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# The counting of rectangular dissections

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**Abstract.** In this paper results are presented from two independently developed computer programs—algorithms RK and CB—on counting and classifying rectangular dissections. A population census is given for all weights less than eleven. In spite of the radically different approaches adopted by the two algorithms in solving this enumeration problem, both sets of results agree completely.

## Definitions

*Rectangular gratings*, defined in a mathematical sense by Newman (1964) and first referred to in the design literature by March (1972), form the basis of the representation of rectangular dissections which is discussed in this paper. In the present context they have been elaborated upon by Mitchell et al (1976) and by Bloch (1976).

An $(l, m)$ rectangular grating is formed by drawing $l-1$ straight line segments parallel to one side of a rectangle and $m-1$ such segments parallel to an adjacent side. The lines go right across the rectangle in such a manner that segmentation divides it into edge-connected rectangular cells, or *two-cells*, arranged in $l$ rows and $m$ columns. Two rectangular gratings will be defined to be equivalent if they have the same number of rows and columns. This is an equivalence relation, and the $(l, m)$ equivalence class, for example, consists of all rectangular gratings with $l$ rows and $m$ columns, irrespective of the different spacings of the line segments. Select as representative of each class the grating in which the line segments are equally spaced: such a representative will be called a *unit grid.* The two-cells of a unit grid are squares.

Suppose a rectangular dissection (containing exactly $p$ rectangles, say) is superimposed on the $(l, m)$ unit grid; then it is said to be $(l, m)$ and *standard* if each line of the grid contains at least one edge of a constituent rectangle, and every edge of a rectangle lies on some grid line. The number of rectangular elements, $p$, is defined here to be the *weight* of the dissection, although elsewhere the terms 'content' and 'order' have also been used for this purpose. If it is assumed, without loss of generality, that $l \leqslant m$, then, for a given $p$, it is known (Bloch, 1976) that the set of unit-grid dimensions $(l, m)$ to which there correspond $(l, m)$ standard rectangular dissections is

$$\left\{(l, m):\ 1 \leqslant l \leqslant \left\lceil \frac{p}{2} \right\rceil,\ \max\left(\left\lceil \frac{p}{l} \right\rceil, l\right) \leqslant m \leqslant p+1-l\right\},$$

where for any real number $s$, $\lceil s \rceil$ denotes the least integer greater than or equal to $s$.

If $l = m$, the symmetry group that leaves the unit grid invariant is clearly $D_4$, the dihedral group of order eight; whereas, if $l < m$, $K_4$, the Klein group of order four, is clearly the group that leaves the unit grid invariant. The symmetry of rectangular dissections is taken into account when enumerating them: that is, an equivalence relation is defined under symmetry.

Arrangements of rectangles have been discussed elsewhere: see, for example, Brooks et al (1940), Bouwkamp (1947), Bouwkamp et al (1960), Tutte (1966), Biggs (1969), and Flemming (1977).

## Characteristics

*Valency*

*Grid points* are the points of intersection of the grid lines. The *internal points* of a dissection are the points of intersection of at least two distinct edges of the rectangular elements, incident at some grid point. The *valency* of an internal point is the number of rectangular elements coincident at that point. For dissections we need only consider 3-way and 4-way points (Biggs, 1969; Combes, 1976). A dissection is *trivalent* if it contains no 4-way points. Since any 4-way point can be replaced by two 3-way points (Biggs, 1969), trivalent dissections may be regarded as being in a sense more fundamental.

*Alignment*

Suppose we are given a dissection in which there are two distinct line segments which are collinear (that is, collinear with the same grid line). These line segments are said to be *aligned* (Earl, 1977). Those dissections that do not contain at least one pair of aligned line segments are referred to as *nonaligned* dissections. It can be shown (Earl, 1978) that the set of nonaligned trivalent dissections of weight $p$ consists of all those on $(l, m)$ unit grids for which $l+m-1 = p$. In keeping with the spirit of the paper by March and Earl (1977), the nonaligned trivalent dissections are defined here to be *fundamental dissections.*

*Grid partition*

Each dissection of weight $p$ on an $(l, m)$ unit grid may be mapped to a partition of the integer $lm$ into $p$ parts, since the $lm$ two-cells are to be distributed into $p$ rectangles (Bloch, 1978). A partition of $lm$ into $p$ parts $q_1, q_2, ..., q_p$ for which at least one standard dissection exists is a *grid partition* of the $(l, m)$ unit grid. (The $q_i$ may be written so that $q_{i+1} \leqslant q_i$, for all $i$, whence

$$lm = q_1 + q_2 + ... + q_p ,$$

where $q_{i+1} \leqslant q_i$ and $1 \leqslant q_i \leqslant lm-p+1$.) Each $q_i$ represents the number of two-cells comprising a rectangular element, or *tile*, and the dimensions of this tile, $(r_{i1}, r_{i2})$, constitute a factorisation of $q_i$. In general there are several possible tiles for each $q_i$, so that a given grid partition may be decomposed into a number of distinct factored representations; not all, but at least one, will be realised as a standard dissection. An equivalence relation can be defined on the set of $(l, m)$ rectangular dissections of weight $p$ by taking two dissections to be equivalent if their grid-partition factored representations are identical.

## Algorithms

Now we briefly outline algorithms RK and CB, both of which draw on concepts defined in the previous section. Algorithm RK makes use of the valency and alignment properties in the formulation of shape rules, whereas algorithm CB utilises grid partition and factored representation as a means of problem decomposition. Both enumerate dissections of a given weight on a given unit grid. Both were implemented in ALGOL68C and run on the Cambridge Computer Laboratory's IBM 370/165.

First, however, we mention a previous attempt at enumerating dissections, namely that of Mitchell et al (1976). Their approach was to generate the dissections of weight $p$ from those of weight $p-1$, making no use of gratings or indeed of any other characteristics as a means of classification. They published results for dissections up to weight 8, but Earl (1977) has demonstrated the inconsistency of their definitions and has shown that their algorithm is not exhaustive.

It should be noted that neither of the following algorithms makes use of any 'list' of dissections of weight $p-1$ in the generation of those of weight $p$.

*Algorithm RK*

The algorithm is based on a technique for enumerating any class of designs that can be encased within a rectangular framework. The algorithm essentially simulates a parameterised shape grammar for dissections. A feature of this shape grammar is that every dissection is produced by a unique sequence of production rules, and furthermore the dissections in the grating are exhaustively generated; that is, duplicates are never generated. The shape rules used here generate the constituent rectangles in the dissection by extending existing rectangles along the horizontal and vertical directions as well as by dividing existing rectangles into smaller rectangles.

The shape rules are translated into equivalent grid-cell colouring rules. The colours are represented by integers. By colouring is meant the obvious colouring, namely, grid cells in the same rectangular element have the same colour and those in different rectangles have different colours. There are of course many possible colourings for a dissection; but of these there is one 'minimum' colouring that allows us to detect symmetry isomorphs without having to resort to any external storage device.

The generating algorithm takes as its input a unit grid. It proceeds recursively over the grid from left to right, bottom to top; all possible shape rules that apply to the current grid cell are selected and for each rule the appropriate colour is allocated to the cell.

Boolean predicates are employed to control the selection of the rule, such as to ensure that the number of colours used does not exceed the weight $p$ and to ensure that only standard dissections are encountered. Alignment checks are also translated from conventional shape rules to Boolean predicates. Trivalent dissections are obtained by eliminating one of the rules. The generation is improved by judicious pruning of the search. For further details the reader is referred to Krishnamurti and Roe (1978).

*Algorithm CB*

First, for the given unit grid and weight, all possible grid partitions and factored representations are generated. The generating algorithm per se accepts as input a unit grid and a single set of tiles. Tiles are 'laid' from the top of the grating downwards, working row by row from left to right. The placing of tiles over unoccupied two-cells in a row effectively partitions the unoccupied segment of cells into as many parts as there are tiles. The unoccupied segments chosen are those lying in the uppermost row containing the unoccupied cells; the partitioning referred to here concerns the division of these segments into horizontal components, that is, by the vertical edges of the tiles so placed. Thus the basic steps in the algorithm are as follows.

1. Locate the uppermost unoccupied segments.
2. Generate a partition of the leftmost segment in the row.
3. For each component of this partition, select a tile from those remaining having one dimension equal to the number of two-cells in the partition component.
4. 'Place' the tile on the grating by colouring the two-cells it covers by an integer.
5. When all components of the partition have been matched to tiles, generate a permutation of these tiles within the confines of the current unoccupied segment.
6. Go to step 2 if there exists another segment in the current row, otherwise go to step 1.

Note that for the resulting dissection to be standard it is necessary but not sufficient that at least one tile be placed per row. For further details the reader is referred to Bloch (1978).

**Computational results**

The computational results are given in the appendix. Table A1 presents the population of dissections, by grating, for weights up to and including $p = 10$; the table also shows the results further subdivided into trivalent, nonaligned, and fundamental dissections. (The results for $p = 10$ were obtained by runs of algorithm RK only.) Figure 1 illustrates these results for $p = 5$. Table A2 gives the distributions of dissections for $p = 6$ and 7 over grid partitions and factored representations of the unit grid.

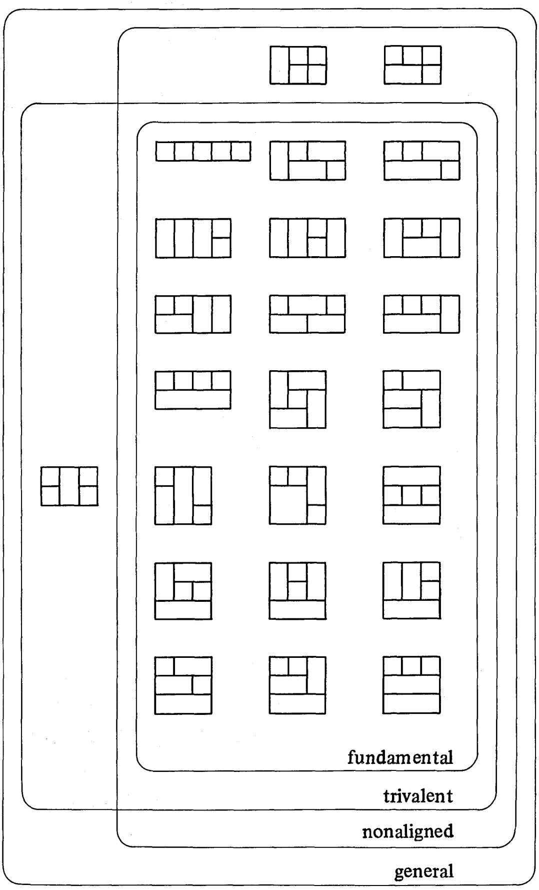


**Figure 1.** The dissections of weight $p = 5$.

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## APPENDIX

**Table A1.** The breakdown of dissections, for weights up to and including $p = 10$, according to their gratings.

| $p$ | Grating | Number of dissections | | | |
|---|---|---|---|---|---|
| | | general | nonaligned | trivalent | fundamental |
| 5 | (1,5) | 1 | 1 | 1 | 1 |
| | (2,3) | 3 | 2 | 1 | – |
| | (2,4) | 9 | 9 | 9 | 9 |
| | (3,3) | 11 | 11 | 11 | 11 |
| | all | 24 | 23 | 22 | 21 |
| 6 | (1,6) | 1 | 1 | 1 | 1 |
| | (2,3) | 1 | 1 | – | – |
| | (2,4) | 11 | 8 | 3 | – |
| | (2,5) | 19 | 19 | 19 | 19 |
| | (3,3) | 13 | 9 | 4 | – |
| | (3,4) | 81 | 81 | 81 | 81 |
| | all | 126 | 119 | 108 | 101 |
| 7 | (1,7) | 1 | 1 | 1 | 1 |
| | (2,4) | 4 | 3 | – | – |
| | (2,5) | 30 | 19 | 11 | – |
| | (2,6) | 35 | 35 | 35 | 35 |
| | (3,3) | 9 | 6 | 1 | – |
| | (3,4) | 181 | 116 | 65 | – |
| | (3,5) | 286 | 286 | 286 | 286 |
| | (4,4) | 269 | 269 | 269 | 269 |
| | all | 815 | 735 | 668 | 591 |

**Table A1** (continued)

| *p* | Grating | Number of dissections | | | |
|---|---|---|---|---|---|
| | | general | nonaligned | trivalent | fundamental |
| 8 | (1,8) | 1 | 1 | 1 | 1 |
| | (2,4) | 1 | 1 | – | – |
| | (2,5) | 20 | 13 | 1 | – |
| | (2,6) | 85 | 55 | 30 | – |
| | (2,7) | 71 | 71 | 71 | 71 |
| | (3,3) | 2 | 1 | – | – |
| | (3,4) | 189 | 97 | 19 | – |
| | (3,5) | 966 | 582 | 384 | – |
| | (3,6) | 968 | 968 | 968 | 968 |
| | (4,4) | 1034 | 610 | 424 | – |
| | (4,5) | 3128 | 3128 | 3128 | 3128 |
| | all | 6465 | 5527 | 5026 | 4168 |
| 9 | (1,9) | 1 | 1 | 1 | 1 |
| | (2,5) | 5 | 3 | – | – |
| | (2,6) | 72 | 41 | 4 | – |
| | (2,7) | 217 | 132 | 85 | – |
| | (2,8) | 135 | 135 | 135 | 135 |
| | (3,3) | 1 | 1 | – | – |
| | (3,4) | 103 | 43 | 2 | – |
| | (3,5) | 1527 | 647 | 196 | – |
| | (3,6) | 4565 | 2597 | 1968 | – |
| | (3,7) | 3135 | 3135 | 3135 | 3135 |
| | (4,4) | 1805 | 710 | 249 | – |
| | (4,5) | 17023 | 9276 | 7747 | – |
| | (4,6) | 16061 | 16061 | 16061 | 16061 |
| | (5,5) | 13422 | 13422 | 13422 | 13422 |
| | all | 58072 | 46204 | 43005 | 32754 |
| 10 | (1,10) | 1 | 1 | 1 | 1 |
| | (2,5) | 1 | 1 | – | – |
| | (2,6) | 31 | 18 | – | – |
| | (2,7) | 259 | 135 | 20 | – |
| | (2,8) | 549 | 332 | 217 | – |
| | (2,9) | 271 | 271 | 271 | 271 |
| | (3,4) | 35 | 17 | – | – |
| | (3,5) | 948 | 325 | 35 | – |
| | (3,6) | 9881 | 3642 | 1513 | – |
| | (3,7) | 19243 | 10443 | 8800 | – |
| | (3,8) | 9936 | 9936 | 9936 | 9936 |
| | (4,4) | 1896 | 542 | 67 | – |
| | (4,5) | 43961 | 14388 | 7890 | – |
| | (4,6) | 117070 | 59915 | 37833 | – |
| | (4,7) | 76622 | 76622 | 76622 | 76622 |
| | (5,5) | 102184 | 51361 | 50823 | – |
| | (5,6) | 195775 | 195775 | 195775 | 195775 |
| | all | 578663 | 423724 | 389803 | 282605 |

**Table A2.** The distribution of dissections for $p = 6$ and 7 over grid partitions and factored representations of the unit grid.

| Grating | Grid partition | Factored representation [a] | Number of dissections |
|---|---|---|---|
| $p = 6$ | | | |
| (2,4) | 3 1 1 1 1 1 | | 1 |
| | 2 2 1 1 1 1 | | 10 |
| (2,5) | 5 1 1 1 1 1 | | 1 |
| | 4 2 1 1 1 1 | (1,4) | 2 |
| | 3 3 1 1 1 1 | | 1 |
| | 3 2 2 1 1 1 | | 6 |
| | 2 2 2 2 1 1 | | 9 |
| (3,3) | 4 1 1 1 1 1 | (2,2) | 1 |
| | 3 2 1 1 1 1 | | 5 |
| | 2 2 2 1 1 1 | | 7 |
| (3,4) | 6 2 1 1 1 1 | (2,3) | 2 |
| | 4 4 1 1 1 1 | (1,4) (1,4) | 2 |
| | | (2,2) (2,2) | 1 |
| | 4 3 2 1 1 1 | (1,4) | 5 |
| | | (2,2) | 9 |
| | 4 2 2 2 1 1 | (1,4) | 11 |
| | | (2,2) | 6 |
| | 3 3 3 1 1 1 | | 4 |
| | 3 3 2 2 1 1 | | 27 |
| | 3 2 2 2 2 1 | | 12 |
| | 2 2 2 2 2 2 | | 2 |
| $p = 7$ | | | |
| (2,4) | 2 1 1 1 1 1 1 | | 4 |
| (2,5) | 4 1 1 1 1 1 1 | (1,4) | 1 |
| | 3 2 1 1 1 1 1 | | 7 |
| | 2 2 2 1 1 1 1 | | 22 |
| (2,6) | 6 1 1 1 1 1 1 | (1,6) | 1 |
| | 5 2 1 1 1 1 1 | | 2 |
| | 4 3 1 1 1 1 1 | (1,4) | 1 |
| | 4 2 2 1 1 1 1 | (1,4) | 6 |
| | 3 3 2 1 1 1 1 | | 3 |
| | 3 2 2 2 1 1 1 | | 10 |
| | 2 2 2 2 2 1 1 | | 12 |
| (3,3) | 3 1 1 1 1 1 1 | | 2 |
| | 2 2 1 1 1 1 1 | | 7 |
| (3,4) | 6 1 1 1 1 1 1 | (2,3) | 1 |
| | 4 3 1 1 1 1 1 | (1,4) | 3 |
| | | (2,2) | 4 |
| | 4 2 2 1 1 1 1 | (1,4) | 18 |
| | | (2,2) | 21 |
| | 3 3 2 1 1 1 1 | | 29 |
| | 3 2 2 2 1 1 1 | | 79 |
| | 2 2 2 2 2 1 1 | | 26 |

**Table A2** (continued)

| Grating | Grid partition | Factored representation [a] | Number of dissections |
|---|---|---|---|
| (3,5) | 8 2 1 1 1 1 1 | (2,4) | 2 |
| | 6 4 1 1 1 1 1 | (2,3) (2,2) | 1 |
| | 6 3 2 1 1 1 1 | (2,3) | 9 |
| | 6 2 2 2 1 1 1 | (2,3) | 6 |
| | 5 5 1 1 1 1 1 | | 2 |
| | 5 4 2 1 1 1 1 | (1,5) (1,4) | 5 |
| | 5 3 3 1 1 1 1 | | 2 |
| | 5 3 2 2 1 1 1 | | 14 |
| | 5 2 2 2 2 1 1 | | 14 |
| | 4 4 3 1 1 1 1 | (1,4) (1,4) | 2 |
| | | (1,4) (2,2) | 2 |
| | | (2,2) (2,2) | 3 |
| | 4 4 2 2 1 1 1 | (1,4) (1,4) | 6 |
| | | (1,4) (2,2) | 9 |
| | | (2,2) (2,2) | 5 |
| | 4 3 3 2 1 1 1 | (1,4) | 15 |
| | | (2,2) | 21 |
| | 4 3 2 2 2 1 1 | (1,4) | 33 |
| | | (2,2) | 30 |
| | 4 2 2 2 2 2 1 | (1,4) | 7 |
| | | (2,2) | 6 |
| | 3 3 3 3 1 1 1 | | 7 |
| | 3 3 3 2 2 1 1 | | 46 |
| | 3 3 2 2 2 2 1 | | 33 |
| | 3 2 2 2 2 2 2 | | 6 |
| (4,4) | 9 2 1 1 1 1 1 | (3,3) | 1 |
| | 6 4 2 1 1 1 1 | (2,3) (1,4) | 5 |
| | | (2,3) (2,2) | 4 |
| | 6 3 3 1 1 1 1 | (2,3) | 2 |
| | 6 3 2 2 1 1 1 | (2,3) | 20 |
| | 6 2 2 2 2 1 1 | (2,3) | 7 |
| | 4 4 4 1 1 1 1 | (1,4) (1,4) (1,4) | 2 |
| | | (1,4) (2,2) (2,2) | 1 |
| | 4 4 3 2 1 1 1 | (1,4) (1,4) | 9 |
| | | (1,4) (2,2) | 22 |
| | | (2,2) (2,2) | 3 |
| | 4 4 2 2 2 1 1 | (1,4) (1,4) | 19 |
| | | (1,4) (2,2) | 14 |
| | | (2,2) (2,2) | 4 |
| | 4 3 3 3 1 1 1 | (1,4) | 5 |
| | | (2,2) | 2 |
| | 4 3 3 2 2 1 1 | (1,4) | 54 |
| | | (2,2) | 21 |
| | 4 3 2 2 2 2 1 | (1,4) | 27 |
| | | (2,2) | 13 |
| | 4 2 2 2 2 2 2 | (1,4) | 4 |
| | | (2,2) | 1 |
| | 3 3 3 3 2 1 1 | | 6 |
| | 3 3 3 2 2 2 1 | | 17 |
| | 3 3 2 2 2 2 2 | | 6 |

[a] Factors are only given for those values of $q_i$ for which more than one factorisation is possible; for example, factors of 1, 2, and 3—(1,1), (2,1), and (3,1)—are not shown, whereas those of 4—(4,1) and (2,2)—are indicated.