PHYSICAL REVIEW D VOLUME 47, NUMBER 8 15 APRIL 1993

Post-Newtonian behavior of the Bondi mass

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(Received 4 January 1993)

We present an algorithm for calculating the Bondi mass, based upon renormalized variables, which converges at second order in grid size. The algorithm is highly effective in exploring both the Newtonian and strong field limits of general relativity. In particular, we study a quasi-Newtonian system with gravitational radiation. The algorithm allows calculation of the Newtonian mass and mechanical energy and joins smoothly to a post-Newtonian expansion of the Bondi mass carried out through terms of $O(\lambda^7)$. At higher λ, the computed mass peels away and shows markedly non-Newtonian behavior. It remains strictly positive, in contrast to the post-Newtonian expansion.

PACS number(s): 04.30.+x, 95.30.Sf

I. INTRODUCTION

This paper explores the mass of a general relativistic system from the point of view of its background Newtonian value using a combination of analytical and computational techniques. Newtonian gravitational physics is essential in determining the importance of general relativistic astrophysical effects. Mainstream astrophysics is couched in Newtonian concepts, some of which have no well-defined meaning in general relativity. In order to provide a sound basis for relativistic astrophysics, it is crucial to develop general relativistic concepts which have well-defined and useful Newtonian limits. The concepts of mass and energy are most fundamental in this regard. Here, starting with a one-parameter family of general relativistic spacetimes which have a Newtonian limit, we compare numerically computed values of the Bondi mass with the corresponding Newtonian mass and binding energy and their post-Newtonian perturbative corrections. One of our accomplishments is the development of an accurate computational algorithm for the Bondi mass which allows such a study to be feasible. Besides its importance for physical interpretation, this algorithm will also provide an important calibration of the numerical evolution of Einstein's equation by using the Bondi mass loss formula to check the accuracy.

Two notable early investigations relating general relativistic and Newtonian energy expressions were carried out by Fock [1] and by Chandrasekhar [2]. They have led to a practical scheme for introducing general relativistic corrections to the Newtonian theory of stellar dynamics. However, there are undesirable theoretical aspects to the various early formalisms. Some examples are (i) the use of coordinate-dependent concepts such as gravitational energy-momentum pseudotensors; (ii) a rather loose application of the notion of asymptotic flatness, particularly in the case of radiative spacetimes; (iii) the appearance of divergent integrals; and (iv) the lack of a clear formulation of the various approximation formalisms such as the weak field or the slow motion expansions.

The geometrization of null infinity [3] with the associated geometrization of Bondi mass [4] and the corresponding results for the ADM mass at spatial infinity [5] have eliminated any need for deficiencies of type (i) or (ii). Advances have also been made in the geometrization of approximation schemes [6,7]. It is now clear that these are of two distinct types, post-Minkowskian and post-Newtonian, each based upon a one-parameter family of spacetimes. In the post-Minkowskian case, the parameter represents a rescaling of the gravitational constant G with the limiting background having the flat geometry of special relativity. In the post-Newtonian case, the parameter rescales the velocity of light $c \to c/\lambda$, with the limiting spacetime as $\lambda \to 0$ yielding the Cartan geometry of Newtonian physics viewed as a spacetime theory. The emergence of Newton-Cartan theory as a limit of general relativity and as a basis for perturbation theory has been developed by Dautcourt [8], by Kunzle [9, 10], and by Ehlers [11]. Recently, Rendall [12] has formulated an axiomatic description of a certain class of post-Newtonian approximations, which includes the scheme used by Chandrasekhar, and has discussed inconsistencies that arise in the treatment of radiation.

The formal description of this limit involves two λ-dependent metric tensors $g_{\mu\nu}(\lambda)$ and $h^{\mu\nu}(\lambda)$, with

$$h^{\mu\nu}(\lambda) = \lambda^2 g^{\mu\nu}(\lambda) \tag{1.1}$$

for $\lambda \neq 0$. In the $\lambda \to 0$ limit, these metrics degenerate into $g_{\mu\nu} \to t_\mu t_\nu$, where $t_\mu = \partial_\mu t$ defines the absolute time slices of Newtonian theory, and $h^{\mu\nu} \to e^{\mu\nu}$, the rank-three Euclidean metric of these time slices. Furthermore, the connection $\Gamma^\sigma_{\mu\nu} \to -e^{\sigma\tau} t_\mu t_\nu \partial_\tau \Phi$, where Φ is the Newtonian potential. An example is the family of Schwarzschild spacetimes

$$ds^2_\lambda = (1 - 2\lambda^2 m/r)\, dt^2 - \lambda^2 (1 - 2\lambda^2 m/r)^{-1} dr^2 - \lambda^2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.2}$$

In the Newtonian limit, this yields the external gravitational field of a point mass [11]. There are some awkward aspects to this formalism which stem from the different length scales used in the general relativistic system and the Newtonian system, i.e., the relative factor of λ between lengths measured by $g_{\mu\nu}(\lambda)$ and $h^{\mu\nu}(\lambda)$. As a result, the [Bondi or Arnowitt-Deser-Misner (ADM)] mass $\tilde{M}$ of these spacetimes (1.2) is $\tilde{M}(\lambda) = \lambda^3 m$ whereas the mass of the Newtonian system obtained as a limit is $M_N = m$. In this formalism, mass densities are λ independent to leading order and the factor of λ^3 arises from the different volume elements. In order to avoid unnecessary confusion in post-Newtonian expansions, we will always speak in terms of the reparametrized Bondi mass M_B which is related to the strict general relativistic version $\tilde{M}_B$ by $M_B = \tilde{M}_B/\lambda^3$.

Two post-Newtonian expansion schemes based upon the Newton-Cartan limit have been formulated in terms of initial value problems. In one case, the initial value hypersurface is spacelike [13, 14] and in the other case null [15, 16]. Persides has formulated another post-Newtonian scheme using a combination of spacelike and null techniques [17]. There are three features by which the null and spacelike approaches especially differ. First, in the null approach, the initial λ-dependent data can be uniquely specified in terms of the initial Newtonian data of the background in a manner which reduces the incoming gravitational radiation contained in the gravitational degrees of freedom. We refer to such null data as quasi-Newtonian. Thus there is no need to average over the gravitational degrees of freedom in examining the validity of the Einstein quadrupole radiation formula, as in the spacelike case. Second, in the spacelike approach standard calculational techniques exist based upon the Green's function for the harmonic coordinate wave operator, whereas in the null approach new calculational techniques are necessary.

Such techniques have been developed to establish the quadrupole formula for quasi-Newtonian null data [18]. In this approach, the Newtonian limit is imposed by requiring that the evolution of the λ-dependent general relativistic system osculate, to third order in time, the evolution of the background Newtonian system. This avoids any inconsistencies with radiation fields while still determining unique gravitational initial data. The quadrupole formula then follows in the form $N = Q_{,ttt}$ where N is the leading λ order of the news function and Q is the transverse Newtonian quadrupole moment, both sides evaluated at the initial time.

The third way in which the spacelike and null approaches substantially differ concerns the appearance of odd powers of λ in the expansions. In the spacelike approach, odd powers first occur at a high order associated with radiation, at which stage the formal symmetry between future and past is broken. In the null case, this symmetry is broken immediately by the use of outgoing initial hypersurfaces and terms of first order in λ appear. Consequently, the description of post-Newtonian effects takes on an unusual form.

This last remark is especially applicable to the Bondi mass M_B. For the particular case of a radiating dust model [19], we find that the loss in Bondi mass due to gravitational radiation appears at the order $\Delta M_B = O(\lambda^7)$. On the other hand, the Bondi mass has a λ expansion

$$M_B = \sum_{n=0}^{7} M_B^{(n)} \lambda^n + O(\lambda^8), \tag{1.3}$$

so that the leading seven terms, $n = 0, 1, \ldots, 6$, in a λ expansion of the Bondi mass are conserved quantities. Of course, $M_B^{(0)}$ should just be the conserved total mass of the Newtonian background. In addition, one might expect $M_B^{(2)}$ to be the conserved Newtonian energy but complications arise here. In a null coordinate system there is in general a nonvanishing first-order term $M_B^{(1)}$ in the expansion. This forces us, at an early stage of the λ expansion, to devise a gauge-invariant technique in order to extract the Newtonian energy.

In this paper, we elucidate these strange features and explore the properties of the Bondi mass, from the Newtonian to the ultrarelativistic regime with the aid of a highly accurate numerical algorithm. Historically, numerical calculations of the Bondi mass have been frustrated by technical difficulties arising from the necessity to pick off nonleading terms in an expansion about infinity. There is a similarity to the experimental task of determining the mass of an object by measurements in its far field. In the nonradiative case this can be accomplished by measuring gravity gradients, but otherwise this approach can be swamped by radiation fields. In the computational case, further complications arise from gauge terms which dominate asymptotically even over the radiation terms. In Sec. IV, we describe two key ingredients which allow us to avoid these problems. The first is the use of Penrose compactification, which allows null infinity to be represented as a finite boundary to the numerical grid. The second is the introduction of renormalized variables in which Bondi's mass aspect appears as the leading asymptotic term. The accuracy of the resulting algorithm enables us to smoothly match the numerical calculation to the perturbation expansion (1.3). These results, presented in Sec. V, constitute the first calculations of the post-Newtonian behavior of the Bondi mass for a radiating system.

The scope of this paper is restricted to axisymmetric models. In Sec. II we present a brief summary of the λ-dependent version of the Bondi hypersurface equations. In Sec. III, we first examine the Bondi mass for spherically symmetric systems in order to reveal what to expect more generally. We adhere to the notation and conventions of our previous papers [15, 16, 19, 20]. In particular, we use units for which $G = c = 1$, adopt the signature $(+ - - -)$, denote λ expansions in the form $f = \sum f^{(n)} \lambda^n$ with $f^{(0)}$ the leading coefficient, and we use lower case Greek letters for spacetime indices and lower case Latin letters for spatial indices.

II. QUASI-NEWTONIAN BONDI EQUATIONS

Here we briefly present an axisymmetric version of that portion of the quasi-Newtonian null cone formalism, developed in Refs. [15, 16], which is pertinent to this paper. Given a Newtonian fluid with initial density ρ, initial velocity v_i and an equation of state $p(\rho)$, this formalism constructs a λ-dependent family of general relativistic spacetimes which also have ρ and v_i as initial data and which yield a Newton-Cartan spacetime in the $\lambda = 0$ limit. This family of spacetimes is overlaid on a single manifold to share a family of outgoing null cones labeled by a common null coordinate $x^0 = u$, which measures proper time along the vertex world-line. In addition, on each cone they share a common luminosity coordinate $x^1 = r$ and ray coordinates $x^2 = \theta$ and $x^3 = \phi$.

This leads to the λ-dependent Bondi metric [15, 19]

$$ds^2_\lambda = (r^{-1}Ve^{2\lambda^2\beta} - \lambda^4 r^2 U^2 e^{2\lambda^2\gamma})\, du^2 + 2\lambda e^{2\lambda^2\beta} du\, dr + 2\lambda^3 r^2 U e^{2\lambda^2\gamma} du\, d\theta - \lambda^2 r^2 (e^{2\lambda^2\gamma} d\theta^2 + e^{-2\lambda^2\gamma} \sin^2\theta d\phi^2), \tag{2.1}$$

where we additionally set $V = r + \lambda^2 W$. The various λ factors are inserted so that, for $\lambda = 0$, we obtain the Newton-Cartan line element $ds_0^2 = dt^2$, where $t = u + \lambda r$. The nonvanishing components of the contravariant metric are

$$g^{01} = \lambda^{-1} e^{-2\lambda^2\beta}, \tag{2.2}$$

$$g^{11} = -\lambda^{-2} r^{-1} V e^{-2\lambda^2\beta}, \tag{2.3}$$

$$g^{12} = e^{-2\lambda^2\beta} U, \tag{2.4}$$

$$g^{22} = -\lambda^{-2} r^{-2} e^{-2\lambda^2\gamma}, \tag{2.5}$$

$$g^{33} = -\lambda^{-2} r^{-2} e^{2\lambda^2\gamma} / \sin^2\theta. \tag{2.6}$$

In the $\lambda \to 0$ limit, $\lambda^2 g^{\alpha\beta} \to e^{\alpha\beta}$, where $e^{\alpha\beta}$ is a spherical coordinate version of the Euclidean metric in accord with the Newton-Cartan limit.

For uniqueness, we choose the vertex world line to be the geodesic which, at the initial time, passes through the center of mass of the background Newton-Cartan system with the center-of-mass velocity. Then (t, r, θ, ϕ) forms a polar version of a freely falling coordinate system. This fixes all essential coordinate freedom in the construction. The energy-momentum tensor has the λ-dependent form

$$T_{\mu\nu} = (\rho + \lambda^2 p)\, w_\mu w_\nu - \lambda^2 p\, g_{\mu\nu}, \tag{2.7}$$

where the four-velocity w_μ has the form $w_\mu = t_{,\mu} + \lambda^2 v_\mu$. Then, on the initial hypersurface, the data for the λ-dependent system consists of the matter data, with p determined by the equation of state. Our practice is to choose the initial values of ρ and v_i to be the λ-independent initial values for the Newton-Cartan background. This has the advantage of providing unique initial data but it is not completely essential. We discuss this point further in Sec. III.

The gravitational null data, in the axisymmetric case, consists of the function γ. Before discussing the prescription for determining a quasi-Newtonian γ, it is convenient to write down the Bondi hypersurface equations in λ-dependent form:

$$\beta_{,r} = 2\pi r\, (\rho + \lambda^2 p)\, (1 + \lambda v_1)^2 + \frac{1}{2}\lambda^2 r (\gamma_{,r})^2, \tag{2.8}$$

$$\left[r^4 e^{2\lambda^2(\gamma-\beta)} U_{,r}\right]_{,r} = 16\pi\lambda r^2 (\rho + \lambda^2 p)\, (1 + \lambda v_1) v_2 + 2r^2 \left[r^2 (r^{-2}\beta)_{,r\theta} - (\sin\theta)^{-2} (\gamma \sin^2\theta)_{,r\theta} + 2\lambda^2 \gamma_{,r}\gamma_{,\theta}\right], \tag{2.9}$$

$$\begin{aligned} V_{,r} = {} & -4\pi\lambda^2 r^2 e^{2\lambda^2\beta} [\rho - \lambda^2 p + \lambda^2 r^{-2} (\rho + \lambda^2 p) e^{-2\lambda^2\gamma} (v_2)^2] - \frac{1}{4}\lambda^4 r^4 e^{2\lambda^2(\gamma-\beta)} (U_{,r})^2 + \frac{1}{2}\lambda^2 (r^4 U \sin\theta)_{,r\theta} / (r^2 \sin\theta) \\ & + e^{2\lambda^2(\beta-\gamma)} \left[1 - \lambda^2 (\beta_{,\theta} \sin\theta)_{,\theta} / \sin\theta + \lambda^2 \gamma_{,\theta\theta} + 3\lambda^2 \gamma_{,\theta} \cot\theta - \lambda^4 (\beta_{,\theta})^2 - 2\lambda^4 \gamma_{,\theta} (\gamma_{,\theta} - \beta_{,\theta})\right]. \end{aligned} \tag{2.10}$$

Given the initial data, Eqs. (2.8)–(2.10) explicitly determine the initial values of the remaining metric quantities β, U, and $V = r + \lambda^2 W$. The evolution equations then explicitly yield the first time derivatives of ρ, v_i, and γ [20].

Quasi-Newtonian gravitational initial data γ is determined by the requirement that the connection of the λ-dependent system yield the connection of the Newton-Cartan background in the $\lambda = 0$ limit. In the present context this is equivalent to requiring that [15], for future times,

$$\Phi^* = \beta^{(0)} + W^{(0)}/2r, \tag{2.11}$$

where Φ^* is the background Newtonian potential in a freely falling frame satisfying $\Phi^* = 0$ and $\nabla_i \Phi^* = 0$ along the central geodesic $r = 0$. In order to implement Eq. (2.11) it is necessary to invert the hypersurface and evolution equations so that future values of $W^{(0)}$ and $\beta^{(0)}$ can be expressed in terms of the initial data. This leads to a sequence of Poisson equations which at successive orders determine the initial $\gamma^{(n)}$ [15, 19]. In the leading order,

$$(r^2 \gamma^{(0)}_{,r})_{,r} = -\sin\theta (\Phi^* / \sin\theta)_{,\theta}. \tag{2.12}$$

For the specific case of the dust model discussed in Sec. V, the calculation has been carried out through the determination of $\gamma^{(3)}$ [19].

It is important to note that the asymptotic dependencies of the quantities γ, β, U, and W obtained here differ from those obtained by Bondi [4]. The reason is that our coordinate conditions are chosen to reduce to a local inertial frame along the vertex world line whereas Bondi's were chosen to lead to an inertial frame at null infinity. As a result, we have the asymptotic behavior [20]

$$\gamma = K + r^{-1}c + O(r^{-2}), \tag{2.13}$$

$$\beta = H + O(r^{-2}), \tag{2.14}$$

$$U = L + O(r^{-1}), \tag{2.15}$$

$$W = r^2(L\sin\theta)_{,\theta}/\sin\theta + re^{2\lambda^2(H-K)}\Big[\lambda^{-2}(1-e^{-2\lambda^2(H-K)}) + 2(H_{,\theta}\sin\theta)_{,\theta}/\sin\theta + K_{,\theta\theta} + 3K_{,\theta}\cot\theta + 4\lambda^2(H_{,\theta})^2 - 4\lambda^2 H_{,\theta}K_{,\theta} - 2\lambda^2(K_{,\theta})^2\Big] - 2e^{2\lambda^2 H}M + O(r^{-1}), \tag{2.16}$$

where $M(\theta)$ corresponds to Bondi's mass aspect. In a Bondi frame K, H, and L all vanish.

The asymptotic two-geometry obtained after rescaling the angular part of the metric by $(-\lambda^{-2}r^{-2})$ is given by

$$d\hat{s}^2 = e^{2\lambda^2 K}d\theta^2 + \sin^2\theta e^{-2\lambda^2 K}d\phi^2. \tag{2.17}$$

This two-geometry also differs from the corresponding unit sphere two-geometry

$$d\hat{s}_B^2 = d\theta_B^2 + \sin^2\theta_B\, d\phi_B^2, \tag{2.18}$$

which would result after introducing a Bondi frame with coordinates θ_B and $\phi_B = \phi$. This complicates the calculation of the Bondi mass in our formalism. We must first find the conformal factor ω which relates the two frames, so that $d\hat{s}_B^2 = \omega^2 d\hat{s}^2$. The simplest approach is to set $y = -\cos\theta$ and $y_B = -\cos\theta_B$. Then

$$\omega^2 = \frac{dy_B}{dy}, \tag{2.19}$$

where

$$y_B = \tanh\left[\int_0^y \frac{dy}{1-y^2}\, e^{2\lambda^2 K}\right]. \tag{2.20}$$

With this ω in hand, the Bondi mass is given by a λ-dependent version of the expression given in [20],

$$M_B = \frac{1}{4}\int_0^\pi \omega^{-1}e^{-2\lambda^2 K}\Big\{ 2e^{2\lambda^2 K}M + \Big[(c\sin^2\theta)_{,\theta}/\sin\theta\Big]_{,\theta}\Big/\sin\theta - 4\lambda^2 c_{,\theta}(H+K)_{,\theta} - \lambda^2 c\Big[4\lambda^2(H_{,\theta})^2 - 8\lambda^2 H_{,\theta}K_{,\theta} - 4\lambda^2(K_{,\theta})^2 + 2(H+K)_{,\theta\theta} + 6(H+K)_{,\theta}\cot\theta\Big]\Big\}\sin\theta\, d\theta. \tag{2.21}$$

To leading order in λ, the integrand in (2.21) reduces to the first two terms, and the second term integrates to zero. This leaves

$$M_B^{(0)} = \frac{1}{2}\int_0^\pi M^{(0)}\sin\theta\, d\theta. \tag{2.22}$$

According to (2.16), the mass aspect is related to W by

$$M = -\tfrac{1}{4}e^{-2\lambda^2 H}[r^2\partial_r][r^2\partial_r](W/r^2)|_\infty. \tag{2.23}$$

The asymptotic behavior of $W^{(0)}$ can be obtained from Newtonian limit condition (2.11) in terms of the asymptotic behavior of the Newtonian potential Φ^* and β. Then (2.23) gives $M^{(0)} = M_N$ so that $M_B^{(0)} = M_N$.

Thus, in the Newtonian limit, the Bondi mass reduces to the mass of the underlying Newtonian system. It would be expected on physical grounds that the Newtonian energy give rise to $O(\lambda^2)$ corrections. However, some subtle complications arise here. The Bondi mass contains an $O(\lambda)$ term which stems from retardation effects between the values of the density ρ on an absolute time slice and on a null cone. In order to extract the Newtonian energy in an invariant way it is necessary to first introduce the mass M_I which measures the relativistic internal energy of the fluid interior,

$$M_I(u) = \lambda^{-3}\int_\Sigma \rho w^\mu dV_\mu = \int_\Sigma (1+\lambda v_1)dm, \tag{2.24}$$

where Σ denotes the outgoing null cone and $dm = \rho r^2\sin\theta\, dr\, d\theta\, d\phi$. M_I also satisfies

$$\lim_{\lambda\to 0} M_I = M_N. \tag{2.25}$$

Note that to first order M_I contains a retardation term arising from the specification of the density on the initial null cone as a λ-independent quantity equal to the initial density of the Newtonian limit system. In the traditional spacelike formulation of Newtonian limits, the initial density is specified to be λ independent on a spacelike hypersurface.

For the same reason, M_B also has exactly the same first-order retardation term as M_I. Since M_I includes the internal energy measured in the local rest frame of the fluid, physical considerations suggest that the Newtonian mechanical energy E_N, i.e., the sum of Newtonian kinetic and potential energies, be given by

$$E_N = \lim_{\lambda\to 0}(M_B - M_I)\lambda^{-2}. \tag{2.26}$$

The validity of (2.26) is established for spherically symmetric systems in the next section and also for the axisymmetric systems considered in Sec. V. The general verification of (2.26) will not be considered here.

We can reexpress the λ dependence of the M_B and M_I in terms of an alternative post-Minkowskian description of the post-Newtonian sequence of spacetimes. Let $u = \lambda\tilde{u}$, $x^i = \tilde{x}^i$, $\tilde{t} = \tilde{u} + \tilde{r}$, $g_{\mu\nu} = \lambda^2\tilde{g}_{\mu\nu}$, and $w_\mu = \lambda\tilde{w}_\mu = \lambda(\tilde{t}_{,\mu} + \lambda v_\mu)$ define post-Minkowskian variables. Then, in these coordinates, the metric $\tilde{g}_{\mu\nu}$ yields a post-Minkowskian sequence of spacetimes which satisfy the Einstein equation

$$\tilde{G}_{\mu\nu} = -8\pi\lambda^2[(\rho + \lambda^2 p)\tilde{w}_\mu\tilde{w}_\nu - \lambda^2 p\tilde{g}_{\mu\nu}]. \tag{2.27}$$

For this post-Minkowskian sequence, it is evident that

$$\tilde{M}[\rho, p, v_i; \lambda] = \tilde{M}[\lambda^2\rho, \lambda^4 p, \lambda v_i; 1], \tag{2.28}$$

for either M_B or M_I. From the coordinate invariance of these masses and their weight under dimensional rescaling of the metric, it then follows that

$$M[\rho, p, v_i; \lambda] = \lambda^{-2}M[\lambda^2\rho, \lambda^4 p, \lambda v_i; 1], \tag{2.29}$$

for the post-Newtonian system. For computational purposes, (2.29) allows considerable simplification by dropping the explicit λ dependence in the metric in favor of rescaled matter data.

III. SPHERICALLY SYMMETRIC SYSTEMS

The λ family of exterior Schwarzschild spacetimes (1.2) satisfy $M_B(\lambda) = M_N = m$ so that the Bondi mass M_B together with its Newtonian limit M_N yield no information about, say, the gravitational potential energy of the Schwarzschild interior. Furthermore, for the analytically extended Schwarzschild spacetime, the relativistic internal energy (2.24) vanishes so that it also provides no physical information. In Newtonian physics, gravitational potential energy is well defined and is nonzero for any smooth interior solution with nonvanishing mass. In order to extract the Newtonian mechanical energy from this λ-dependent family of spacetimes we must provide a λ-dependent fluid interior matching the Schwarzschild exterior. Indeed, we would then expect (2.26) to hold. We now show how this emerges from the quasi-Newtonian formalism which we have presented.

In the spherically symmetric case, $\gamma = U = v_2 = 0$ and $\omega = 1$. The governing equations then reduce to

$$\beta_{,r} = 2\pi r(\rho + \lambda^2 p)(1 + \lambda v_1)^2, \tag{3.1}$$

$$W_{,r} = -4\pi r^2 e^{2\lambda^2\beta}(\rho - \lambda^2 p) + \lambda^{-2}(e^{2\lambda^2\beta} - 1), \tag{3.2}$$

$$M_B = M, \tag{3.3}$$

and in the exterior of the fluid (3.1) and (3.2) are replaced by

$$\beta = H, \tag{3.4}$$

$$W = \lambda^{-2}r(e^{2\lambda^2 H} - 1) - 2e^{2\lambda^2 H}M + O(r^{-1}), \tag{3.5}$$

where the constants H and M are determined by continuity at the fluid boundary.

Next consider the Bondi mass. From (2.23), the mass aspect M can be reexpressed in terms of $W_{,r}$ by

$$M = e^{-2\lambda^2 H}\left[-\frac{r^4}{4}\left(\frac{W_{,r}}{r^2}\right)_{,r} - \frac{1}{2}\int_0^r W_{,r}dr\right]\Bigg|_\infty. \tag{3.6}$$

Using (3.2) to substitute for $W_{,r}$ and then integrating by parts, using (3.1) to substitute for $\beta_{,r}$, we obtain

$$\begin{aligned} M &= e^{-2\lambda^2 H}\left\{-\frac{r^4}{4}\left(\frac{e^{2\lambda^2 H} - 1}{\lambda^2 r^2}\right)_{,r} - \frac{1}{2}\int_0^r \left[-4\pi r^2 e^{2\lambda^2\beta}(\rho - \lambda^2 p) + \frac{e^{2\lambda^2\beta} - 1}{\lambda^2}\right] dr\right\}\Bigg|_\infty \\ &= e^{-2\lambda^2 H}\int_0^\infty \left\{2\pi r^2 e^{2\lambda^2\beta}(\rho - \lambda^2 p) + re^{2\lambda^2\beta}\beta_{,r}\right\}dr \\ &= \int_0^\infty \left\{4\pi r^2\rho + 4\pi r^2\rho\,[e^{2\lambda^2(\beta - H)} - 1] + 2\pi\lambda r^2 e^{2\lambda^2(\beta-H)}(\rho + \lambda^2 p)\,v_1\,(2 + \lambda v_1)\right\}dr. \end{aligned} \tag{3.7}$$

For $\lambda = 0$, we see that $M_B^{(0)} = M_N$. Proceeding with further integration by parts (and now setting $dm = 4\pi\rho r^2 dr$), we obtain

$$M_B = M_N + \int_0^{M_N} \lambda^2\left(-\frac{m}{r} + \frac{1}{2}v_1^2\right) dm + \lambda\int_0^{M_N} v_1\, dm \tag{3.8}$$

$$\begin{aligned} &+ \int_0^\infty \Big\{-4\pi\lambda^2 m\rho r\,[\lambda e^{2\lambda^2(\beta-H)}v_1(2 + \lambda v_1) + e^{2\lambda^2(\beta-H)} - 1] \\ &\quad -4\pi\lambda^4 mpre^{2\lambda^2(\beta-H)}(1 + \lambda v_1)^2 + 2\pi\lambda\rho r^2[e^{2\lambda^2(\beta-H)} - 1]v_1(2 + \lambda v_1) \\ &\quad +2\pi\lambda^3 pr^2 e^{2\lambda^2(\beta-H)}v_1(2 + \lambda v_1)\Big\}\, dr, \end{aligned} \tag{3.9}$$

so that

$$M_B = M_N + \lambda^2 E_N + \lambda \int v_1\, dm + O(\lambda^3), \tag{3.10}$$

where the Newtonian energy E_N is the sum of the kinetic and gravitational potential energy. Comparing (2.24) and (3.9), the desired result (2.26) is established for spherical quasi-Newtonian systems. Note that the occurrence of a first-order term

$$M_B^{(1)} = \int v_1\, dm \tag{3.11}$$

leads to no difficulty. This term is a gauge effect of the null-cone formalism which appears in both M_I and M_B but does not appear in E_N which is gauge invariant.

In the spherically symmetric case, there is no gravitational radiation so that $dM_B/du = 0$. Thus each coefficient $M_B^{(n)}$ in the expansion of the Bondi mass must be a conserved quantity. Two of these, M_N and E_N, are well-known Newtonian constants of the motion. The remaining terms cannot apparently be given gauge-invariant meaning. To understand this, let us consider the first-order term more closely. At the initial time, our practice is to choose ρ and v_1 to be independent of λ so that retardation effects lead to (3.11). Now $M_B^{(1)}$ is clearly not a conserved Newtonian quantity. It is conserved in the λ-dependent system because at later times, ρ and v_1 develop λ dependence according to the general relativistic fluid evolution equations. Thus, from expanding (3.10),

$$M_B^{(1)} = \int 4\pi r^2 [\rho^{(1)} + \rho^{(0)} v_1]\, dr, \tag{3.12}$$

at later times. The conserved quantity $M_B^{(1)}$ is of a post-Newtonian nature associated with the growth of $\rho^{(1)}$. Similar considerations apply to the higher-order conserved quantities. From the point of view of a Newtonian limit based upon spacelike hypersurfaces, $\rho^{(1)}$ would be interpreted as a correction to the fluid density due to retardation. However, only in very simple cases could such an interpretation be extended to higher orders because of the tenuous relationship between null and spacelike foliations.

At this point, it should be noted that the λ independence of the initial data ρ and v_1 has the advantage of leading to the unique data of the quasi-Newtonian scheme. However, it is not completely clear whether relaxation of this requirement might lead to any substantial improvement in the formalism. One case in which one might want to relax this requirement is in the study of post-Newtonian effects on fluid equilibrium models. In that case, for a static spherical fluid, $\xi^\alpha \partial/\partial x^\alpha = \partial/\partial u$ is a Killing vector and w^α and ξ^α are parallel so that $w^1 = 0$. Thus

$$1 = g_{\alpha\beta} w^\alpha w^\beta = g_{00} w^0 w^0 = g_{00} g^{01} g^{01} w_1 w_1, \tag{3.13}$$

so that

$$1 = e^{-2\lambda^2 \beta}(1 + \lambda^2 W/r)(1 + \lambda v_1)^2. \tag{3.14}$$

From this last equation, although $v_1^{(0)} = 0$ it is apparent that v_1 cannot be λ independent at higher orders for a λ family of static fluids. If v_1 were initially set equal to the value v_{N1} for a static Newtonian background then the evolution would not lead to a static λ family of quasi-Newtonian spacetimes. This suggests that the covariant w^1, rather than w_1 (or, equivalently, v_1), might be more useful as velocity data for quasi-Newtonian systems. However, more generally such a modification does not seem to have any distinct advantage.

The Oppenheimer-Snyder spacetime is an example of particular physical interest whose evolution can be expressed in analytic form. In a null-cone coordinate system, the case of a $k = 0$ collapsing Friedmann interior has dust density

$$\rho = (1/6\pi) u^{-2} (1-\alpha)^{-6}, \tag{3.15}$$

and four-velocity components

$$w_\mu = (1-\alpha)(1-3\alpha)^{-1}[(1-\alpha)^2, 1, 0, 0], \tag{3.16}$$

where $r = -3u\alpha(1-\alpha)^2$, with the singularity at $u = 0$. As the boundary separating the dust from its Schwarzschild exterior, we take the geodesic $\alpha = -A(3u)^{-1/3}$. This system can be embedded as the $\lambda = 1$ case of a one-parameter family of Oppenheimer-Snyder spacetimes approaching a Newtonian limit [21,22]. The limiting Newtonian system has density $\rho_N = (1/6\pi)u^{-2}$, $v_{N1} = -(2r/3u)$ and boundary $R_N = 3u^{2/3}A$.

In this case, the masses M_B and M_I can be calculated exactly, using the general relativistic rules, and also the Newtonian mass M_N and energy E_N, using the Newtonian rules. We find $M_B = M_I = M_N = 2A^3$ and $E_N = 0$. These results are in agreement with physical expectations, the Bondi mass is constant in time because there is no radiation, and the internal energy is constant because there is no pressure. In the initial dispersed state, in the limit $u \to -\infty$, there is no potential energy and no kinetic energy in the $k = 0$ case. Thus these general relativistic masses are initially equal and remain equal. Also, since the initial dispersed state satisfies the physical criteria to be a Newtonian limit, it is no surprise that the Bondi mass exactly equals the conserved mass of the Newtonian background.

IV. COMPUTING THE BONDI MASS

It is not possible to calculate the Bondi mass analytically even for the simplest nonspherical system of physical interest. A computational approach is necessary. An accurate computational algorithm for the Bondi mass is not only crucial for our present purpose of exploring post-Newtonian behavior. It is vital for using the Bondi mass loss equation as a check on the accuracy of numerical evolution schemes to calculate gravitational radiation. For these purposes, it is essential to have an algorithm which gives the Bondi mass to second-order accuracy in grid size.

Some difficult technical issues are involved here. These stem from the role that the mass aspect M plays in the integral (2.21). The mass aspect must be extracted as the

$O(1)$ term in the asymptotic expansion of W (2.16) whose first term is of order r^2 and diverges at null infinity. Thus the accuracy of any straightforward numerical calculation of M would be swamped by the noise in the calculation of the coefficient of r^2. The same difficulties arise from the term of order r in W. It is thus necessary to introduce renormalized variables to eliminate this source of noise. For simplicity, we set set $\lambda = 1$ in describing this process.

We have found a new set of renormalized variables which are globally smooth and allow the mass aspect to be extracted to second-order accuracy in grid size. These new variables replace U and $V = r + W$ according to the definitions

$$2\tau = (1-y^2)^{-1/2} r^3 e^{2(\gamma-\beta)} U_{,r} + 2r\beta_{,y} - r^2(1-y^2)^{-1} e^{2\gamma}[(1-y^2)e^{-2\gamma}]_{,ry} \tag{4.1}$$

and

$$2\mu = -V + r^2[(1-y^2)^{1/2}U]_{,y} - r^3 e^{2\beta}(\mathcal{K}/r)_{,r} + e^{2\beta}[(1-y^2)e^{-2\gamma}\tau]_{,y}, \tag{4.2}$$

where $y = -\cos\theta$ and

$$\mathcal{K} = -\frac{1}{2}[(1-y^2)e^{-2\gamma}]_{,yy}. \tag{4.3}$$

Note that $\mathcal{K}$ is the Gaussian curvature of the angular metric

$$e^{2\gamma} d\theta^2 + e^{-2\gamma}\sin^2\theta\, d\phi^2. \tag{4.4}$$

They satisfy the radial equations

$$(r\tau)_{,r} = 8\pi r^2(\rho + p)(1-y^2)^{-1/2} w_1 w_2 + 4\pi r^3[(\rho+p)(w_1)^2]_{,y} + r(1-y^2)^{-1}e^{2\gamma}[(1-y^2)e^{-2\gamma}(r^2\gamma_{,r})_{,r}]_{,y} \tag{4.5}$$

and

$$2\mu_{,r} = e^{2\beta}\Big\{ 4\pi[r^2(\rho-p) + e^{-2\gamma}(\rho+p)(w_2)^2] + r^{-2}(1-y^2)e^{-2\gamma}\psi^2 + 2\beta_{,r}\{[(1-y^2)e^{-2\gamma}\tau]_{,y} - r^3(\mathcal{K}/r)_{,r}\} - \{(1-y^2)e^{-2\gamma}[2\psi\gamma_{,r} - 2r\beta_{,ry} - 8\pi r(\rho+p)(1-y^2)^{-1/2}w_1 w_2]\}_{,y}\Big\}, \tag{4.6}$$

where

$$\psi = \tau + \frac{1}{2} r^2 (1-y^2)^{-1} e^{2\gamma}[(1-y^2)e^{-2\gamma}]_{,ry}. \tag{4.7}$$

With these renormalizations, τ vanishes at null infinity, either as $O(1/r)$ for asymptotically flat spacetimes or as $O(\ln r/r)$, for logarithmically asymptotically flat spacetimes. (We will deal with an example of the latter type in the next section.) In either case, (4.5) may be integrated by standard numerical techniques to determine τ globally to second-order accuracy in grid size. In doing so, null infinity is introduced as a finite grid boundary by introducing the new radial coordinate

$$x = r/(1+r), \tag{4.8}$$

with the grid uniformly spaced in Δx. Similarly, μ is finite at null infinity and can be determined to global second-order accuracy by numerical integration of (4.6).

The asymptotic value of μ plays the role of a generalized mass aspect, with the formula (2.21) for the Bondi mass taking the simple form

$$M_B = \frac{1}{4\pi}\oint \omega^{-1} e^{-2H}\mu|_{x=1}\sin\theta\, d\theta\, d\phi. \tag{4.9}$$

We have carried out extensive convergence tests on our numerical implementation of the above scheme for calculating M_B. As analytic testbeds, we have used boost-rotation symmetric vacuum solutions [23], spherically symmetric matter solutions and the axially symmetric post-Newtonian dust solution to be discussed in the next section. All these tests confirm that our algorithm for the Bondi mass is second-order accurate in grid size.

V. RADIATING DUST

We now apply this computational algorithm to study the mass of a post-Newtonian radiating sphere of dust. The model is generated by the Newtonian initial data consisting of an initially homogeneous ball centered at the origin,

$$\rho = \begin{cases} k & \text{for } r < R, \\ 0 & \text{for } r > R, \end{cases} \tag{5.1}$$

moving with the quadrupolar radial velocity flow

$$v_1 = vr^3(3\cos^2\theta - 1). \tag{5.2}$$

The Newtonian mass and energy are

$$M_N = \frac{4}{3}\pi k R^3, \qquad E_N = \frac{8}{45}\pi k v^2 R^9 - \frac{16}{15}\pi^2 k^2 R^5. \tag{5.3}$$

For this model, the relativistic internal energy (2.24) is simply $M_I = M_N$. In order to fix the dimensional freedom in this model, all our numerical results will be based upon the choice $R = 1$.

In a previous study of the Bondi news function for this model [19], the post-Newtonian expansion for the initial value of γ was calculated through terms of order λ^3. The expansion was truncated at this order since this is sufficient to remove incoming radiation from the data which would otherwise negate the generalization of the Einstein quadrupole formula for the outgoing radiation. This truncated data and other pertinent post-Newtonian properties of the model are given in the Appendix.

Using computer assisted algebra, it is fairly straightforward, although challenging, to calculate the post-Newtonian expansion M_λ of the Bondi mass (1.3) based upon these data. The results are given through terms of $O(\lambda^7)$ in the Appendix. For this model, $M_B^{(1)} = 0$, so that $M_B^{(0)}$ and $M_B^{(2)}$ are equal to M_N and E_N, respectively.

For small λ, where radiation fields are small, all the essential physical effects are already present in the spherically symmetric case. So, to provide some bearing on the effects of radiation, first consider the nonradiative case with $v = 0$. Figure 1 compares the corresponding post-Newtonian expansion of the Bondi mass for $k = 0.01$ with its numerically computed values, as a function of λ. For small λ these two graphs are in good agreement as the binding energy $-E_N$ serves to decrease the total energy. However, as λ enters the strong field regime, the post-Newtonian expansion becomes negative whereas the computed value remains positive. This shows how keeping even seven terms in a post-Newtonian expansion can lead to contradictions with the positivity of the Bondi mass. The graph of the post-Newtonian expansion rapidly peels away from the computed graph as the strictly defined general relativistic mass, $\tilde{M}_B = \lambda^3 M_B$, approaches $R/2$. This is the regime in which any post-Newtonian expansion is destined to fail.

It is also instructive, for this example, to plot the Bondi mass as a function of k, setting $\lambda = 1$, as shown in Fig. 2. This generates a post-Minkowski family of systems. Note that in this case the Bondi mass saturates at the value $M_B = 1 = R$. From a post-Minkowski viewpoint, the null data develops an antitrapped surface at high density, with the matter inside a white hole of radius $2R$. This strong field limit as $k \to \infty$ can be verified analytically by an asymptotic method developed for a Klein-Gordon source [24]. In the present case, the last expression for M_B in (3.7) gives

$$M_B = 4\pi k \int_0^1 dr\, r^2 e^{2(\beta - H)}, \tag{5.4}$$

where $\beta = \pi r^2 k$ and $H = \pi k$. Application of the method of Laplace [25] to this integral then yields $M \sim 1$ for large k.

Next consider the radiative case, again setting $k = 0.01$, but now with $|v| = v_0/10$, where $v_0 = (6\pi k)^{1/2}$ is the value of v for which the Newtonian binding energy vanishes. For positive v the dust sphere evolves toward a pancake shape; and for negative v, toward a cigar shape. Figure 3 graphs the Bondi mass for this system, as well as its post-Newtonian expansion. Since the background Newtonian system is bound, the Bondi mass initially decreases below the Newtonian mass for small λ. However, as λ increases, the graphs sharply reverse and the Bondi mass rapidly increases. Note that the behavior of the Bondi mass in this regime depends on the sign of v, as opposed to the independence of Newtonian kinetic energy on the direction of velocity. This dependence on the sign of v also appears in the post-Newtonian expansion. Since the highest order coefficient $M_B^{(7)}$, given in (A14), is negative when $v > 0$, the post-Newtonian expansion

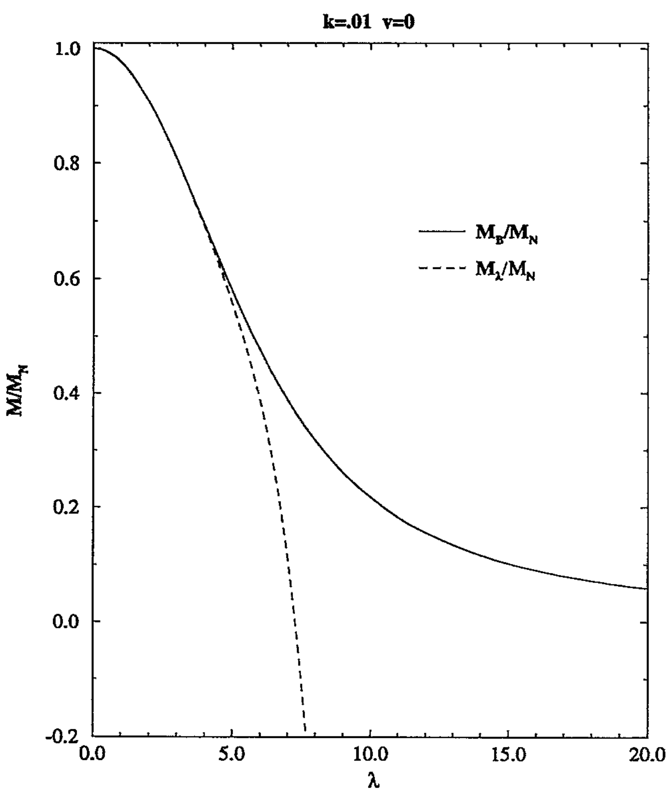

FIG. 1. The Bondi mass M_B and its post-Newtonian expansion M_λ, as a function of λ for $k = 0.01$. Both curves are in units of M_N, the Newtonian mass.

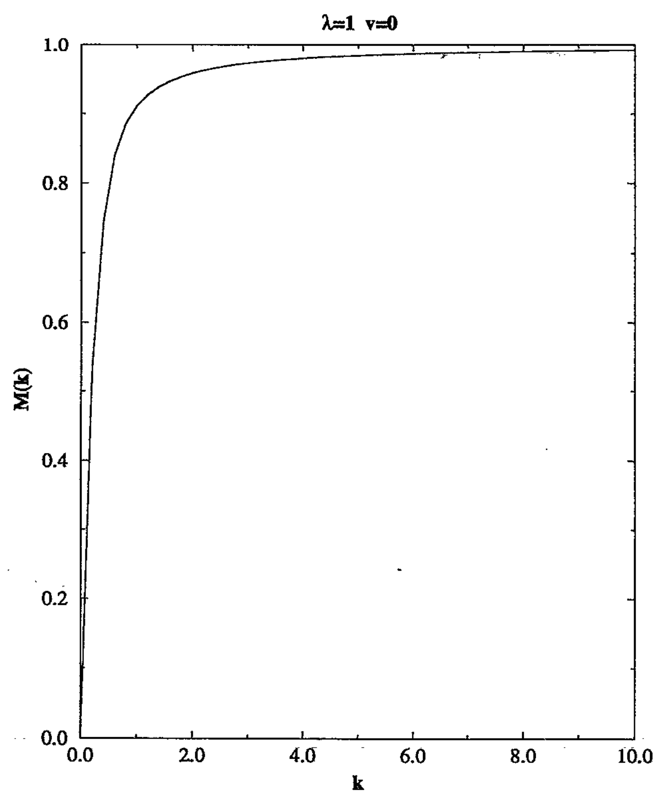

FIG. 2. The Bondi mass M_B as a function of k, the central density, for $\lambda = 1$. Note that the Bondi mass saturates at the value $M_B = 1 = R$.

for the pancake case always leads to negative values of the Bondi mass at sufficiently large λ.

In the high-λ limit, the purely gravitational contribution to the Bondi mass dominates the contribution from matter. This is evident in comparing Fig. 1 with Fig. 3. In the spherically symmetric case where there is only a matter contribution, shown in Fig. 1, the Bondi mass remains bounded for high λ. This is in contrast to the infinite limit apparent in the radiative case of Fig. 3. In a separate study of strong field limits [26], we have shown that the Bondi mass increases linearly with the amplitude of the gravitational data, as opposed to the quadratic dependence in the weak field case. This results from an exponentially strong redshift effect which suppresses all but the far field contribution to the mass. However, this linearity only holds when the gravitational data is expressed in a standard Bondi frame, in which $K = 0$. [See Eq. (2.13).] Because $K \neq 0$ in the present frame, an exponential dependence on λ arises in Fig. 3 from the conformal factor ω.

Figure 4 graphs the Bondi mass for $k = .01$ in the transitional case $|v| = v_0$. Because the Newtonian binding energy vanishes, the graphs are almost horizontal for small λ with just a small negative slope corresponding to the negative value of the coefficient $M_B^{(4)}$, in (A11). At larger λ, the gravitational contributions again dominate. For the case $v = 2v_0$, shown in Fig. 5, the kinetic energy is the dominant factor for small λ and the graphs are everywhere increasing.

The foregoing results incorporate the quasi-Newtonian formalism in an essential way to determine the appropriate data for the gravitational field. Newtonian matter data inserted naively into Einstein's theory would lead to different behavior. For example, a change in the value of $\gamma^{(0)}$ would lead to a change in the quadratic coefficient $M_B^{(2)}$ in the expansion of the Bondi mass. As a result the validity of (2.26) for extraction of the Newtonian mechanical energy E_N depends upon using the quasi-Newtonian formalism to determine the proper gravitational data. When the Newtonian matter density is spherically symmetric $\gamma^{(0)} = 0$, but otherwise setting $\gamma^{(0)} = 0$ would in general be inconsistent with (2.26). For our quasi-Newtonian dust model, with $k = 0.01$, $|v| = v_0/10$ and a grid of 256 angular and 256 radial points, our computed value via (2.26) is $E_N = -1.0411 \times 10^{-3}$ which agrees with the exact value $E_N = -1.0422 \times 10^{-3}$.

VI. CONCLUSION

We have developed an algorithm for calculating the Bondi mass based upon renormalized variables which has been tested to converge at second order in grid size. We have shown this algorithm to be highly effective in exploring both the Newtonian and strong field limits of general

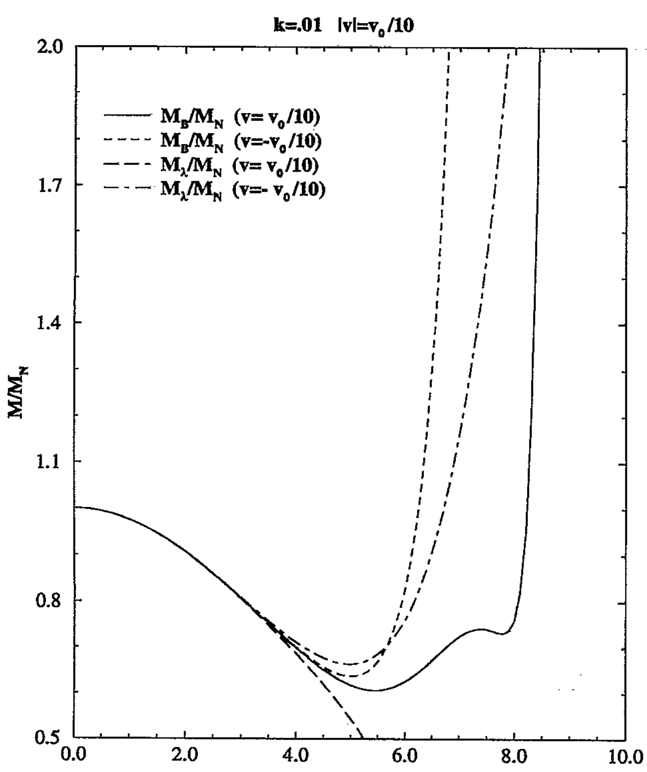

FIG. 3. The Bondi mass M_B and its post-Newtonian expansion M_λ, as a function of λ for $k = 0.01$ and $|v| = v_0/10$, in units of M_N, the Newtonian mass. Note that the behavior of the Bondi mass in this regime depends on the sign of v, unlike the Newtonian kinetic energy, which does not depend on the direction of velocity.

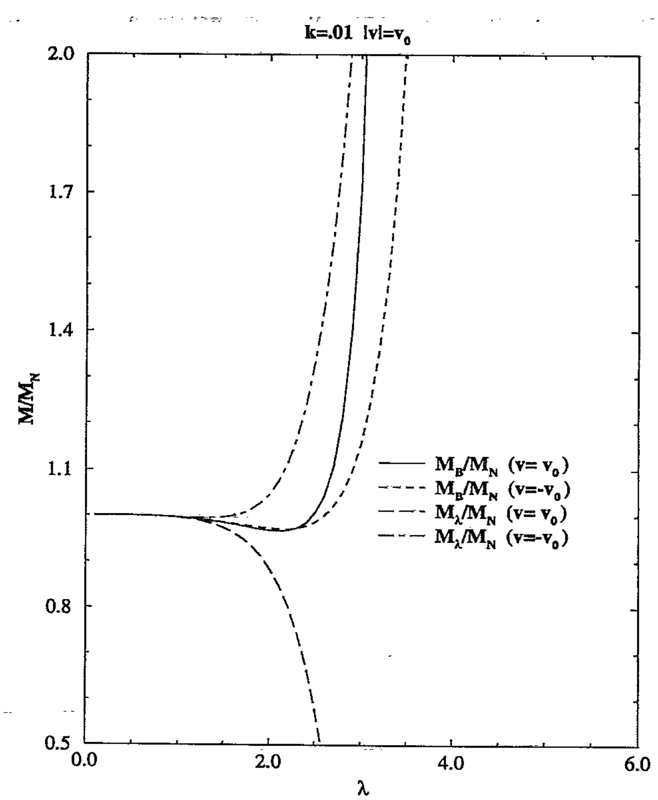

FIG. 4. The Bondi mass M_B and its post-Newtonian expansion M_λ, as a function of λ for $k = 0.01$ and $|v| = v_0$, in units of M_N, the Newtonian mass. The graphs are almost horizontal for small λ because of the vanishing of the Newtonian binding energy.

relativity. In particular, the algorithm joins smoothly and accurately to a post-Newtonian expansion of the Bondi mass carried out through terms of $O(\lambda^7)$. At higher lambda, the computed mass peels away and shows markedly non-Newtonian behavior. It remains strictly positive, in contrast to the post-Newtonian expansion.

The algorithm extends the role of the Bondi mass as a basic theoretical concept in the theory of gravitational radiation to also serving as a highly accurate tool of computational relativity. Computational checks of the Bondi mass loss formula can provide a global check on the preservation of the Bianchi identities. The mass loss rates themselves have important astrophysical significance. Our results establish that computational approaches, based strictly upon the geometrical definition of mass in general relativity, can be used to calculate radiation losses in highly nonlinear processes where perturbation calculations fail.

ACKNOWLEDGMENTS

We thank the National Science Foundation for research support under NSF Grant No. PHY92-08349 and the Pittsburgh Supercomputing Center for making computer time available. R.A.I. and J.W. would like to thank the Max Planck Institute for Astrophysics in Garching, whose hospitality was key to the initiation of this work.

APPENDIX

For the Newtonian matter data (5.1) and (5.2), the quasi-Newtonian gravitational data are given by

$$\gamma = \lambda\gamma^{(1)} + \lambda^2\gamma^{(2)} + \lambda^3\gamma^{(3)}, \tag{A1}$$

with $\gamma^{(0)} = 0$ in this special case. The values of the coefficients in the interior ($r < 1$) are [19]

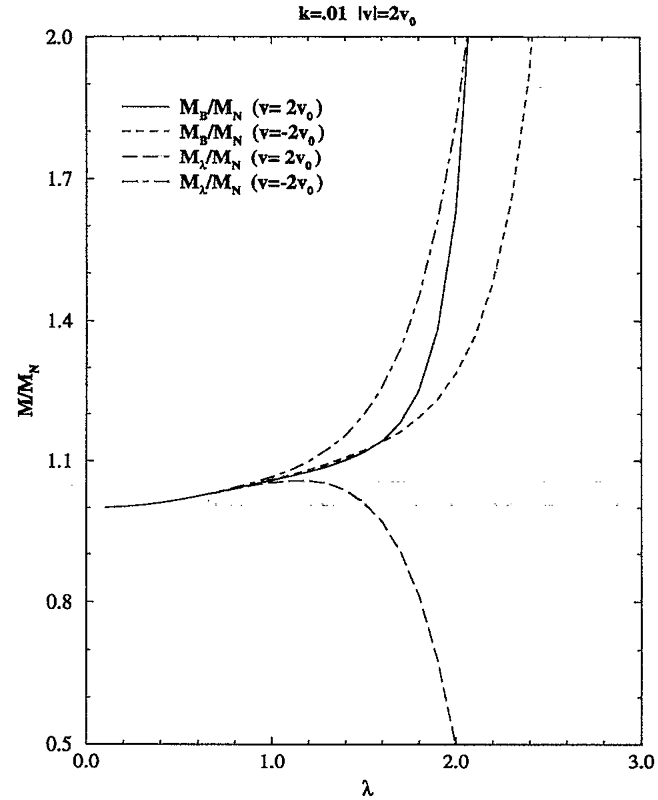

FIG. 5. The Bondi mass M_B and its post-Newtonian expansion M_λ, as a function of λ for $k = 0.01$ and $|v| = 2v_0$, in units of M_N, the Newtonian mass. Since the kinetic energy is the dominant factor for small λ, the slope of the graphs of M_B and M_λ is always positive.

$$\gamma^{(1)} = \pi\,\eth^2 Y_2\, k\left(-\frac{4\,r^2}{45} + \frac{r^3}{6} - \frac{7\,r^5}{180}\right), \tag{A2}$$

$$\gamma^{(2)} = \pi\,\eth^2 Y_4\, k\left(\frac{13\,r^4}{4900} - \frac{8\,r^5}{225} + \frac{16\,r^6}{385} - \frac{3499\,r^8}{360360}\right) v^2 + \pi\,\eth^2 Y_2\, k\left(-\frac{4\,r^2}{105} - \frac{8\,r^3}{105} + \frac{24\,r^4}{245} - \frac{31\,r^8}{3465}\right) v^2, \tag{A3}$$

$$\begin{aligned}\gamma^{(3)} = {} & \frac{\pi^2\,\eth^2 Y_2\, k^2}{2}\left(-\frac{3068\,r^2}{7875} + \frac{79\,r^3}{35} - \frac{78\,r^4}{25} + \frac{2386\,r^5}{1575} - \frac{2524\,r^7}{18375}\right) v \\ & + \frac{\pi\eth^2 Y_2\, k}{2}\left(-\frac{64\,r^2}{315} + \frac{8\,r^3}{7} - \frac{576\,r^4}{343} + \frac{16\,r^5}{21} - \frac{40\,r^{11}}{3087}\right) v^3 \\ & + \frac{\pi\eth^2 Y_4\, k}{2}\left(-\frac{816\,r^4}{18865} + \frac{32\,r^5}{175} - \frac{768\,r^6}{3025} + \frac{72\,r^7}{605} - \frac{438\,r^{11}}{94325}\right) v^3 \\ & + \frac{\pi\,\eth^2 Y_6\, k}{2}\left(-\frac{96\,r^6}{4235} + \frac{7452\,r^7}{77077} - \frac{144\,r^8}{1001} + \frac{36\,r^9}{455} - \frac{26\,r^{11}}{2805}\right) v^3,\end{aligned} \tag{A4}$$

and in the exterior,

$$\gamma^{(1)} = \pi\, \eth^2 Y_2\, k\, \left(\frac{4}{15} + \frac{1}{20\, r^3} - \frac{5}{18\, r}\right) v, \tag{A5}$$

$$\gamma^{(2)} = \pi\, \eth^2 Y_2\, k\, \left(\frac{1}{21} + \frac{232}{8085\, r^3} - \frac{32}{315\, r}\right) v^2 + \pi\, \eth^2 Y_4\, k\, \left(\frac{9}{1400} + \frac{149}{25025\, r^5} - \frac{1}{126\, r^4} + \frac{4}{2695\, r^3} - \frac{11}{1575\, r}\right) v^2, \tag{A6}$$

$$\begin{aligned}\gamma^{(3)} = {} & \pi^2\, \eth^2 Y_2\, k^2\, \left(-\frac{5783}{22050} + \frac{1}{6\, r^4} - \frac{7789}{15750\, r^3} + \frac{1028}{1575\, r} - \frac{32\, (\log r)}{315\, r^3}\right) v \\ & + \pi\, \eth^2 Y_6\, k\, \left(\frac{2}{1617\, r^7} - \frac{24}{7735\, r^6} + \frac{27}{11011\, r^5} - \frac{32}{55055\, r^4}\right) v^3 \\ & + \pi\, \eth^2 Y_4\, k\, \left(\frac{51}{21175\, r^5} - \frac{104}{21175\, r^4} + \frac{48}{18865\, r^3}\right) v^3 + \frac{8\, \pi\, \eth^2 Y_2\, k\, v^3}{1715\, r^3},\end{aligned} \tag{A7}$$

where

$$\eth^2 Y_2 = 6\, \sin^2\theta, \tag{A8}$$

$$\eth^2 Y_4 = 60\, \left(-1 + 7\, \cos^2\theta\right) \sin^2\theta, \tag{A9}$$

$$\eth^2 Y_6 = 210\, \left(1 - 18\, \cos^2\theta + 33\, \cos^4\theta\right) \sin^2\theta\,. \tag{A10}$$

This leads to the expansion coefficients for the Bondi mass, $M_B^{(0)} = M_N$, $M_B^{(2)} = E_N$, $M_B^{(1)} = M_B^{(3)} = 0$, and

$$M_B^{(4)} = \frac{64\, \pi^3\, k^3}{105} - \frac{3424\, \pi^2\, k^2\, v^2}{3465}, \tag{A11}$$

$$M_B^{(5)} = -\frac{513776\, \pi^2\, k^2\, v^3}{4244625}, \tag{A12}$$

$$M_B^{(6)} = -\frac{256\, \pi^4\, k^4}{945} + \frac{375691088\, \pi^3\, k^3\, v^2}{118243125} + \frac{16357504\, \pi^2\, k^2\, v^4}{938062125}, \tag{A13}$$

$$M_B^{(7)} = -\frac{8\, \pi^4\, k^4\, v}{45} - \frac{274691273174\, \pi^3\, k^3\, v^3}{28969565625} - \frac{2984192\, \pi^2\, k^2\, v^5}{491365875}. \tag{A14}$$

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