therefore given by *single frequency* solutions of the Maxwell equations. For each of these solutions $f_a(x,t)$ with frequency ω, namely if $f_a(x,t)=\exp\{i\omega t\}\ f_a(x)$, we have $(\Delta)^{1/2}f_a=-\omega f_a$ so that, from (3.7) we have

$$H\Psi_f[u,v]=\hbar\omega\Psi_f[u,v]=\mathcal{E}\Psi_f[u,v]\ ; \tag{3.10}$$

$\Psi_f[u,v]$ is then an eigenstate of the Hamiltonian with an eigenvalue

$$\mathcal{E}=\hbar\omega\ . \tag{3.11}$$

Thus we have recovered the Einstein-Planck relation, the key result of the quantization of the free Maxwell field.

The calculation for n-particle states is essentially analogous. For example, a two-particle state is determined by a function $f_{ab}(x,y)$, and is given by

$$\Psi_{f_{ab}}[u,v]=\int d^3x\int d^3y\,\varepsilon^{abc}\partial_b u(x)\partial_c v(x) \times\varepsilon^{def}\partial_e u(y)\partial_f v(y)f_{ad}(x,y)\ . \tag{3.12}$$

The evolution of these states is determined by the fact that $f_{ab\ldots}(x,y,\ldots)$ must be the sum of products of positive-frequency solutions of the Maxwell equations.

B. Relation with the loop representation

There is a remarkable similarity between the expression of the Fock states of the Maxwell field in the Euler representation (i.e., $\Psi[u,v]$), and in the loop representation (i.e., $\Psi[\alpha]$) (see Refs. [6,7]; the loop representation of the Maxwell field is discussed in Ref. [8]). We sketch here the relation between the two. (We avoid the issue of whether or not the curves, given by u and v constant, are really "loops.") One-photon states in the loop representation are given by

$$\Psi_f[\alpha]=\int_\alpha f:=\int f_a(\alpha(s))\frac{d\alpha^a(s)}{ds}ds\ , \tag{3.13}$$

where $f_a(x)$ is an arbitrary one-form. Now, let us consider a congruence $\mathcal{C}$, and let $u(x)$, $v(x)$ be the corresponding Euler potentials and $\alpha(u,v)$ be the corresponding family of loops. Consider the value of the one-particle state $\Psi_f[u,v]$ on the fields $u(x),v(x)$. From Eq. (3.2), we have [see Eq. (2.4)]

$$\Psi_f[u,v]=\int d^3x\,\varepsilon^{abc}f_a\partial_b u\partial_c v=\int d\underline{u}\int d\underline{v}\int_{\alpha(\underline{u},\underline{v})}f\ . \tag{3.14}$$

Therefore the one-photon states in the two representations are related by the remarkable expression

$$\Psi_f[u,v]=\int d\underline{u}\int d\underline{v}\Psi_f[\alpha(\underline{u},\underline{v})]\ . \tag{3.15}$$

The value of the quantum state Ψ_f on the $[u,v]$ fields that define the congruence $\mathcal{C}$ is the integral on all the loops α in $\mathcal{C}$ of the values of the quantum state on α. In terms of the Dirac notation $\Psi_f[u,v]=\langle uv|\Psi_f\rangle$, and $\Psi_f[\alpha]=\langle\alpha|\Psi_f\rangle$, we can write

$$\langle uv|=\int d\underline{u}\int d\underline{v}\langle\alpha(\underline{u},\underline{v})| \tag{3.16}$$

where $\alpha(\underline{u},\underline{v})$ are the loops in the congruence determined by $u(x)$ and $v(x)$.

C. Momentum operator

In order to make the relationship with the standard Fock space basis more explicit, let us find the momentum operator, and study its eigenvalues, namely the states corresponding to photons of given k. (We first point out that for an arbitrary finite region R on which we are working there will be no momentum eigenstates; for the moment, we consider that our region R is a single cell in periodici-

ty lattice.) A straightforward calculation gives the classical momentum observable

$$P_a=\varepsilon_{abc}E^bB^c=\tfrac{1}{4}\varepsilon_{abc}E^{+b}B^{-c}=\partial_a u\ p_u+\partial_a v\ p_v\ . \quad (3.17)$$

The corresponding quantum operator acts on the one-particle states as

$$P_a\Psi_f[u,v]=(\partial_a u\ p_u+\partial_a v\ p_v)\int \varepsilon^{dbc}f_d\partial_b u\partial_c v$$
$$=i\hbar\int \varepsilon^{dbc}\partial_a f_d\partial_b u\partial_c v\ . \quad (3.18)$$

The last step is obtained by integration by parts and using the periodicity of f_d. We thus have that the momentum eigenstates with momentum eigenvalue p_a are determined by

$$i\hbar\partial_a f_d(x)=p_a f_d(x) \quad (3.19)$$

or

$$f_d(x)=\varepsilon_d\exp\{-(i/\hbar)p_a x^a\} \quad (3.20)$$

where ε_b is an arbitrary constant vector. If ε_b is parallel to p_a then $f_d(x)$ is an exact one-form, and $\Psi_f[u,v]$ vanishes; therefore there are only two independent momentum states. These correspond to the two polarizations of the photon.

D. Scalar product

Finally, we address the issue of the scalar product. Since we have the standard Fock structure of the quantum Maxwell theory, we can obtain the scalar product on the full state space by just defining a scalar product on the one-particular sector. We introduce the scalar product by exploiting the reality conditions, according to the ideas discussed, e.g., in Refs. [9,7]: The scalar product can be determined by requiring that the operators corresponding to real observables be self-adjoint in the quantum theory. In order to satisfy this requirement, we must ask that the reality conditions be implemented in the quantum theory as operator equations, where complex conjugation is replaced by the adjoint operation. Thus, we require that the reality conditions (2.25) on $E^{+a}(x)$ and $A^{-a}(x)$ be implemented in the quantum theory as quantum reality conditions [9,7]. Namely, the operators must satisfy

$$[E^{+a}(x)]^\dagger=i\Delta^{1/2}A^{-a}(x)\ , \quad (3.21)$$

where the dagger indicates the adjoint operation. The adjoint operation depends on the scalar product, and therefore Eq. (3.21) can be seen as a condition on the choice of the scalar product. We now show that (3.21) determines the one-particle scalar product (and therefore the full Fock space scalar product). First we recall that in the representation that we are constructing the vacuum Ψ_0 is given by $\Psi_0[u,v]=1$ and is annihilated by the $A^{-a}(x)$ operator. Thus, from Eq. (3.21) we have that

$$\begin{aligned}\langle\Psi_f|\Psi_g\rangle&=\langle\int\varepsilon^{abc}f_a\partial_b u\partial_c v|\int\varepsilon^{abc}g_a\partial_b u\partial_c v\rangle\\&=\langle\int f_a E^{+a}\Psi_0|\int g_a E^{+a}\Psi_0\rangle\\&=\int d^3x\,f_a(x)\int d^3y\,g_b(y)\langle E^{+a}(x)\Psi_0|E^{+b}(y)\Psi_0\rangle\\&=\int d^3x\,f_a(x)\int d^3y\,g_b(y)\langle\Psi_0|i(\Delta)^{1/2}A^{-a}(x)E^b(y)\Psi_0\rangle\\&=\int d^3x\,f_a(x)\int d^3y\,g_b(y)i(\Delta)^{1/2}[A^{-a}(x),E^b(y)]\langle\Psi_0|\Psi_0\rangle\\&=\int d^3x\,f_a(x)\int d^3y\,g_b(y)\hbar(\Delta)^{1/2}\delta^a{}_b\delta(x,y)\\&=\hbar\int d^3x\,f_a(\Delta)^{1/2}g_a\ .\end{aligned} \quad (3.22)$$

By computing the scalar product of two one-photon states, one can recognize this scalar product as the conventional one [10]. The extension to the n-photon states is straightforward.

Finally, we add a speculative remark on the possibility of writing the scalar product directly on the functional space of the Euler representation states, namely the $\Psi[u,v]$ states. Let us tentatively write the scalar product by means of a functional integral:

$$(\Psi,\Phi)=\int d\mu[u,v,u^*,v^*]\Psi^*[u,v]\Phi[u,v]\ . \quad (3.23)$$

The asterisk indicates the complex conjugate. The problem is to determine the measure $d\mu[u,v,u^*,v^*]$. We can write it in the conventional form

$$d\mu[u,v,u^*,v^*]=[du][dv][d\,u^*][dv^*]\mu[u,v,u^*,v^*] \quad (3.24)$$

where $[du]\,[dv]\,[d\,u^*][dv^*]$ is the "infinite-dimensional Lebesgue measure" (namely the factor that is not affected by integration by part); the measure factor $\mu[u,v,u^*,v^*]$ will be determined. Using again the quantum reality conditions (3.22) for determining the scalar product, and hence μ, we can write

$$\int d\mu[u,v,u^*,v^*][E^{+a}(x)]\Psi[u,v]^*\Phi[u,v]=\int d\mu[u,vu^*,v^*]\Psi^*[u,v][i(\Delta)^{1/2}A^{-a}(x)]\Phi[u,v]\ . \tag{3.25}$$

Since $A^{-a}(x)$ is a first order derivative operator, we assume we can integrate it by parts. We obtain

$$\int [du][dv][du^*][dv^*]\mu[u,v,u^*,v^*][\varepsilon^{abc}\partial_b u(x)\partial_c v(x)\Psi[u,v]]^*\Phi[u,v]$$
$$=\int [du][dv][du^*][dv^*][-i(\Delta)^{1/2}A^{-a}(x)\mu[u,v,u^*,v^*]]\Psi^*[u,v]\Phi[u,v]\ , \tag{3.26}$$

or

$$\varepsilon^{abc}\partial_b u^*\partial_c v^*\mu[u,v,u^*,v^*]=-i(\Delta)^{1/2}A^{-a}(x)\mu[u,v,u^*,v^*]\ . \tag{3.27}$$

We have now to solve this equation for μ. Let us tentatively write μ in the form

$$\mu[u,v,u^*,v^*]=\exp\{\int f_a(u^*,v^*)\varepsilon^{abc}\partial_b u\partial_c v d^3x\}\ ; \tag{3.28}$$

then, by recalling Eq. (3.6), we have that

$$A^{-}{}_a(x)\mu[u,v,u^*,v^*]=\hbar f_a(u^*,v^*)\exp\left\{\int f_d(u^*,v^*)\varepsilon^{dbc}\partial_b u\partial_c v d^3x\right\}\ , \tag{3.29}$$

so that Eq. (3.18) is satisfied if

$$\varepsilon_a{}^{bc}\partial_b u^*\partial_c v^*=-i\hbar\Delta^{1/2}f_a(u^*,v^*)\ , \tag{3.30}$$

namely, $f_c=(1/\hbar)(\Delta)^{-1/2}[\varepsilon_a{}^{bc}\partial_b u^*\partial_c v^*]$. We conclude that

$$\mu[u,v,u^*,v^*]=\exp\left\{-\frac{1}{\hbar}\int[\varepsilon^{abc}\partial_b u^*\partial_c v^*]\Delta^{-1/2}[\varepsilon^{ade}\partial_d u\partial_e v]d^3x\right\}$$
$$=\exp\left\{-\frac{1}{\hbar}\int\partial_{[a}u^*\partial_{b]}v^*\Delta^{-1/2}[\partial^a u\partial^b v]d^3x\right\}\ . \tag{3.31}$$

Equations (3.23) and (3.31) give us a formal definition of the scalar product that implements the reality conditions. We expect that from this form of the scalar product one can recover the scalar product previously determined. To check this, let us consider the scalar product of two one-particle states:

$$(\Psi_f,\Psi_g)=\int[du][dv][du^*][dv^*]\exp(-(1/\hbar)\int\partial_{[a}u\partial_{b]}v\Delta^{-1/2}[\partial^a u^*\partial^b v^*])\Psi^*{}_f[u,v]\Psi_g[u,v]\ . \tag{3.32}$$

This integral can probably be performed by means of formal manipulations. We sketch here a possible means of derivation, leaving a more accurate analysis for future investigations. A formal change of integration variable from (u,v) to the divergenceless vector field $E^a=\varepsilon^{abc}\partial_a u\partial_c v$ (and similarly for the conjugate variables) transforms the functional integral into a Gaussian integral, where $\Delta^{-1/2}$ is the covariance of the quadratic form that defines the Gaussian measure. The integral could then be performed, hopefully yielding Eq. (3.22).

IV. CONCLUSION

We have reformulated the classical and quantum Maxwell theory in terms of Euler potentials $u(x)$ and $v(x)$, and their conjugate momenta $p_u(x)$ and $p_v(x)$. These two scalar fields are directly related to the lines of force of the electric field. This formulation provides a fully gauge-invariant canonical formalism, which is defined entirely in configuration space and without the need of gauge fixing. No background metric is used in extracting the physical degrees of freedom of the Maxwell field. The classical dynamics becomes nonlinear and nonlocal; global issues and boundary conditions are not easy to control. However, from this structure, we have constructed the quantum theory in terms of the Euler potentials of the positive-frequency component of the electric field (Euler representation). Several aspects of this formulation are quite cumbersome; for instance, the classical equations of motion are nontrivial. However, there are also some appealing aspects of the construction described here, which may become useful and interesting in view of the quantization difficulties in non-Abelian theories and in general relativity.

In Yang-Mills theory, the difficulty of extracting the gauge-invariant degrees of freedom nonperturbatively is well known. Since gauge-invariant variables analogous to $u(x)$ and $v(x)$ are available in Yang-Mills theory [3], we hope that some of the techniques developed here could be extended to the nonperturbative treatment of these theories too. In general relativity, the key aspect of the nonperturbative theory is the absence of a background metric, which, in particular, prevents us from using the powerful technique of the Fourier transform, as is generally used in extracting the gauge-independent degrees of freedom of Maxwell field. Thus, it is important to inquire to what extent one could single out the physical degrees of freedom and treat the quantum Maxwell field en-

tirely in coordinate space. We have shown that this is indeed possible in a straightforward way in terms of the $u(x)$ and $v(x)$ fields. An even stronger result is given by the fact that $u(x)$ and $v(x)$ capture the two gauge-invariant degrees of freedom of the Maxwell field in an entirely metric-independent way. Therefore we can treat the kinematical aspects of the theory without recourse to the metric structure of spacetime. This is very valuable in view of the fact that quantum general relativity has to be entirely constructed in the absence of background metric structure. We recall that gauge- *and diffeomorphism*-invariant variables analogous to $u(x)$ and $v(x)$ *are* available in general relativity theory [3]. Whether these variables could play a role in the nonperturbative quantization of the theory is an entirely open issue. A suggestive result in this direction is the close relation between the representation developed here and the loop representation [7].

ACKNOWLEDGMENTS

This work was supported by the NSF through Grants Nos. PHY-5-39634 and PHY-92-05109.

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