Environment and Planning B: Planning and Design, 1992, volume 19, pages 585-603

Shape recognition in three dimensions

R Krishnamurti
Department of Architecture, Carnegie Mellon University, Pittsburgh, PA 15213, USA
C F Earl
Department of Mechanical, Materials and Manufacturing Engineering, University of Newcastle upon Tyne NE1 7RU, England
Received 17 December 1991

Abstract. The subshape recognition problem for three-dimensional shapes under linear transformations is considered. The problem is analysed in a series of cases, some that provide a determinate number of solutions and others that have indeterminately many solutions. Procedures for its solution for general shapes are developed. Difficulties posed by strict adherence to rational transformations are examined. As a corollary, an outline of a procedure for determining the symmetries of a shape is presented.

Introduction

A central computational problem associated with shape grammar theory is discussed, namely, under which transformations is one shape a subshape of another? Further, the question is examined whether these transformations—if they exist—between shapes that can be rationally described also can be described by rational coefficients. In other words, can the subshape recognition problem for rational shapes always be resolved using exact arithmetic?

The importance of the subshape problem can be gauged from the fact that its resolution is a prerequisite for the composition of shapes by the application of spatial rules. A spatial relation (α, β) between shapes α and β can be considered as a spatial rule that *applies* to a shape γ if we can find a similarity (in general, a linear transformation) τ such that $\tau(\alpha)$ occurs as a shape in γ in which case $\tau(\beta)$ replaces $\tau(\alpha)$ in γ under *rule application*.

The subshape recognition problem in the form considered here has direct application to shape grammars (Stiny, 1980a; 1990), and to the ways in which spatial relations between shapes can be used to specify shape rules (Earl, 1986; Earl and Krishnamurti, 1984; Stiny, 1980b). Although acquaintance with shape grammar concepts would be an advantage, it is not crucial. The properties of shapes and definitions of spatial terms necessary for the arguments in this paper will be developed as needed.

The treatment for two-dimensional shapes given in Krishnamurti (1980; 1981) is the basis for implementations of shape generation systems (Chase, 1989; Krishnamurti, 1982; Krishnamurti and Giraud, 1986). A discussion of three-dimensional shapes is presented in Earl (1986), and an independent approach both for two-dimensional and for three-dimensional shapes is given in Stiny (forthcoming). This paper completes the details of the three-dimensional case.

The following ideas are basic.

A *shape* is a finite set of maximal straight lines of finite, nonzero length, where each line is specified by the coordinates of its endpoints. A shape is *rational* if its lines have endpoints given by rational coordinates, and *real* otherwise. The lines in a shape are *maximal* in the sense that colinear lines are separated by a gap, but otherwise lines may touch or intersect. A *subshape* of a shape is a shape the lines of which each have endpoints that are coincident with a line in the original shape. The verification of the subshape relation for any two shapes s and s' can be approached as follows.

Let s denote a shape and the set of maximal lines that describes it. Shape s may be structured by an equivalence relation that partitions the set into equivalence classes of colinear maximal lines. That is, each line l is associated with an unambiguous *descriptor*, say $co(l)$, such that colinear lines have identical descriptors.

An equivalence class of colinear lines can be organised by the order relation, $<$, extended to n-tuples as follows. Let A and B denote two arbitrary n-tuples of numbers, $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$. Then $A < B$, if for some $j \in \{1, 2, \ldots, n\}$, $a_k = b_k$, $k < j$, and $a_j < b_j$. Equality between A and B holds whenever the components of A and B are identical.

The endpoints of a line segment l can be ordered such that one is designated as the $tail(l)$ and the other as the $head(l)$ where $tail(l) < head(l)$. A line l may be thus described by the triple $\langle co(l), tail(l), head(l)\rangle$. A set of colinear maximal lines can be arranged as a sequence of lines in (increasing) order of head or tail values.

A further structuring of a shape can be carried out. Suppose that the descriptor function co is expressed as a tuple of numbers, then the equivalence classes of colinear lines can be arranged as a sequence in order of their descriptor values.

Two shapes can then be checked for equality, denoted $s = s'$, by comparing the sequences in their description term by term. Observe that by arranging the description of a shape in the manner suggested above, shape equality can be tested for in a time linear in the number of lines in shapes s or s'.

In a similar fashion, the subshape relation $s \leqslant s'$ can be verified. Here the description of s must be wholely contained within the description of s'. That is, for any two colinear lines, $l = \langle co(l), tail(l), head(l)\rangle$ in s, and $l' = \langle co(l'), tail(l'), head(l')\rangle$ in s', l is *contained* in l' if and only if:

$$co(l) = co(l'), \quad tail(l) \geqslant tail(l'), \quad \text{and } head(l) \leqslant head(l') .$$

The subshape relation can be decided in a time linear in the number of lines in s or s'.

Line geometry in three dimensions

Points are described in *homogeneous coordinates* (x, y, z, w) in order to facilitate the description of points with rational coordinates as 4-tuples of integers. For real shapes we take $w = 1$. For rational shapes, any three-dimensional point (x, y, z) is expressed as a 4-tuple of relatively prime integers (xw, yw, zw, w) for some integer w chosen such that the integers are relatively prime. A simple way to do this is to set $w = w'/a$, where w' is the product of the denominators of the nonzero x, y, and z, and $a = \gcd(xw', yw', zw', w')$, the greatest common divisor. We will assume that all coordinates are given in their (in the case of rational shapes, to their *reduced* relatively prime) homogeneous coordinate form.

Points can be defined by vectors (which are indicated by boldface). A point p with coordinates (x, y, z, w) can be represented as $\boldsymbol{p} = (x, y, z, w)$. The operations of vector arithmetic, scalar, and cross products are defined for any two vectors $\boldsymbol{p} = (x_p, y_p, z_p, w_p)$ and $\boldsymbol{q} = (x_q, y_q, z_q, w_q)$ as:

$$\lambda \boldsymbol{p} = (\lambda x_p, \lambda y_p, \lambda z_p, w_p) , \tag{1}$$

$$\boldsymbol{p} \pm \boldsymbol{q} = (x_p w_q \pm x_q w_p, y_p w_q \pm y_q w_p, z_p w_q \pm z_q w_p, w_p w_q) , \tag{2}$$

and

$$\boldsymbol{p}\cdot\boldsymbol{q} = \frac{1}{w_p w_q}(x_p x_q + y_p y_q + z_p z_q)\,, \tag{3}$$

$$\boldsymbol{p}\times\boldsymbol{q} = (y_p z_q - y_q z_p,\, z_p x_q - z_q x_p,\, x_p y_q - x_q y_p,\, w_p w_q)\,. \tag{4}$$

Line descriptor

The descriptor function for a line is now established. Let p and p' be endpoints of a line segment l, $p = (x, y, z, w)$ and $p' = (x', y', z', w')$. The line l can be described by a pair of vectors $(\boldsymbol{L}, \boldsymbol{L}_0)$ representing the direction of the line and its moment about the origin respectively (figure 1). The vectors are given by the following expressions:

$$\boldsymbol{L} = \boldsymbol{p}-\boldsymbol{p}' = (xw'-x'w,\, yw'-y'w,\, zw'-z'w,\, ww')\,, \tag{5}$$

$$\boldsymbol{L}_0 = \boldsymbol{p}\times\boldsymbol{p}' = (yz'-y'z,\, zx'-z'x,\, xy'-x'y,\, ww')\,. \tag{6}$$

The components of $\boldsymbol{L}$ are denoted by $\boldsymbol{L} = (L, M, N, W)$ and those of $\boldsymbol{L}_0$ by $\boldsymbol{L}_0 = (L_0, M_0, N_0, W)$.

For a line segment $(\boldsymbol{q}, \boldsymbol{q}')$ colinear with $(\boldsymbol{p}, \boldsymbol{p}')$, the positions of $\boldsymbol{q}$ and $\boldsymbol{q}'$ can be expressed as $\boldsymbol{q} = \lambda\boldsymbol{p}+\mu\boldsymbol{p}'$ and $\boldsymbol{q}' = \rho\boldsymbol{p}+\nu\boldsymbol{p}'$, where $\lambda+\mu = \rho+\nu = 1$. Equations (5) and (6) reduce to

$$\boldsymbol{q}-\boldsymbol{q}' = (\lambda\rho-\mu\nu)(\boldsymbol{p}-\boldsymbol{p}') = (\lambda\rho-\mu\nu)\boldsymbol{L}\,,$$

$$\boldsymbol{q}\times\boldsymbol{q}' = (\lambda\rho-\mu\nu)(\boldsymbol{p}\times\boldsymbol{p}') = (\lambda\rho-\mu\nu)\boldsymbol{L}_0\,.$$

Thus colinear line segments have descriptions $(k\boldsymbol{L}, k\boldsymbol{L}_0)$, $k \neq 0$.

Suppose the descriptors are *normalised* according to the schemes: (a) for real line segments, set $1/k^2 = L^2+M^2+N^2$, and $W = 1$; and (b) for rational line segments, divide $\boldsymbol{L}$ and $\boldsymbol{L}_0$ by their greatest common denominator (gcd) by setting $1/k = \mathrm{gcd}(L, M, N, L_0, M_0, N_0)$, and $W = 1$. Further, the first nonzero entry of $\boldsymbol{L}$ is assumed positive by setting the sign of k the same as the sign of the first nonzero entry of $\boldsymbol{L}$. This, in effect, assigns an orientation to the line segment. Then, colinear line segments have the same description provided all lines belong to the same class (real or rational), and thus have a consistent method of description. Note that parallel lines share the same normalised direction descriptor $\boldsymbol{L}$ to within a scalar factor.

The definition of the line descriptor means that vectors $\boldsymbol{L}$ and $\boldsymbol{L}_0$ are perpendicular and their scalar product $\boldsymbol{L}\cdot\boldsymbol{L}_0$ is zero:

$$LL_0+MM_0+NN_0 = 0\,. \tag{7}$$

Conversely, a vector pair $(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{0}$, is a line descriptor if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$. This may be shown as follows. Consider vectors $\boldsymbol{x}$ satisfying $\boldsymbol{b} = \boldsymbol{x}\times\boldsymbol{a}$. These have the

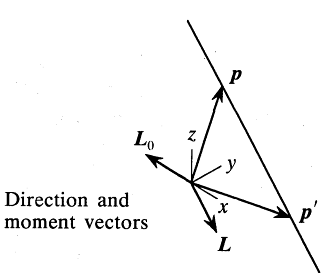


Figure 1. The direction and moment vectors of a line.

general vector form:

$$x = \lambda a + \frac{a \times b}{a \cdot a},$$

where λ is an arbitrary scalar. A line segment with endpoints x and x' of this form has descriptor (ka, kb) where k is a scalar. The descriptor (L, L_0) is also referred to as the *Plücker* (Brand, 1947) coordinates of the infinite line on which the line segment lies. Note that a point $p = (x, y, z, w)$ lies on the line (L, L_0) if and only if $L_0 = p \times L$.

Figure 2 illustrates examples of spatial relations between pairs of line segments and *distinguished* points that result from the relations. These points are functions of the relations between the lines. They remain unaltered by Euclidean and scale transformations of the shape in the sense that the image of a point of a given kind under a Euclidean transformation is another point of the same kind. For instance, figure 2(a) shows lines that intersect at points of intersection which are distinguished points; figure 2(b) shows a line, a designated point, and the perpendicular from this point to the line the foot of which is a distinguished point; and figure 2(c) illustrates skew lines and the feet of their common perpendicular which are distinguished points. In determining the transformations under which the subshape relation holds it is often necessary to construct distinguished points. In the next section we examine some of these distinguished points which are used subsequently in determining the transformations. The distinguished points considered all have the property that they are rational if constructed on rational shapes.

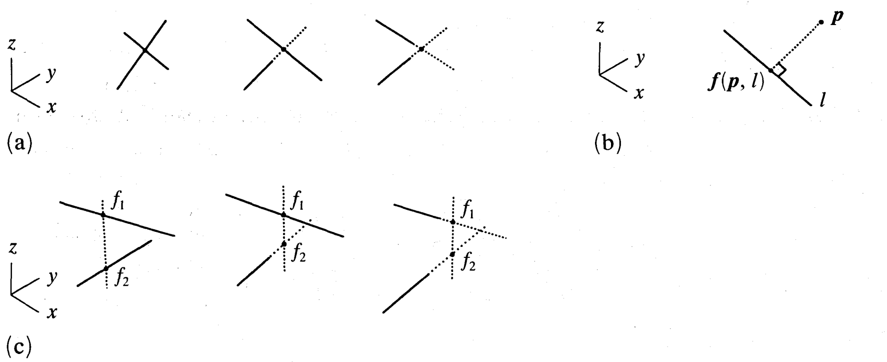


Figure 2. Spatial relations between pairs of lines and distinguished points: (a) point of intersection of two lines, (b) the foot of the perpendicular from a point to a line, (c) the feet of the common perpendicular between two skew lines.

Intersecting lines

The point of intersection of two noncolinear lines is the point that is coincident with both lines or their extensions. Parallel line segments may be considered to intersect on a plane at infinity (according to the homogeneous coordinate description of points).

Suppose $l(L, L_0)$ and $l'(L', L'_0)$ are two noncolinear line segments. The segments intersect if and only if they are coplanar, and they are coplanar if and only if the volume of the tetrahedron with the segments as opposite edges is zero. This geometric condition can be expressed in a number of ways. If the endpoints of the segments are (p, q) and (p', q'), respectively, then the condition for intersection is expressed by the vanishing determinant Δ, where

$$\Delta = \begin{vmatrix} x_p & y_p & z_p & w_p \\ x_q & y_q & z_q & w_q \\ x_{p'} & y_{p'} & z_{p'} & w_{p'} \\ x_{q'} & y_{q'} & z_{q'} & w_{q'} \end{vmatrix} . \tag{8}$$

The condition $\Delta = 0$ expresses the linear dependence or equivalently the coplanarity of the four given points. This condition can be expressed succinctly in vector form as:

$$\boldsymbol{L} \cdot \boldsymbol{L}_0' + \boldsymbol{L}' \cdot \boldsymbol{L}_0 = 0 . \tag{9}$$

The point of intersection $\boldsymbol{c}$, $\boldsymbol{c} = (x_c, y_c, z_c, w_c)$, of the two lines is the common solution of the two equations

$$\boldsymbol{c} \times \boldsymbol{L} = \boldsymbol{L}_0, \quad \boldsymbol{c} \times \boldsymbol{L}' = \boldsymbol{L}_0' .$$

The vector product of these two equations gives

$$\boldsymbol{c} = \frac{\boldsymbol{L}_0 \times \boldsymbol{L}_0'}{\boldsymbol{L}_0 \cdot \boldsymbol{L}'} = \frac{\boldsymbol{L}_0' \times \boldsymbol{L}_0}{\boldsymbol{L}_0' \cdot \boldsymbol{L}} . \tag{10}$$

In coordinates this corresponds to

$$\frac{x_c}{w_c} = \frac{M_0 N_0' - M_0' N_0}{L_0 L' + M_0 M' + N_0 N'}, \quad \frac{y_c}{w_c} = \frac{N_0 L_0' - N_0' L_0}{L_0 L' + M_0 M' + N_0 N'},$$
$$\frac{z_c}{w_c} = \frac{L_0 M_0' - L_0' M_0}{L_0 L' + M_0 M' + N_0 N'} . \tag{11}$$

Thus the intersection of two intersecting lines with rational descriptors is a rational point.

Perpendiculars to lines

Let $\boldsymbol{p}$, $\boldsymbol{p} = (x_p, y_p, z_p, w_p)$, be a point not coincident with line $l(\boldsymbol{L}, \boldsymbol{L}_0)$. Let $\boldsymbol{f}(\boldsymbol{p}, l)$ be the foot of the perpendicular from $\boldsymbol{p}$ to the line l, $\boldsymbol{f}(\boldsymbol{p}, l) = (x_f, y_f, z_f, w_f)$. Perpendicularity and the coincidence of $\boldsymbol{f}$ and $l(\boldsymbol{L}, \boldsymbol{L}_0)$ implies that

$$\boldsymbol{f} \times \boldsymbol{L} = \boldsymbol{L}_0, \quad (\boldsymbol{f} - \boldsymbol{p}) \cdot \boldsymbol{L} = 0 .$$

Taking the vector product of the first equation with $\boldsymbol{L}$ gives

$$\boldsymbol{f} = \frac{(\boldsymbol{p} \cdot \boldsymbol{L})\boldsymbol{L}}{\boldsymbol{L} \cdot \boldsymbol{L}} - \frac{\boldsymbol{L}_0 \times \boldsymbol{L}}{\boldsymbol{L} \cdot \boldsymbol{L}} . \tag{12}$$

This can be expressed in coordinate form. The x-coordinate is given by

$$\frac{x_f}{w_f} = \frac{(x_p L + y_p M + z_p N)L - (M_0 N - M N_0) w_p}{(L^2 + M^2 + N^2) w_p} . \tag{13}$$

The other coordinates are expressed similarly. Thus for line l and point $\boldsymbol{p}$ both rational, the foot of the perpendicular from $\boldsymbol{p}$ to l is a rational point.

Skew lines

The arguments above can be extended to apply to the case of two skew lines $l(\boldsymbol{L}, \boldsymbol{L}_0)$ and $l'(\boldsymbol{L}', \boldsymbol{L}_0')$. Skew lines are neither coplanar nor parallel. There is a unique common perpendicular which meets the line segments or their extensions at $\boldsymbol{f} = (x_f, y_f, z_f, w_f)$ and $\boldsymbol{f}' = (x_f', y_f', z_f', w_f')$, respectively. The feet of the common perpendiculars are given by

$$f = \frac{\boldsymbol{L}_0 \times (\boldsymbol{L} \times \boldsymbol{L}'_0 - \boldsymbol{L}' \times \boldsymbol{L}_0)}{\boldsymbol{L}_0 \cdot (\boldsymbol{L} \times \boldsymbol{L}')} , \quad f' = \frac{\boldsymbol{L}'_0 \times (\boldsymbol{L} \times \boldsymbol{L}'_0 - \boldsymbol{L}' \times \boldsymbol{L}_0)}{\boldsymbol{L}'_0 \cdot (\boldsymbol{L} \times \boldsymbol{L}')} , \tag{14}$$

and the descriptor of the common perpendicular line is $(\boldsymbol{L} \times \boldsymbol{L}', \boldsymbol{L} \times \boldsymbol{L}'_0 - \boldsymbol{L}' \times \boldsymbol{L}_0)$. The common perpendicular of two skew rational lines is a rational line.

Transformations

A transformation, τ, of a shape s is the shape $\tau(s)$. The transformation τ acts on the lines in s so that a line with endpoints p and q in s is transformed to a line with endpoints $\tau(p)$ and $\tau(q)$ in $\tau(s)$. The transformation τ: $p \rightarrow p'$ acting on general points $p = (x, y, z, w)$ to give points $p' = (x', y', z', w')$ can be represented by a matrix $\mathbf{T}$ of point coordinates:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \mathbf{T}(x, y, z, w)^{\mathrm{T}} . \tag{15}$$

Note that $E \neq 0$. A transformation is determined if the correspondence between two sets of four noncoplanar points is given. The treatment will be restricted to Euclidean transformations augmented by scale. The appropriate correspondences are thus the similarity transformations of the tetrahedra formed by the sets of points.

Consider the correspondence between two sets of noncoplanar points (p_1, p_2, p_3, p_4) and (p'_1, p'_2, p'_3, p'_4) in which p_i and p'_i correspond for $i = 1, 2, 3, 4$. The points are expressed in homogeneous coordinates of the forms (x_i, y_i, z_i, w_i) and (x'_i, y'_i, z'_i, w'_i). It is assumed that the two sets of noncoplanar points are similar tetrahedra under the correspondence. If the tetrahedra have symmetry, then there are further correspondences involving the two sets of points which give rise to distinct transformations related by the symmetry.

In order to determine the transformation it is necessary to find the coefficients of the matrix $\mathbf{T}$ which satisfy equation (15) for the points (x_i, y_i, z_i, w_i), (x'_i, y'_i, z'_i, w'_i), $i = 1, 2, 3, 4$. The matrix equation (15) provides three groups of four equations in four unknowns: $\{a_x, b_x, c_x, d_x\}$, $\{a_y, b_y, c_y, d_y\}$, or $\{a_z, b_z, c_z, d_z\}$. Each group can be solved provided the equations are nonsingular. This is guaranteed by the requirement that the points are noncoplanar.

However, we note that the solution is not quite as straightforward in the formulation proposed for the matrix representation. The points are represented in homogeneous coordinates which means that points (x, y, z, w) and (kx, ky, kz, kw) are the same. The matrix will transform these onto the same point. The difficulty arises because the four pairs of points in the correspondence will not in general be expressed in a way which allows direct use in a system of linear equations. Essentially, the value of E for each pair of points will be different. This is dealt with by using the scalar freedom in the homogeneous point representation. We notice that, if each of the matrix elements is multiplied by the same scalar factor, then the transformation remains unchanged. The freedom to express points and transformations in homogeneous coordinates can be incorporated in the equations derived from the matrix representation of the transformation as follows.

The matrix gives the following sets of equations

$$\left.\begin{aligned} a_x x_i + b_x y_i + c_x z_i + d_x w_i &= k_i x_i' \,, \\ a_y x_i + b_y y_i + c_y z_i + d_y w_i &= k_i y_i' \,, \\ a_z x_i + b_z y_i + c_z z_i + d_z w_i &= k_i z_i' \,, \\ E w_i &= k_i w_i' \,, \end{aligned}\right\} \tag{16}$$

for $i = 1, 2, 3, 4$. The value of E can be chosen as convenient, giving a value for k_i and thus three sets of four equations in four unknowns. Essentially what is happening here is that we are choosing a scale factor for all the matrix entries. This does not affect the transformation because homogeneous coordinates are being used.

Define the determinant

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{vmatrix} .$$

The equations (16) have a solution provided they are nonsingular. This is guaranteed by the condition $\Delta \neq 0$, which is equivalent to the condition that the four points are noncoplanar. Assign $E = \Delta$, which sets values for k_i, $i = 1, 2, 3, 4$. With this assignment, the solution of the equations are found by standard results to be

$$\left.\begin{aligned} a_x &= \Delta_{x'yzw}, & b_x &= \Delta_{xx'zw}, & c_x &= \Delta_{xyx'w}, & d_x &= \Delta_{xyzx'}, \\ a_y &= \Delta_{y'yzw}, & b_y &= \Delta_{xy'zw}, & c_y &= \Delta_{xyy'w}, & d_y &= \Delta_{xyzy'}, \\ a_z &= \Delta_{z'yzw}, & b_z &= \Delta_{xz'zw}, & c_z &= \Delta_{xyz'w}, & d_z &= \Delta_{xyzz'}. \end{aligned}\right\} \tag{17}$$

$\Delta_{x'yzw}$ is the determinant derived from Δ by replacing the first column by the corresponding x' entries. The other determinants are defined similarly. For example, the determinant $c_y = \Delta_{xyy'w}$ is the determinant derived from Δ by replacing the third column by y' entries.

For real shapes, the transformation matrix is most conveniently given by dividing each element by Δ and setting $E = 1$. For rational shapes, each coefficient of the transformation can be calculated by using integer arithmetic. The points are expressed as integer coordinates and thus determinants are calculated by integer arithmetic.

The transformation can be considered graphically in a number of ways. One of these is described below (figure 3). There are five steps in all.

(1) Translation τ_1 taking p_1 to the origin, that is, $\tau_1(p_1) = \boldsymbol{0}$.

(2) Translation τ_2 taking p_1' to the origin, that is, $\tau_2(p_1') = \boldsymbol{0}$.

(3) Rotation and scale τ_3 about the common normal of the lines $\tau_1(\{p_1, p_c\})$ and $\tau_2(\{p_1', p_c'\})$ such that the lines $\tau_3\tau_1(\{p_1, p_c\})$ and $\tau_2(\{p_1', p_c'\})$ coincide and the points $\tau_3\tau_1(p_c)$ and $\tau_2(p_c')$ coincide, where p_c and p_c' denote the centroids of the triangles formed by the points p_2, p_3, p_4 and p_2', p_3', p_4', respectively.

(4) Rotation and scale τ_4 about the line through $\tau_3\tau_1(p_c) = \tau_2(p_c')$ normal to the plane formed by $\tau_3\tau_1(p_2)$, $\tau_3\tau_1(p_3)$, and $\tau_3\tau_1(p_4)$ such that $\tau_4\tau_3\tau_1(p_2)$ and $\tau_2(p_2')$ coincide, $\tau_4\tau_3\tau_1(p_3)$ and $\tau_2(p_3')$ coincide, and $\tau_4\tau_3\tau_1(p_4)$, and $\tau_2(p_4')$ coincide.
[For similarity (isometric) transformations, these latter correspondences will always hold provided the two tetrahedra are initially similar (congruent).]

(5) Translation $(-\tau_2)$ taking the origin to p_1'.

The composition $\tau = (-\tau_2)\tau_4\tau_3\tau_1$ gives the desired transformation.

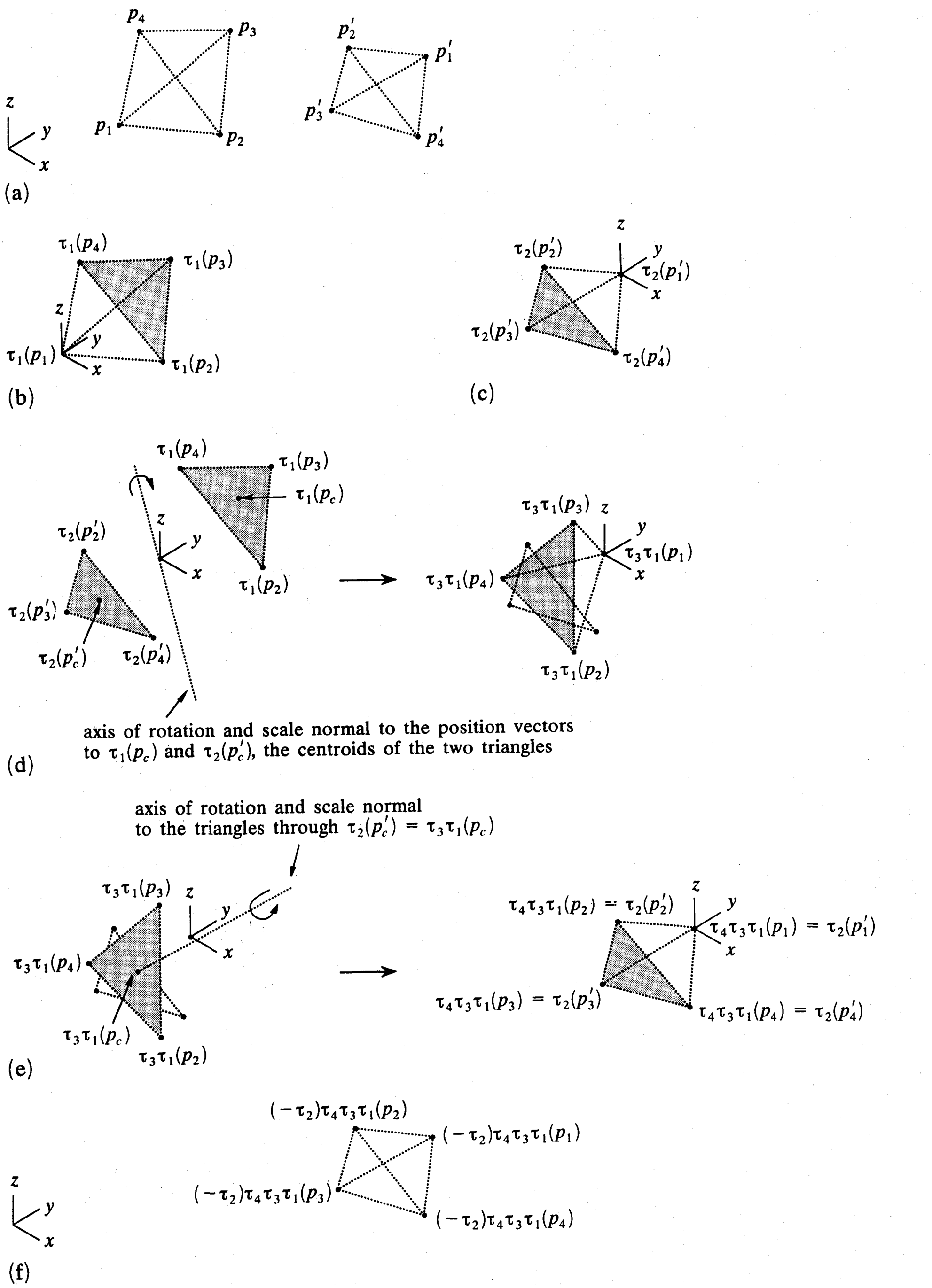


Figure 3. Graphical construction of the transformation between a pair of similar tetrahedra: (a) correspondence $p_i \rightarrow p_i'$ between similar tetrahedra; (b) step 1: translation τ_1, p_1 to the origin; (c) step 2: translation τ_2, p_1' to the origin; (d) step 3: rotation and scale τ_3 about the common normal to position vectors $\tau_1(p_c)$ and $\tau_2(p_c')$; (e) step 4: rotation and scale τ_4 about the common normal through $\tau_2(p_c')$; (f) step 5: translation $-\tau_2$ back to p_1'.

Subshape recognition under similarity transformations

When composing shapes by the recursive application of shape rules, it is imperative to know when two shapes are equal under a linear transformation. Of particular interest are the similarity transformations that are made up from finite compositions of the Euclidean transformations of *translation*, *rotation*, and *reflection*, augmented by a *scale* transformation. In the sequel, we restrict the discussion to the similarity transformations.

Under similarity, transformed shapes maintain the same spatial relation between the lines composing the shape. That is, angles and separation among lines are preserved.

When considering the problem of determining when a shape is a subshape of another, difficulties arise because there are general cases where there are an indeterminate number of valid transformations under which the subshape relation will be satisfied. This difficulty is compounded by the fact that endpoints of lines cannot be used as distinguished points because endpoints are not necessarily preserved in a subshape relation. Nevertheless, it is possible to demonstrate that, in such cases where there are an indeterminate number of valid transformations, one can isolate a 'base' transformation from which the other transformations can be generated.

Before this problem is addressed, the augmentation of shapes by labelled points is considered. In the generation of shapes, labelled points are used to guide both the choice of shape rules and the locations where they are applied. A labelled point is a point p with an associated label set A, and is denoted by $p:A$. The convention is adopted that no two distinct labelled points in a shape share the same coordinates. That is, if there are two labelled points $p:A_1$ and $p:A_2$ with $A_1 \neq A_2$, they may be replaced by a single maximally labelled point $p:A_1 \cup A_2$, where $\cup$ denotes set union. The label set A does not carry any geometrical import in the sense that under a linear transformation τ, the labelled point $p:A$ maps onto the labelled point $\tau(p):A$.

Labelled points are distinguished in the sense that the spatial relations of the lines and labelled points in a labelled shape are preserved under a similarity transformation. A labelled point $p:A$ is a subshape of $p':A'$ whenever $p = p'$ and $A \leqslant A'$.

The procedures to determine the set, T, of possible similarity transformations τ of a shape s augmented with a set of labelled points P, denoted by $\sigma = (s, P)$, such that $\tau(\sigma) = [\tau(s), \tau(P)]$ is a subshape of the second shape $\sigma' = (s', P')$ are enumerated in a series of cases which depend on certain conditions being satisfied in σ. Precisely, one of these cases must hold for σ. The subshape expression $\tau(\sigma) \leqslant \sigma'$ is used as a shorthand for the pair of subshape expressions $\tau(s) \leqslant s'$ and $\tau(P) \leqslant P'$.

To determine whether or not the subshape relation between σ and σ' holds, it is more than sufficient to examine the mapping between corresponding sets of four noncolinear distinguished points in the two shapes. We start by selecting labelled points as candidates for distinguished points. If there are an insufficient number of these, other points are selected as possible candidates. If this is still not possible, we have a situation where there are potentially indeterminately many ways of satisfying the subshape relation between σ and σ'.

In the next two sections we exhaustively enumerate the determinate and indeterminate cases, respectively. The cases are enumerated in decreasing order of the number of labelled points in σ. That is, case 1 prefixes the case when there are four labelled points, case 2 corresponds to cases when there are three labelled points and so on. The case numbering is preserved over the two sections.

In some of the cases, we separate the discussion into shapes when they are rational and into shapes when they are real. In all other cases, the constructions apply equally both to rational and to real shapes.

The determinate cases

Case 1: There are at least four noncoplanar labelled points in P. Let the labelled points be $p_i: A_i, p_i = (x_i, y_i, z_i, w_i)$, $1 \leqslant i \leqslant 4$. The four points are noncoplanar if and only if $\Delta \neq 0$ (as defined in the previous section). For each set of labelled points $p_i' : A_i'$ in P', satisfying $A_i \leqslant A_i'$ and with $\{p_i\}$ similar to $\{p_i'\}$, $1 \leqslant i \leqslant 4$, a transformation τ can be generated by the procedure described in the previous section. If $\tau(\sigma) \leqslant \sigma'$ is satisfied, then τ is added to the set of valid transformations. Distinct correspondences of labelled points generate distinct transformations. Distinct correspondences on the same sets of points are possible if the labelled tetrahedra possess symmetries.

If all possible correspondences are examined, then the procedure is exhaustive.

If the labelled points in both sets are rational, the valid transformations will have rational coefficients.

Case 2: There are three labelled points in P and they are not all colinear. Let the three labelled points be $p_i: A_i, p_i = (x_i, y_i, z_i, w_i)$, $i = 1, 2, 3$. If there is a line which is not in the plane of the points and not perpendicular to the plane, then it is possible to construct distinguished points as feet of the perpendiculars from p_i to the line which is not in the plane of the points. The procedure in case 1 is then used.

If there is no such line, then define the points $c^+(p_1, p_2, p_3)$ and $c^-(p_1, p_2, p_3)$ as illustrated in figure 4 and given by the expression,

$$c^{\pm}(p_1, p_2, p_3) = c(p_1, p_2, p_3) \pm k[(\boldsymbol{p}_1 - \boldsymbol{p}_2) \times (\boldsymbol{p}_1 - \boldsymbol{p}_3)],$$

where

$$c(p_1, p_2, p_3) = \tfrac{1}{3}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)$$

is the centroid of the points p_1, p_2, and p_3, and k is a scale factor equal to the inverse of the magnitude of $(\boldsymbol{p}_1 - \boldsymbol{p}_2)$. The vector product $(\boldsymbol{p}_1 - \boldsymbol{p}_2) \times (\boldsymbol{p}_1 - \boldsymbol{p}_3)$ represents a line normal to the plane of the three labelled points. Fix $p_4 = c^+(p_1, p_2, p_3)$.

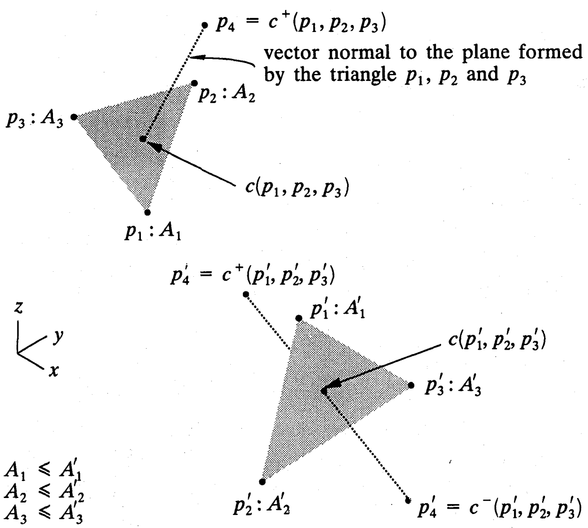


Figure 4. Constructing possible correspondences from similar triangles.

For each set of three labelled points $p_i' : A_i'$, $i = 1, 2, 3$, in P', satisfying $A_i \leqslant A_i'$, and with triangles $\{p_i\}$ and $\{p_i'\}$ similar under the correspondence p_i to p_i', $i = 1, 2, 3$, construct the points $c^+(p_1', p_2', p_3')$ and $c^-(p_1', p_2', p_3')$. Potential transformations τ can be generated by setting p_4' to each of these points in turn. The construction of the fourth point guarantees that the tetrahedra are similar if the original triangles are similar. If $\tau(\sigma) \leqslant \sigma'$ is satisfied, then τ is added to the set of valid transformations. Distinct correspondences of labelled points generate distinct transformations. Distinct correspondences on the same sets of points are possible if the labelled triangles possess symmetries.

If all possible correspondences are examined, then the procedure is exhaustive.

The above argument demonstrates that the correspondence between two sets of three noncolinear distinguished points is sufficient to generate possible transformations.

This above procedure will not always succeed for rational shapes in the sense that the constructed points may not be rational and thus the transformation may not be rational. However, if the constructed points in σ and σ' are both irrational, then the transformation may be rational. This can only occur if the values of k and k' used to construct $c^{\pm}(p_1, p_2, p_3)$ and $c^{\pm}(p_1', p_2', p_3')$, respectively, have k/k' rational. Clever choice of the constructed points is irrelevant and one construction is as good as another for the purpose of determining the transformation. If only one of the constructed points in σ or σ' is rational, then the transformation is irrational as the following examples indicate.

Examples

(1) Let $p_1 = (0, 0, 0)$, $p_2 = (0, 2, 0)$, and $p_3 = (2, 0, 0)$ be the (x, y, z)-coordinates of the three labelled points. Let $p_f = f[p_1, (p_2, p_3)] = (1, 1, 0)$. Two possible choices for p_4 based on p_f are $(1, 1, \pm 2^{1/2})$. That is, p_4 is irrational. Note that p_4 is not constructed according to the above procedure but is given by an equivalent construction corresponding to a right triangle $\{p_1, p_f, p_4\}$ orthogonal to the triangle $\{p_1, p_2, p_3\}$. Let $p_1' = (0, 0, 0)$, $p_2' = (-1, 1, 0)$, and $p_3' = (1, 1, 0)$ be the corresponding labelled points. Let $p_f' = f[p_1', (p_2', p_3')] = (0, 1, 0)$. The corresponding choices for p_4' are $(0, 1, \pm 1)$. That is, p_4' is rational. The similarity transformation that maps between the two tetrahedra has a scale factor of $2^{-1/2}$ and, hence, must have irrational coefficients.

(2) Let p_1, p_2, and p_3 be the same points as above. Let p_1', p_2' and p_3' be their mirror images in the yz-plane. Then, p_4 and p_4' are not rational but the similarity transformation between the two tetrahedra has rational coefficients.

(3) Let $p_1 = (0, 0, 0)$, $p_2 = (0, 1, 0)$, and $p_3 = (2, 0, 0)$. Then, a possible choice for p_4 is $(2/5, 4/5, 1/5^{1/2})$ which is clearly not rational. Let $p_1' = (0, 0, 0)$, $p_2' = (1, -1, 0)$, and $p_3' = (-2, -2, 0)$. The corresponding choices for p_4' are $(2/5, -6/5, \pm 2^{1/2}/5^{1/2})$ which are clearly not rational. The two tetrahedra are similar with a scale factor of $2^{1/2}$.

Thus, it is not always possible to find valid transformations with rational coefficients that map a rational shape as a subshape of another rational shape.

Case 3: There are two labelled points in P. Let the two labelled points be $p_1 : A_1$ and $p_2 : A_2$. There are a number of subcases which are conveniently enumerated by considering the lines in the shape σ if they are present. The maximum number of lines considered is limited to one. In general, it may be possible to use a number of lines in σ for the construction of distinguished points, thus allowing the application of cases already considered. However, this is not always a sensible strategy to be considered in checking subshape because the number of possible constructions in σ' increases exponentially with the number of extra lines. As will be demonstrated below, the use of a single extra line, if it exists, is sufficient to determine the possible transformations. This guideline is augmented by the use of distinguished points if they are available within the configuration selected in σ. This is relevant to case 3 because it is possible to specify a general construction for two labelled points and a noncoincident line, but determining which of the resulting transformations are valid requires checking the subshape relation in all cases. However, with the use of distinguished points, an initial similarity check can be done before transformation generation. This reduces the operations required in subshape recognition. The result of these observations is that the minimum of extra lines are used in constructing the distinguished points, and all the distinguished points available are used. In all there are three determinate subcases to be considered.

Subcase 3(a): There is a line l not coplanar and not perpendicular to (p_1, p_2) The condition on the line descriptor and the point for this condition is straightforward. The feet of the perpendiculars from p_1 and p_2 to the line (figure 5) are distinct and can be used as two additional points to construct possible transformations as in the procedure for case 1. All possible constructions of the additional points must be made in σ' to enumerate the transformations exhaustively. However, repetitions of valid transformations will be generated if there are two or more colinear lines in σ' used in the construction of the additional points. This is avoided by using only one line from each colinear class of lines in σ' during the construction.

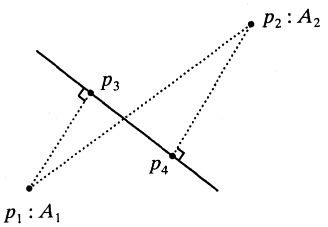


Figure 5. Constructing distinguished points on a line not coplanar and not perpendicular to the line specified by two labelled points.

Subcase 3(b): There is a line l: (1) perpendicular and not coplanar to (p_1, p_2), or (2) coplanar, not perpendicular, and not coincident with (p_1, p_2). In this subcase the foot of the perpendicular from p_1 or p_2 defines a third point p_3 noncolinear with p_1 and p_2 (figure 6). Using this additional point the procedure for constructing the transformations is as case 2 above. All possible constructions of the additional points must be made in σ' based upon pairs of labelled points (p_1', p_2'), $p_i' : A_i'$, $A_i \leqslant A_i'$. Associated correspondences are then used as the basis for generating possible transformations. Further, the constructions in σ' should follow the choice of p_1 or p_2 in the construction of the third point in σ.

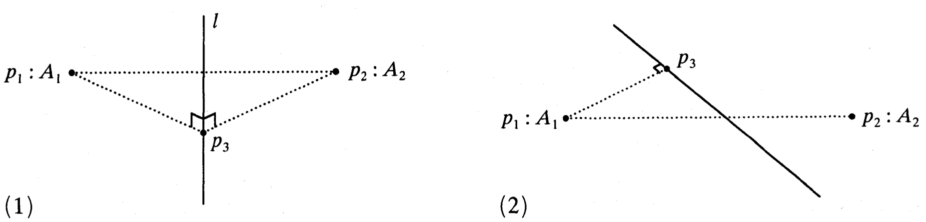


Figure 6. Constructing a distinguished point on a line which is exclusively (1) perpendicular, or (2) coplanar with (and distinct from) the line specified by two labelled points.

Subcase 3(c): There is a line l coplanar and perpendicular to (p_1, p_2). There are several ways of considering this case (figure 7). One of these will be examined. Let the point of intersection of l and (p_1, p_2) be c. Construct points $a^{\pm}$ on l at a distance from c equal to the length of (p_1, p_2), where a^+ lies along the positive direction of l as determined by the descriptor. Further, construct the line m through c and perpendicular to l and (p_1, p_2) and points $b^{\pm}$ on m at a distance from c equal to the length of (p_1, p_2), where b^+ lies along the positive direction of m as determined by its descriptor. Assign $p_3 = a^+$ and $p_4 = b^+$.

For each set of two labelled points $p_i' : A_i'$, $i = 1, 2$, in P', satisfying $A_i \leqslant A_i'$ and lines l' in s' perpendicular and coplanar with (p_1', p_2') such that the intersection point c' on l' and (p_1', p_2') divides (p_1', p_2') in the same ratio as c divides (p_1, p_2), construct points $a'^{\pm}$ and $b'^{\pm}$. For each pairwise selection of the constructed points, assigned to p_3', p_4', generate the transformation for the associated correspondence between $\{p_1, p_2, p_3, p_4\}$ and $\{p_1', p_2', p_3', p_4'\}$. Transformations with $\tau(\sigma) \leqslant \sigma'$ represent valid transformations. Add these to the list of transformations. All possible constructions of the additional points must be made for the exhaustive enumeration of potential transformations. The procedure can generate repetitions of the transformations if there are two or more colinear lines in σ' which are used in the generation. To avoid this only one line from each colinear class of lines in σ' is considered in the construction of the additional points.

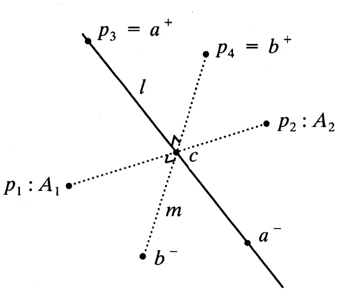


Figure 7. Distinguished points from a configuration consisting of a line which is coplanar and perpendicular to the line specified by two labelled points.

Case 4: There is one labelled point in P.
Subcase 4(a): There is a line l in s not coincident with the labelled point p_1. Let $p_2 = f(p_1, l)$ be the foot of the perpendicular from p_1 to l (figure 8). The case is similar to case 3(c) with all lines in σ' not coincident with a corresponding labelled point p_1', satisfying $A_1 \leqslant A_1'$. Note that the constructed transformations may produce transformations that do not have rational coefficients.

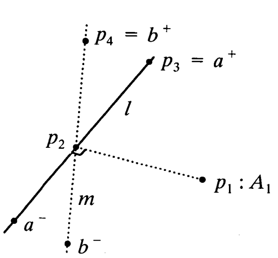


Figure 8. Distinguished points constructed from a line not coincident with a given labelled point.

We illustrate by a simple example a situation where rational shapes are mapped onto irrational shapes. Figure 9(a) illustrates a shape rule with the left-hand shape consisting of single rational line and a single rational labelled point. Figure 9(b) is a subshape of the 'current' shape to which it applies. The appropriate transformation that maps the left-hand shape of the rule to the current shape has irrational coefficients and when applied to the right-hand side of the rule will produce an irrational shape.

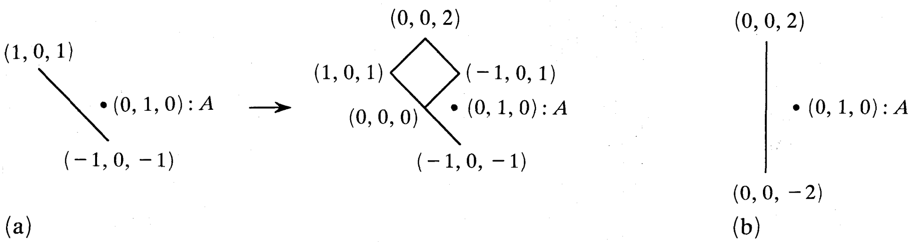


Figure 9. A simple example to show that a rational shape rule (a) applied to a rational shape (b) may produce irrational shapes.

Case 5: There are no labelled points in P. That is, $P = \varnothing$. There are two determinate cases depending on the configuration of lines in σ.

Case 5(a): There are two skew lines in s. Skew lines are not parallel by definition. Let the lines be l_1 and l_2. Construct their common perpendicular and denote the corresponding intersections with l_1 and l_2 by p_1 and p_2 (figure 10). This case may be considered similarly to case 3(c) using just one of those lines. However, this ignores some of the geometrical information in the shape. A variant of the treatment is considered. Construct points $a_i^{\pm}$, $i = 1, 2$ on l_1 at a distance from p_i equal to the length of (p_1, p_2), where a_i^+ lies along the positive direction of l_i as determined by the descriptor. Assign $p_3 = a_1^+$ and $p_4 = a_2^+$.

For each pair of skew lines l_1' and l_2' in σ' construct the common perpendicular and denote the feet by p_1' and p_2'. Further construct the points $a_1'^{\pm}$ and $a_2'^{\pm}$. For each pairwise selection from $\{a_1'^{\pm}, a_2'^{\pm}\}$ assigned to p_3', p_4' generate the transformation from the associated correspondence between $\{p_1, p_2, p_3, p_4\}$ and $\{p_1', p_2', p_3', p_4'\}$. A preliminary check is made to ensure similarity of the correspondence. This effectively checks that the skew lines have equal twist angle. All possible constructions of the additional points must be made for the exhaustive enumeration of potential transformations. Colinear lines can give rise to repetitions of the constructions and the corresponding transformations. As with previous cases only one line from each colinear class in σ' is considered.

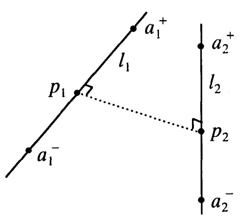


Figure 10. Distinguished points from two skew lines.

Case 5(b): There are three coplanar lines in s not all parallel and not all concurrent at a common point. The situation considered here is that of three lines forming the sides of a triangle [figure 11(a)] or of parallel lines with transversal [figure 11(b)]. These two situations are similar to cases 2 and 3(b), respectively. However, if the transversal is perpendicular to the parallel lines [figure 11(c)], then a situation similar to case 3(c) arises.

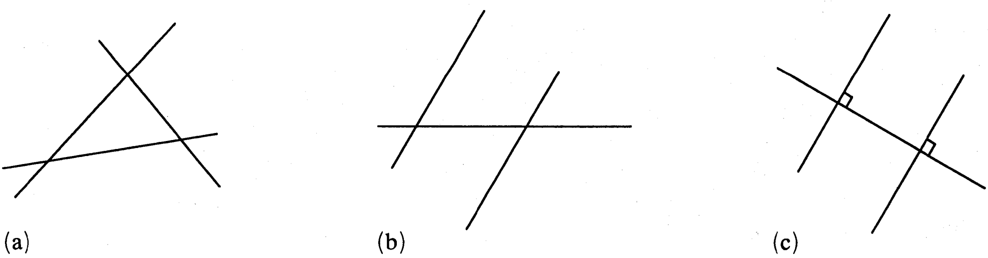


Figure 11. The determinate cases when there are no labelled points and three coplanar lines in σ.

The indeterminate cases

The cases in which there are an indeterminate number of transformations are now considered. In these cases the number of valid transformations effecting subshape relation may not be finite, but the nature of the indeterminacy is specified. The first instance arises in case 3.

Case 3: There are two labelled points in P.

Subcase 3(d): All labelled points in P and all lines in s are colinear. This case subsumes the case when s is empty. For each set of two labelled points $p_i' : A_i'$, $i = 1, 2$ in P', satisfying $A_i \leqslant A_i'$, one possible transformation is generated by a combination of translations, rotations, and scale taking (p_1, p_2) to (p_1', p_2'). The particular transformation is not critical. The complete set of transformations is generated by composing this with rotations about the axis (p_1', p_2') and with a reflection in a plane containing p_1' and p_2'. The first transformation may be considered as a base transformation for the complete set.

The base transformation can be constructed as follows (figure 12). At p_1 construct the lines l_1 and l_2, where l_1 has the descriptor $(\boldsymbol{L}_1, \boldsymbol{L}_{01})$, $\boldsymbol{L}_1 = \boldsymbol{p}_1 \times \boldsymbol{p}_2$ and $\boldsymbol{L}_{01} = \boldsymbol{p}_1 \times (\boldsymbol{p}_1 \times \boldsymbol{p}_2)$, and l_2 has the descriptor $(\boldsymbol{L}_2, \boldsymbol{L}_{02})$, $\boldsymbol{L}_2 = (\boldsymbol{p}_1 - \boldsymbol{p}_2) \times \boldsymbol{p}_1 \times \boldsymbol{p}_2$ and $\boldsymbol{L}_{02} = \boldsymbol{p}_1 \times [(\boldsymbol{p}_1 - \boldsymbol{p}_2) \times \boldsymbol{p}_1 \times \boldsymbol{p}_2]$. The expressions for $\boldsymbol{L}_{01}$ and $\boldsymbol{L}_{02}$ are not used. Construct the points $a_i^{\pm}$ on lines l_i, $i = 1, 2$ at a distance from p_1 equal to the length of (p_1, p_2). Select $p_3 = a_1^+$ and $p_4 = a_2^+$. For each pairwise selection $\{a_1'^{\pm}, a_2'^{\pm}\}$

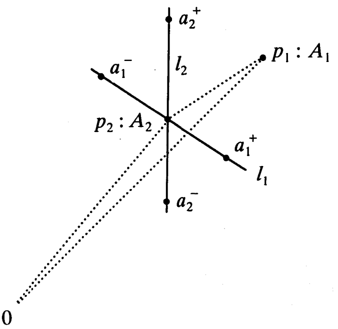
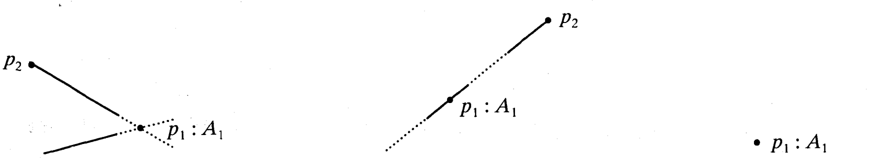


Figure 12. The distinguished points generated from two labelled points.

assigned to p_3', p_4' generate the transformation from the associated correspondence between $\{p_1, p_2, p_3, p_4\}$ and $\{p_1', p_2', p_3', p_4'\}$. These base transformations will incorporate possible reflections. We can construct the complete set of transformations by taking the composition of these base transformations with rotations about the axis (p_1', p_2'). Note that there is a degree of redundancy in this specification, arising from the generation of four base transformations three of which incorporate reflections. Only one of these three is required when composing with the general rotation.

Case 4: There is exactly one labelled point, $p_l : A_l$, in P. There are three indeterminate situations. These are illustrated in figure 13.



subcase 4(b) subcase 4(c) subcase 4(d)

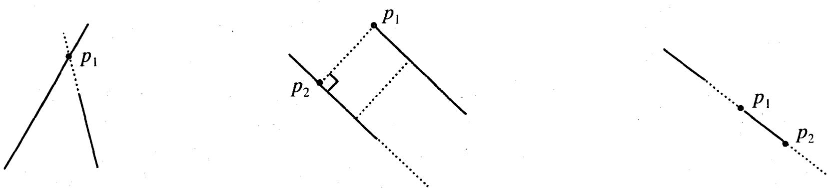
Figure 13. The three indeterminate subshape conditions with one labelled point in σ.

Subcase 4(b): p_l is coincident with all lines in s and there are two noncolinear lines in s. Let p_2 be an endpoint of a line in s distinct from p_1. For each labelled point $p_1' : A_1'$ in P', satisfying $A_1 \leqslant A_1'$, consider each endpoint $p_2' \neq p_1'$ of lines through p_1'. For each pair (p_1', p_2') and each line through p_1' but not through p_2', generate the transformation as in cases 3(b) or 3(c) above. The full set of transformations is obtained by composing this transformation with scales keeping p_1' as the fixed centre of the scale.

Subcase 4(c): p_l is coincident with all lines in s and all lines are colinear. Let p_2 be an endpoint of a line in s. For each labelled points $p_1' : A_1'$ in P', satisfying $A_1 \leqslant A_1'$, consider each endpoint $p_2' \neq p_1'$ of lines through p_1'. For each pair (p_1', p_2') generate the base transformation as in case 3(d) above. The full set of transformations is obtained by composing these transformations with rotations about (p_1', p_2') and with scale having p_1' as the fixed centre of the scale.

Subcase 4(d): There are no lines in s. For each labelled point $p_1' : A_1'$ in P', satisfying $A_1 \leqslant A_1'$, construct the translation that takes p_1 into p_1'. The full set of transformations consists of this base transformation composed with rotations and scales having p_1' as a fixed point and a reflection in a plane through p_1'.

Case 5: There are no labelled points in P. There are three possible indeterminate cases as shown in figure 14.



subcase 5(c) subcase 5(d) subcase 5(e)

Figure 14. The three indeterminate subshape conditions with no labelled points in σ.

Subcase 5(c): All lines in s are coincident at a common point and are not all colinear. Let p_1 be the common point and construct the transformations as in case 4(b).

Subcase 5(d): All lines in s are parallel and are not all colinear. Choose two lines l_1 and l_2. Let p_1 be an endpoint of l_1, and p_2 the foot of the perpendicular from p_1 to l_2. For each pair of parallel lines (l_1', l_2') in s' construct in a similar way the two pairs of points (p_1', p_2') corresponding to the choice of endpoint of l_1'. The base transformations are then generated as for case 3(c). The full set of transformations is generated by composing these with translations along the direction of the lines l_1' and l_2' together with a reflection in a plane perpendicular to the parallel lines. This case may have either a determinate or an indeterminate number of valid transformations.

Subcase 5(e): All lines in s are colinear. Select endpoints $\{p_1, p_2\}$ of a line in s. Find all pairs of endpoints $\{p_1', p_2'\}$ of all lines in s' and generate transformations for each pair (p_1', p_2') as in the generation of the base transformations in case 3(d). The full set of possible transformations is found by composing the base set with rotations about (p_1', p_2'), appropriate scale reductions, translations along (p_1', p_2'), and reflections in a plane containing p_1' and p_2' and perpendicular to (p_1', p_2').

This completes the enumeration of the determinate and indeterminate cases.

The symmetries of a shape

We have considered the problem of, given two shapes, determining all possible similarity transformations of one shape such that the transformed shape is a subshape of the second shape. In this section we consider the related problem of determining the possible similarity transformations which render two unlabelled shapes equal. Procedures will now be given to determine all possible similarity transformations which maintain shape equality. A corollary of these procedures is that, if we are given two equal shapes, then the possible similarity transformations that take a shape into itself (other than the identity transformation) may be determined. That is, each transformation defines a symmetry of the shape. The symmetries of a given shape may thus be determined by these transformations.

In general, the set of transformations that preserve equality of shape is finite. There are three exceptional cases to consider: namely, when two unlabelled shapes each consist of single sets of finite colinear lines, and when the shape consists of no more than two labelled points and has no lines. In all cases, the shapes are always equal and there are an indeterminate number of potential transformations that yield equality. For the unlabelled shapes with single sets of finite colinear lines and the labelled shapes with two labelled points and no lines, one particular transformation can be singled out, from which the other transformations can be derived. This is the screw transformation along the common perpendicular with a translation along the direction of one of the lines—the two labelled points may be treated as a line—and a scale transformation. Further, note that the heads and tails are preserved.

To determine the potential transformation for shape equality four noncoplanar distinguished points in each shape are mapped onto one another. The coordinates of these points are sufficient to calculate the coefficients of a linear transformation. For rational points that map onto rational points, the coefficients are also rational. If distinguished points of the shapes are arbitrarily chosen, say the endpoints of line segments, or their points of intersection or the feet of the common perpendicular of two skew lines, then many potential transformations may have to be considered. Thus, labelled points are used whenever possible.

In situations where there are insufficient labelled points, other suitable distinguished points are used. If there are still insufficient distinguished points, then transformations of point and line descriptors are invoked by constructing additional points. The only case where this is not possible are the three cases mentioned above.

The procedure for determining the symmetries of a shape is summarised as follows. In principle, four distinguished points in a shape are kept fixed. Then, each possible transformation corresponds to a permutation of four distinguished points in the shape such that corresponding points are identical in type or label. That is, a labelled point in the shape must map onto another labelled point with the same label set in the shape; an endpoint of a line must map onto an endpoint of another line in the shape; and so on. Moreover, the tetrahedra formed by the two sets of corresponding points must be congruent. Once a correspondence between distinguished points in the shape has been constructed, the shape equality procedure can be applied to compare the shape and its transformation.

Conclusion

We have shown the existence of procedures for solving the recognition problem for three-dimensional line shapes. We have also shown that valid transformations that map between rational shapes may not have rational coefficients; consequently, it is not always possible to resolve the shape recognition problem using exact arithmetic. This would inexorably lead to round-off and other numerical errors that would have to be taken care of in practice. However, if computations for shape recognition are restricted to integers, we would then have to take greater care in the way in which we specify shape rules. Principally, this means that the shape rules have to be sufficiently dense to ensure that there are always four noncoplanar labelled or other distinguished points(1).

Nonetheless, we believe, we have established that a grammatical approach to the construction of form-led three-dimensional spatial designs is a viable proposition. In doing so, we have accomplished one half of a two-part objective, laid out and argued for by Earl (1986), namely, to develop a constructive computational paradigm for the creation of worlds of designs. To fulfill this ambition and to provide a practical computational framework we would need to implement the theory described here, taking into consideration the numerical problems that are introduced by arithmetic on reals, or by the constraints that would have to be imposed on the specification of shape rules as a result of limiting arithmetic to integers. That, however, remains the object of another endeavour, the actions of another incarnation, and the subject of another report.

Acknowledgement. We would like to thank George Stiny for his critical comments and for suggestions on improving the presentation of the material in this paper. We would like to thank Rudi Stouffs for verifying the equations and their solutions in *Mathematica*™ (Wolfram, 1988).

(1) There are other issues that surface when we restrict shapes to rational descriptors. More often than not, we are interested in shapes that have rational or integral dimensions rather than have rational endpoints. Rational shapes raise issues that are more of a theoretical nature, whereas real shapes raise practical issues that demand accuracy of numerical computation. One permits exact subshape testing and hence, accurate shape recognition; the other greater flexibility in shape representation, for example, the unit cube rotated through 45° about the z-axis inevitably has nonrational descriptors. A fuller discussion of the rational–real divide requires greater depth and is beyond the scope of this paper.

References

Brand L, 1947 *Vector and Tensor Analysis* (John Wiley, New York)
Chase S C, 1989, "Shapes and shape grammars: from mathematical model to computer implementation" *Environment and Planning B: Planning and Design* **16** 215-242
Earl C F, 1986, "Creating design worlds" *Environment and Planning B: Planning and Design* **13** 177-188
Earl C F, Krishnamurti R, 1984, "Spatial relations, kinematics and assembly", in *Proceedings of the International Symposium on Design and Synthesis* (Japan Society of Precision Engineering, Tokyo) pp 589-593
Krishnamurti R, 1980, "The arithmetic of shapes" *Environment and Planning B* **7** 463-484
Krishnamurti R, 1981, "The construction of shapes" *Environment and Planning B* **8** 5-40
Krishnamurti R, 1982, "SGI: a shape grammar interpreter", technical report, Design Discipline, The Open University, Milton Keynes
Krishnamurti R, Giraud C, 1986, "Towards a shape editor: the implementation of a shape generation system" *Environment and Planning B: Planning and Design* **13** 391-404
Stiny G, 1980a, "Introduction to shape and shape grammars" *Environment and Planning B: Planning and Design* **7** 343-351
Stiny G, 1980b, "Kindergarten grammars: designing with Froebel's building gifts" *Environment and Planning B: Planning and Design* **7** 409-462
Stiny G, 1990, "What is a design?" *Environment and Planning B: Planning and Design* **17** 97-103
Stiny G, forthcoming *Shape: A Primer in Algebra, Grammar, and Description* manuscript, Graduate School of Urban Planning and Architecture, University of California at Los Angeles, CA
Wolfram S, 1988 *Mathematica™: A System for Doing Mathematics on Computer* (Addison-Wesley, Reading, MA)