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High amplitude limit of scalar power

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For spherically symmetric, massless Einstein-Klein-Gordon fields, we show that the scalar radiation power saturates at high amplitude at the universal value $\pi/32$, independent of the interior structure of the field. This provides a framework for understanding the energetics of self-gravitating effects in which our high-amplitude limits complement the low-amplitude predictions of linearized theory. The gravitational collapse of a high-amplitude initial state to a black hole is studied.

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I. INTRODUCTION

Einstein's equation for the gravitational field reduces to a simple linear system in the weak-field limit, which has led to a very complete physical understanding of that regime. Here we investigate a simplification of a different mathematical nature that arises in the high-amplitude limit of asymptotically flat spacetimes. Results of this type were first found in the context of a spherically symmetric, self-gravitating, massless scalar field. If at a given retarded time the scalar field possesses a monopole moment Q,

$$\Phi = Qr^{-1} + O(r^{-2}), \tag{1.1}$$

then the Bondi mass has the high-amplitude asymptotic behavior

$$M \sim \pi|Q|/\sqrt{2} \tag{1.2}$$

at that time [1]. Rather than the quadratic relationship between the Bondi mass and the field which holds for small amplitudes, this surprising formula not only relates the Bondi mass linearly to the field but determines it solely in terms of the monopole moment, which represents the radiation amplitude of the field. In this limit, the mass is completely independent of the interior structure.

The gravitational redshift is the physical mechanism responsible for this effect. The redshift between any worldline at fixed luminosity distance and null infinity increases exponentially with increasing field amplitude. In the extreme high-amplitude limit, the contribution of the interior to the Bondi mass is redshifted away when compared with the contribution from the far field. In the compactified Penrose picture, the dominant contribution to the total mass lies in a narrow boundary layer at null infinity.

Such a basic mechanism as the redshift might be expected to have a universal effect on all high-amplitude fields including vacuum gravitational fields. Subsequent work [2] for axially symmetric fields confirmed this expectation for the pure gravitational case. In the strong-field limit, it was found that the Bondi mass does scales linearly with field amplitude, although in this case no simple asymptotic formula analogous to Eq. (1.2) emerges. Besides a contribution from the radiation amplitude, an additional contribution arises from the angular-momentum-dipole-moment aspect of the field.

These results furnish some perspective on the physics that arises in the presence of intense gravitational fields. The two asymptotes, low amplitudes and high amplitudes, provide a bracket which serves to gauge the nonlinearity of intermediate states. In addition, these results would appear to have important implications for the contribution of high energies to the sums over states that occur in statistical mechanics or in the Feynman approach to quantization.

In this paper, we return to the case of a spherically symmetric, general relativistic Klein-Gordon field to investigate the properties of radiation power in the high amplitude limit. Our main analytic result, derived in Sec. III, is that the radiation power saturates at the universal value $\mathcal{P} = \pi/32$, provided the field has a nonvanishing monopole moment. In the linear regime, the rate of energy emission as scalar radiation is proportional to the rate entering as initial conditions. In the high-amplitude limit, the emission rate is independent of what goes in. In Sec. IV, we use a numerical evolution algorithm [1] to study the gravitational collapse of high-amplitude scalar fields as they shed their monopole moment to form a black hole in accordance with the no-hair theorem [3–5]. Numerical studies of this system have also led to the discovery of interesting critical phenomena [6] in the intermediate amplitude regime characterized by initial data on the borderline of black hole formation.

II. NULL CONE DESCRIPTION OF SCALAR WAVES

In a Bondi coordinate system [7], the Einstein-Klein-Gordon equations for a zero rest mass scalar field are quite simple in the spherically symmetric case [1,3]. The line element has the form

$$ds^2 = e^{2\beta} du \left(\frac{V}{r} du + 2dr \right) - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

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where u is the proper time along the central geodesic $r = 0$, with u=constant on the outgoing null cones, and r is the luminosity distance on these null cones, so that $4\pi r^2$ is the area of the spheres of symmetry. The field equations are

$$\beta_{,r} = 2\pi r(\Phi_{,r})^2, \tag{2.2}$$

$$V_{,r} = e^{2\beta} \tag{2.3}$$

and the scalar wave equation $\Box\Phi = 0$, which takes the form

$$2(r\Phi)_{,ur} = r^{-1}(rV\Phi_{,r})_{,r}. \tag{2.4}$$

At the origin, we adopt the boundary conditions

$$V(u,r) = r + O(r^3) \quad \text{and} \quad \beta(u,0) = 0, \tag{2.5}$$

so that the metric reduces to a Minkowski (null polar) form along the central world line. The resulting metric does not take an asymptotic Minkowski form in the limit $r \to \infty$ of null infinity J^+. We set $H(u) = \beta(u,\infty)$. Then Bondi time $\tilde{u}$ at J^+ is related to proper time u along the central geodesic by

$$\frac{d\tilde{u}}{du} = e^{2H}. \tag{2.6}$$

The coordinates $\tilde{u}$, r, θ and ϕ constitute a standard Bondi frame whose line element is given by (2.1) with the replacements $V \to \tilde{V} = e^{-2H}V$ and $\beta \to \tilde{\beta} = \beta - H$. Bondi time $\tilde{u}$ is the physically relevant time for distant observers whereas central time u is relevant to the internal dynamics governing gravitational collapse. A horizon forms in a finite central time $u = u_{\mathcal{H}}$ but an infinite Bondi time $\tilde{u}_{\mathcal{H}} = \infty$, with the central redshift determined by Eq. (2.6).

Initial null data for evolution consists of $\Phi(u_0, r)$ at a given retarded time $u = u_0$. By radial integration of the hypersurface equations (2.2, 2.3) using the boundary conditions (2.5), this data uniquely determines, in turn, $\beta(u_0,r)$ and $V(u_0,r)$. Formal evolution then proceeds by determining $\partial_u\Phi(u_0,r)$ from the radial integral of the wave equation (2.4) which gives, after integration by parts,

$$2r\Phi_{,u} = V\Phi_{,r} + \int_0^r \frac{V}{r}\Phi_{,r}dr. \tag{2.7}$$

The Bondi mass is

$$M = \frac{1}{2}e^{-2H}r^2\left(\frac{V}{r}\right)_{,r}\Bigg|_{r=\infty} \tag{2.8}$$

and the scalar news function is

$$N = e^{-2H}r\Phi_{,u}\Bigg|_{r=\infty} = Q_{,\tilde{u}}, \tag{2.9}$$

where Q is the scalar monopole moment and the factors of e^{-2H} arise from the relation of central time to Bondi time. These quantities satisfy the Bondi mass-loss equation

$$\mathcal{P} = -M_{,\tilde{u}} = 4\pi N^2. \tag{2.10}$$

The news function can be expressed as a null cone integral,

$$N = \frac{1}{2}e^{-2H}\int_0^\infty \frac{V}{r}\Phi_{,r}dr, \tag{2.11}$$

as follows immediately from (2.7). The mass may also be expressed as the integral

$$M = 2\pi\int_0^\infty e^{2(\beta-H)}r^2(\Phi_{,r})^2dr. \tag{2.12}$$

III. HIGH AMPLITUDE LIMIT OF THE NEWS

Given initial null data satisfying the asymptotic flatness condition $\Phi(u_0,r) = O(1/r)$, the substitution $\Phi(u_0,r;\lambda) \to \lambda\Phi(u_0,r)$ generates a one-parameter family of asymptotically flat spacetimes ranging from the linearized regime (λ small) to the high-amplitude limit ($\lambda \to \infty$). It is essential here to define this family in terms of a luminosity distance r so that amplitude scaling automatically preserves asymptotic flatness. (An analogous family based upon an affine parameter would for sufficiently large λ lead to an initial null cone with trapped surfaces so that it would lie interior to a horizon.)

For this one parameter family, the hypersurface equations (2.2) and (2.3) give

$$\beta(u_0,r;\lambda) = \lambda^2\beta(u_0,r) \tag{3.1}$$

and

$$V(u_0,r;\lambda) = \int_0^r e^{2\lambda^2\beta(u_0,s)}ds. \tag{3.2}$$

The key factor in the high-amplitude behavior is the exponential $e^{2(\beta-H)}$. The divergence of the outgoing null cone equals $e^{-2\beta}/r$ and as a result of the optical equations β is a monotonically increasing function of r for coupling to any matter field satisfying a positive energy condition. As a result this exponential factor is very small except close to null infinity, where $\beta = H$. Accordingly, the contribution to the λ-dependent mass integral (2.12)

$$M(\lambda) = 2\pi\lambda^2\int_0^\infty e^{2\lambda^2(\beta-H)}r^2(\Phi_{,r})^2dr, \tag{3.3}$$

is very small except in a region where the gravitational field can be represented by leading terms in a $1/r$ expansion and the integral has the appropriate structure for extracting its large-λ dependence by the method of Laplace [8].

This method gives the asymptotic large-τ behavior of the integral

$$I(\tau, a, b) = \int_a^b f(x) e^{\tau h(x)} dx, \tag{3.4}$$

where $h(b) > h(x)$ for $a \le x < b$, by (i) introducing the truncated integral $I(\tau, b-\epsilon, b)$, (ii) replacing $f(x)$ and $h(x)$ in $I(\tau, b-\epsilon, b)$ by the leading terms in their Taylor expansions about $x = b$, and (iii) evaluating the resulting integral in the limit $\epsilon \to \infty$. Justification of this technique depends upon showing that the higher order terms discarded in the Taylor expansion and the extension of the integral to infinity both lead to weaker contributions in λ. In some cases, the integral must first be prepared by a judicious integration by parts to insure this.

The application of this method to $M(\lambda)$ depends upon the large-r behavior of Φ. Suppose Φ has a nonvanishing monopole moment so that $\Phi = Q/r + O(1/r^2)$. Then β attains its maximum H asymptotically at infinity, with

$$\beta = H - (\pi Q^2/r^2) + O(1/r^3), \tag{3.5}$$

according to the expansion of (2.2). By introducing the new integration variable $x = r/(1+r)$, the integration limits are compactified between $x = 0$ and $x = 1$. The method of Laplace then yields [1] the asymptotic mass formula (1.2).

We now carry out an analogous derivation of the high-amplitude asymptotic behavior of the news function, on the assumption of a nonvanishing monopole moment. For technical simplicity we will also assume that the field has a $1/r$ expansion

$$\Phi = Qr^{-1} + Q_2 r^{-2} + Q_3 r^{-3} + \cdots \tag{3.6}$$

although the essential ingredient is a smooth $O(r^{-2})$ remainder term. First, the λ-dependent version of the integral formula (2.11) must be integrated by parts, by expressing $dr = -r^3 d(1/r^2)/2$. The resulting boundary term vanishes and, after using (2.3),

$$N(u_0;\lambda) = I_1(u_0;\lambda) + I_2(u_0;\lambda), \tag{3.7}$$

where

$$I_1(u_0;\lambda) = \frac{\lambda}{4}\int_0^\infty e^{2\lambda^2(\beta-H)} \Phi_{,r} dr \tag{3.8}$$

and

$$I_2(u_0;\lambda) = \frac{\lambda}{4} e^{-2\lambda^2 H} \int_0^\infty \frac{V}{r^2}(r^2\Phi_{,r})_{,r} dr. \tag{3.9}$$

The strategy behind this separation is that the term $(r^2\Phi_{,r})_{,r} = 2Q_2 r^{-2} + O(r^{-3})$ has no contribution from Q in its far-field behavior. As a result, we will show that I_1 dominates I_2 in the high-amplitude limit, as suggested by the physical picture of a boundary layer at null infinity.

In terms of the Hawking mass $m(u, r;\lambda) = (r - e^{-2\lambda^2\beta}V)/2$, I_2 is given by

$$I_2(u_0;\lambda) = \frac{\lambda}{4}\int_0^1 \frac{e^{2\lambda^2(\beta-H)}(1-x)}{x} \times \left(1 - \frac{2m(1-x)}{x}\right)(x^2\Phi_{,x})_{,x} dx, \tag{3.10}$$

where we have introduced the compactified integration variable x, as above. This now has the necessary form to apply the method of Laplace. Using the expansion

$$\beta - H = -\pi Q^2 (x-1)^2 (2\pi/3)(4QQ_2 + 3Q^2) \times (x-1)^3 + O((x-1)^4) \tag{3.11}$$

which follows from (2.2), and the asymptotic value $4\pi\lambda|Q|/\sqrt{2}$ of the Hawking mass, which follows from the λ-dependent version of (1.2), we obtain

$$I_2(u_0;\lambda) \sim \frac{Q_2}{8\pi\lambda Q^2}(1 + \pi/2). \tag{3.12}$$

Thus this contribution goes asymptotically to 0 in the high-amplitude limit.

Expressing I_1 in terms of the integration variable x and applying the method of Laplace now gives

$$I_1(u_0;\lambda) = \frac{\lambda}{4}\int_0^1 e^{2\lambda^2(\beta-H)}\Phi_{,x} dx \tag{3.13}$$

$$\sim \frac{-\lambda}{4}\int_{-\infty}^1 e^{-2\pi\lambda^2 Q^2 (x-1)^2} Q dx + O(1/\lambda) \tag{3.14}$$

$$\sim -\frac{\sqrt{2}}{16}\frac{Q}{|Q|} + O(1/\lambda), \tag{3.15}$$

so that

$$N(u_0;\lambda), \sim -\frac{\sqrt{2}}{16}\frac{Q}{|Q|} + O(1/\lambda) \tag{3.16}$$

in the high-amplitude limit. Only the sign of the news function, not the magnitude, depends upon the monopole moment. Referring to (2.10), this implies a radiation power $\mathcal{P} \sim \pi/32$. The news function and power are dimensionless quantities so that such universal limits are physically possible.

This result should be compared to the weak-field limit, for which $V = r$ and (2.11) integrates to give $N(u) = -\frac{1}{2}\Phi(u, 0)$. In that case, the news function depends linearly upon the signal arriving at the origin. As the field amplitude increases, the effects of backscattering and redshfting become prominent.

Some examples illustrate how the news function makes the transition between the extremes of low and high amplitude. Consider first the one-parameter set of initial data

$$\Phi(u_0, r;\lambda) = \frac{\lambda}{(1+r)}. \tag{3.17}$$

Figure 1 gives a graph of the numerically computed values of the news function versus λ. For small λ, the graph displays the linear λ dependence of the news function in the weak field case. For large λ, the graph is asymptotic to the high-amplitude limit (3.16). The maximum value of $|N|$ occurs at $\lambda \approx 0.65$, at about one-half the critical value $\lambda_c \approx 1.317$, above which the system evolves to form a horizon. The critical value λ_c lies in the transition region in which the news goes from a linear dependence on amplitude to its universal limit. The graph extends up to the value $\lambda = 18$, past which terms of magnitude $e^{2\lambda H} \approx 10^{295}$ lead to numerical underflow.

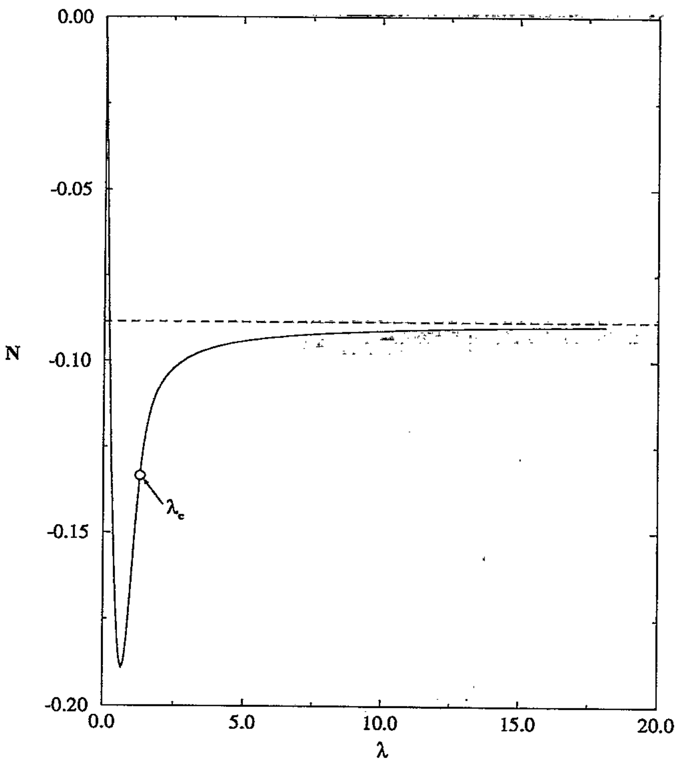

FIG. 1. News versus amplitude for the initial data (3.17). For small λ (weak field), the news is linear. For large λ, the graph is asymptotic to the high-amplitude limit (3.16) indicated by the dashed line.

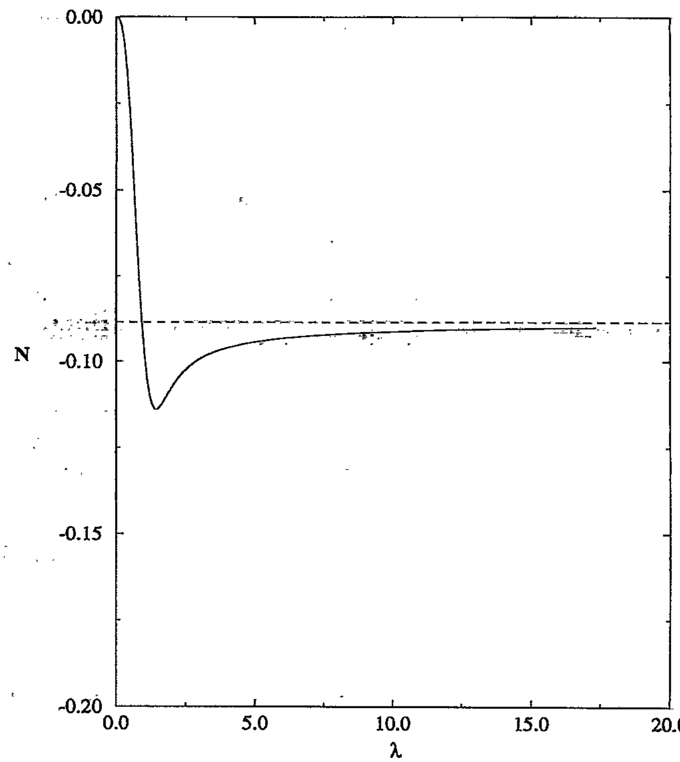

FIG. 3. News versus amplitude for the superposition of the initial data (3.17) and (3.18). The behavior for intermediate λ differs from that shown in Fig. 1 but the asymptotic behavior is still given by (3.16).

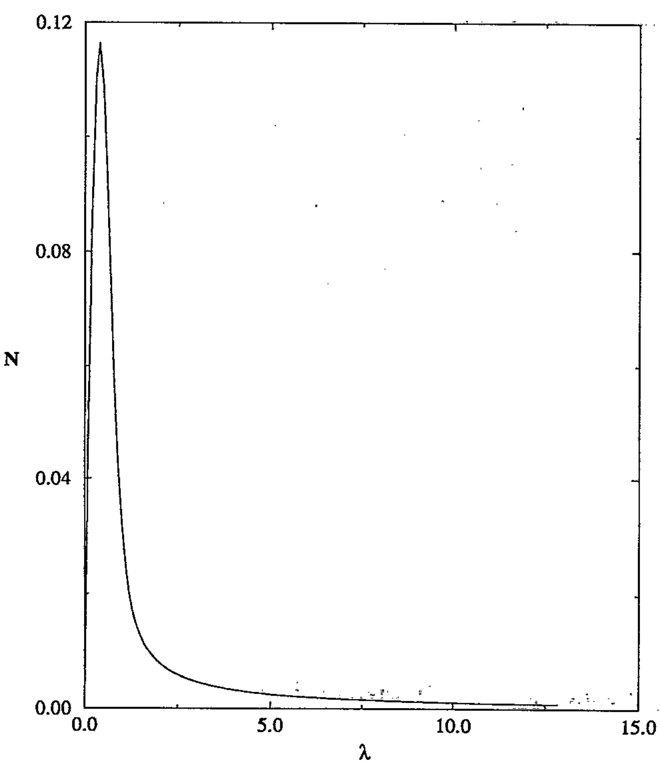

FIG. 2. News versus amplitude for the compact data (3.18). For large λ, the news decays to zero with the $O(1/\lambda)$ dependence predicted by (3.16).

As the next example, Fig. 2 is a graph of the computed values of the news function versus λ for the data

$$\Phi(u_0, r; \lambda) = \begin{cases} -\lambda(1-r)^2 & \text{for } r \leq 1 \\ 0 & \text{for } r \geq 1\,, \end{cases} \tag{3.18}$$

which describes a pulse of compact support. For small λ, the graph again displays a linear λ dependence, as in Fig. 1. However, the news function is asymptotic to zero in the large-λ limit, in accord with the vanishing monopole moment of this data. The decay to zero has the $O(1/\lambda)$ dependence predicted by (3.16). The study of an analytically integrable model [9] with compact null data also found that the news function is completely redshifted away, $N(\lambda) \to 0$ as $\lambda \to \infty$.

Figure 3 is a graph of the news function versus λ for data consisting of the addition of (3.17) and (3.18). In this case, there is no longer a linear dependence on λ in the weak-field regime because the superposition principle leads to exact cancellation between the two signals reaching $r = 0$. Although the behavior for intermediate λ differs from Fig. 1, it is apparent that the news function remains asymptotic to the universal value $-\sqrt{2}/16$ in the extremely nonlinear high-amplitude limit. This illustrates how the high-amplitude limit of the news function is independent of interior structure.

IV. EVOLUTION OF HIGH AMPLITUDE DATA

Amplitude scaling of the data at time u_0 is not preserved under evolution, $\Phi(u_1, r; \lambda) \neq \lambda\Phi(u_1, r)$ except in the small-λ limit, in which Φ behaves as a linear field in Minkowski space. Strong fields evolve to form a black hole with vanishing monopole moment but nonvanishing mass. For a nonvanishing initial monopole moment, the high-amplitude formulas (1.2) and (2.10) imply $|Q_{,\tilde{u}}/Q| = 2|M_{,\tilde{u}}/M|$ so that the monopole moment decays at twice the rate as the mass.

In the initial high-amplitude state, the exponential factor $e^{2(\beta-H)}$ creates a boundary layer between $r \approx Q$ and ∞ (or between $x \approx 1 - 1/Q$ and 1) which dominates the integrals for the mass and news function. On the other hand, as the system evolves to form a horizon with Bondi mass $M_{\mathcal{H}}$, Christodoulou's [5] no-hair theorem implies that this exponential factor has the asymptotic late-time step-function behavior

$$e^{2(\beta-H)} \to \begin{cases} 0 & \text{for } r < 2M_{\mathcal{H}}\ , \\ 1 & \text{for } r > 2M_{\mathcal{H}}, \end{cases} \tag{4.1}$$

which requires that the scalar field goes to zero outside $r = 2M_{\mathcal{H}}$.

We illustrate these features in terms of the numerical evolution of the initial data (3.17), choosing $\lambda = 10$. Figure 4 is a graph of the news function versus central time. Initially, N is close to its asymptotic limit, in accord with Fig. 1. It then rapidly decays to zero as a black hole is formed. This takes place in the exceedingly short central time $\Delta u \approx 7\times 10^{-90}$. The higher the amplitude, the quicker black hole formation takes place. Of course, from the reference point of an external observer this takes an infinite amount of Bondi time. Initially the redshift factor is $e^{2H} \approx 10^{91}$ for this example. There is additional structure in the final decay of the news function which is lost in the graphical resolution of Fig. 4. Recent results indicate [10] that at very high redshifts the field decays similarly to a scalar perturbation of a Schwarzschild background. Figure 5 is a graph of the Bondi mass. It starts close to its asymptotic limit $\pi|Q|/\sqrt{2}$ and then decays at a remarkably constant rate with respect to central time, until it levels off at the final black hole mass which is roughly half the initial mass. The exponential factor $e^{2(\beta-H)}$ is graphed in Fig. 6. The initial boundary layer steepens in time to form the final step function (4.1).

These results are representative of the evolution of other high-amplitude initial data with nonvanishing monopole moment. This confirms that it is the energy stored in the asymptotic field that governs the high-amplitude dynamics, regardless of the details of the interior structure. Indeed, a black hole forms in such a rapid time that there is negligible evolution in the interior region. For instance, in the above example, the velocity of light near the central world line is $du/dr \approx 1$ in our coordinates so that there is negligible wave propagation in the short time $\Delta u \approx 7\times 10^{-90}$ prior to horizon formation.

It is interesting to compare these results with the high-amplitude behavior of a system whose monopole moment initially vanishes. For data of compact support filling a sphere of luminosity distance R, the Bondi mass saturates at the high-amplitude limit [1] $M \sim R/2$. In this

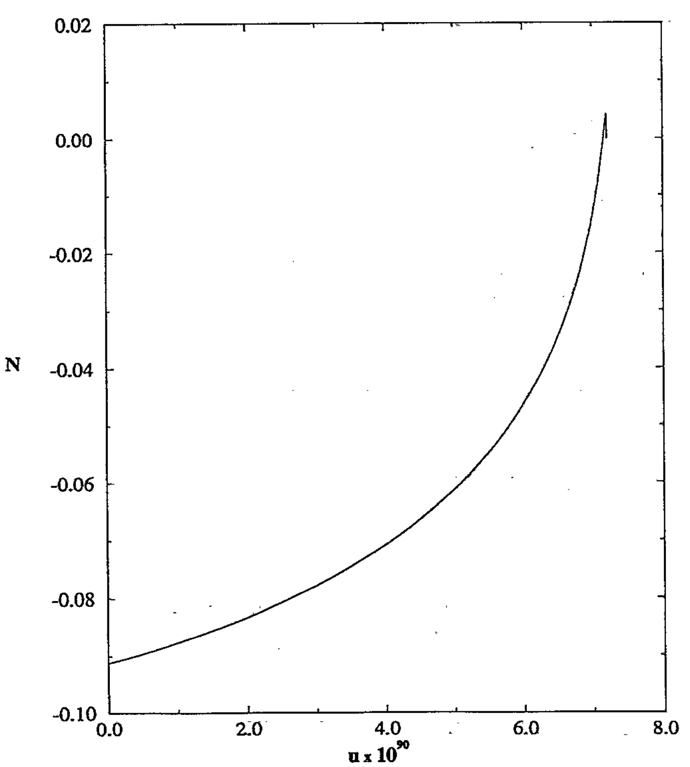

FIG. 4. News versus central time u for the initial data (3.17), with $\lambda = 10$. The news is initially close to its high-amplitude limit and then decays rapidly to zero as a black hole is formed.

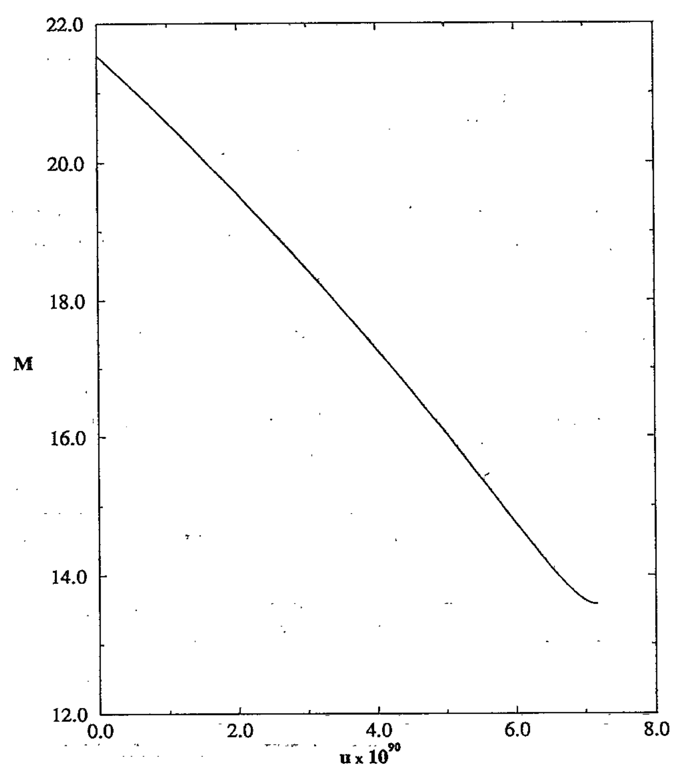

FIG. 5. The Bondi mass versus central time u for the same data as in Fig. 4. It starts close to its high-amplitude limit and then decays at a remarkably constant rate with respect to central time. The final black hole mass is roughly half the initial mass.

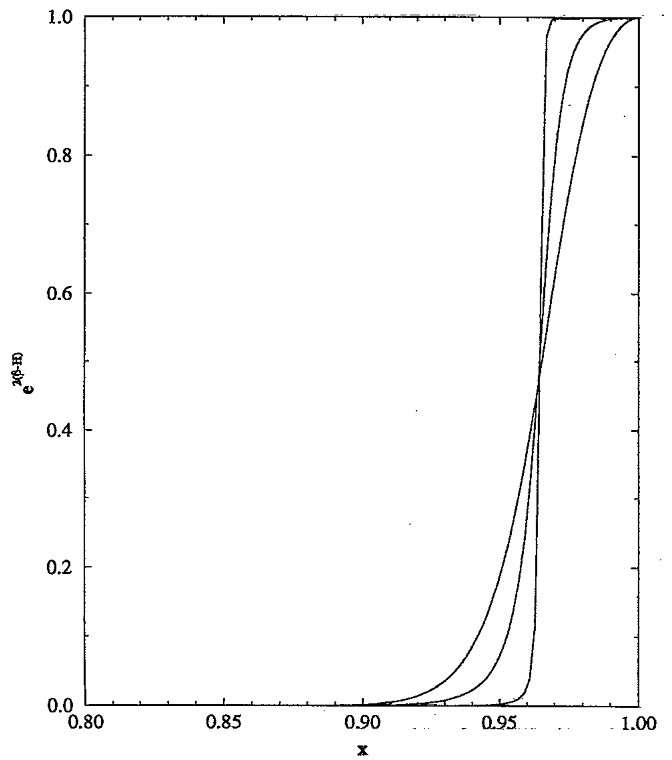

FIG. 6. Profiles of the exponential factor $e^{2(\beta-H)}$ versus x, at $u=0$ and two later times, for the same data as in Fig 4. The initial boundary layer gets even steeper to form the final step function (4.1).

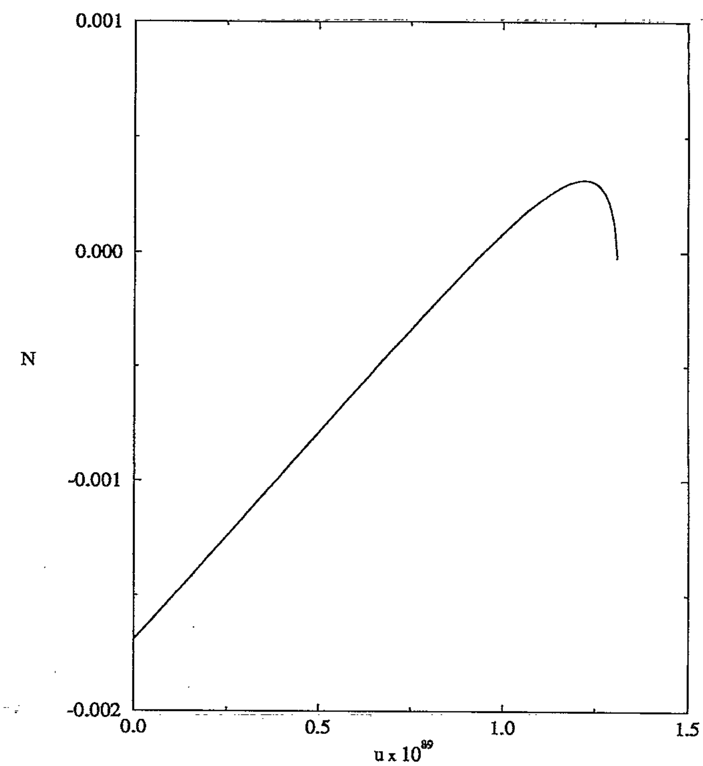

FIG. 7. News function versus central time, for the compact data (3.18), with $\lambda = 7$. It's maximum value, at $u = 0$, is $N \approx 1.7 \times 10^{-3}$, well below the high-amplitude limit of $\sqrt{2}/16$ for data with nonvanishing monopole moment.

limit, the scalar field is again at the brink of black hole formation. (This is a peculiarity of the Einstein-Klein-Gordon system. For dust of compact support, the mass again saturates at the value $R/2$ in the high-density limit but in this case a white hole surrounds the matter on the initial outgoing null cone [11].) For data of compact support, the news function initially vanishes in the high-amplitude limit and there is little time before horizon formation for it to build up to produce any substantial mass loss. As an illustration, consider the numerical evolution of the compact pulse (3.18), for $\lambda = 7$. Figure 7 is a graph of the news function versus central time. Its maximum absolute value, attained at the initial time, is $|N| \approx 1.7 \times 10^{-3}$, well below the high-amplitude limit of $\sqrt{2}/16$ for data with nonvanishing monopole moment. The initial mass for this example is $M \approx .45$ and less than 0.01% is radiated in the short central time required for black hole formation, as opposed to the roughly 50% mass loss in the nonvanishing monopole case.

V. CONCLUSION

Our results provide a useful framework for understanding the self-gravitating effects of scalar waves. The high-amplitude behavior predicted for the mass and radiation flux complement the low-amplitude behavior predicted by linearized theory. Our results also portray an extremely stiff behavior of systems in the high-amplitude regime. Such systems rapidly decay to a final black hole equilibrium state.

All the fundamental ingredients for these results remain intact in the nonspherically symmetric case, either in the presence of other matter fields or for pure vacuum fields. The focusing equation still requires β to be a monotonically increasing function along the outgoing null rays. The redshift still has an exponential dependence on $\beta - H$. Thus one should expect similar saturation effects on the radiation power, although the calculation of the high-amplitude limit will undoubtedly be much more challenging.

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