# INVESTIGATING CONFIGURATIONS OF POLYHEDRA IN 3-DIMENSIONAL MODELING ENVIRONMENTS 

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#### Abstract

One of the seminal interests in Architecture is that of spatial relationships. Often architectural concepts are centred on the locus and characteristics of a spatial system. To this end we investigate the particular spatial relationships inherent in the topological characteristics of polyhedral shapes. Formal models are composed of polyhedra represented by the five regular Platonic solids. Spatial generation algorithms focus on the formal relationships in a set of modular configurations and their subsequent interactions with the various symmetry properties of polyhedra. These two generational elements work in concert to build lattices in 3-space representing structured compositions of polyhedra. The study demonstrates how the regularity, rich formal structure, and aesthetic content of polyhedra serve to produce a capacious modeling environment for use in architectural ideation and concept exploration.


## Introduction

Effective construction of spatial configurations is essential to the art of architectural design. Systems of spatial constructions proposed by architects are well documented (Calatrava 1996, Frampton 1991, Sorkin 1999). Since such spatial systems are rarely created within strict computational environments, our investigation responds by describing an experiment driven entirely by algorithmic propositions. We employ the characteristics of mathematical constructs inherent in classes of polyhedra, through algorithms, to build geometrical forms distributed about 3 -space lattices.

From insights gained from this experiment we intend to further explore computing systems that respond to both the formal aesthetics component of architecture and the techniques architects engage in concept design. Within this formal context our quest is to identify traits of computation that will serve as useful techniques to drive Ideation - the generation of forms that stimulate cognitive investigation of ideas. Creating formal constructs and compositions as visual stimulators of ideas is a formal method of expressing ideation.

Firstly, we suggest that the symmetry groups of polyhedra offer characteristics that can be dynamically configured and reconfigured in special ways (Kappraff 1991). Specifically, combinations of polyhedra subjected to computational manipulations express ideas useful for instantiating concepts within an architectural context. Secondly, these transformations, mathematically derived and based on the structural characteristics of
polyhedra, are associated with numerous generative algorithms. The discussion describes the algorithms, their computational features, and presents examples of formal constructs. As an introduction to the kinds of constructs possible, Figure 1 provides an illustration derived from one of the algorithms.


Figure 1. An aggregation of scaled cuboids in c4 configuration
Our investigation was carried out on a form modeler, referred to as 'Sketcher'. Sketcher was chosen for its metaphorical reference to both the architects' physical tools and their cognitive modus operandi. Besides form generation, an essential feature of Sketcher is in the convenience of allowing the architect to react to ideas suggested in a composition through direct modification and visualisation of the modeler's forms. By means of exploring arbitrary views a designer may effectively build a volumetric understanding of the construct. Providing for this sort of reactive response accommodates the kinds of cognitive thought processes, described by Akin (1986), Lawson (1990), Schoon (1992), and Simon (1969), that are considered tightly associated with design. The expectation is that architects will react to a computed design media in ways similar to experiences encountered while using conventional design media.

## Polyhedra

We focus on the Platonic solids. Our choice is governed by the following characteristics: regularity, symmetry, and lines of construction implied in the axes, edges and faces. These characteristics imply suggestions of structure, axiality, and planar extension, all of which are components available for mixing within the design process.

Polyhedra also inherently portray aesthetic content, which plays an important role in the interpretation of compositions. Perception of aesthetic purpose is based in part in the ways of generating compositions. According to Hildebrand (1999) for example, order and complexity are phenomena essential to human aesthetics. Order and complexity are integral to polyhedra. Order is expressed in the structural symmetry and extended organisations implied by internal axial qualities. Each solid exhibits a unique order of complexity. For example, the tetrahedron might be considered the least complex owing to its minimal ordering of face-edge relationships, while the icosahedron expresses a great deal of complexity in its higher order of components and symmetries. Proportional distribution and angular relationships also play important roles, consistently perceived as desirable and sought after aesthetic expressions.

Any geometrical study of shape and form in a mathematical sense is an investigation of patterns (Devlin 1994), a context in which shape and symmetry play a central role in pattern articulation. Shape is a cognitive phenomenon of the natural world, where the shapes that are in constant evidence interpret characteristics of the environment. A search for shape and pattern, and more so, the forms that such patterns might suggest, is a search for meaning. The purpose for using the five Platonic solids for this study then is to provide content-rich atoms from which to construct aggregate forms that will inherit a robust formal content. Meanings implied by the forms correspond to those implied by both the formal organisations and the geometric characteristics of the solids.

Three isometry groups are associated with the five Platonic solids: the tetrahedral group, $\mathbf{T}$, of order 12, the octahedral group, $\mathbf{O}$, of order 24 and the icosahedral group, $\mathbf{I}$, of order 60 . Cromwell (1997) provides a complete analysis of the three groups. Of concern here are the specific kinds of symmetries included in each group, some of which will provide the structural motivation for constructing algorithms.

As adjuncts to the symmetry characteristics, each of the five solids posses unique relationships between their face and edge components: for example, the cube has 6 faces and 12 edges that align in three mutually orthogonal directions. These directions are parallel to three of the axes of symmetry and orthogonal to others. By reduction to sets of parallels, the cube offers 13 axial orientations, which include the three edge and three face orientations. The 13 axes, in pair, form 78 planar orientations, which by imposing an orthogonality restriction on the axes, further reduces to 12 distinct non-isomorphic planar orientations. With further restrictions the planes combine to form 4 non-isomorphic 3 -sets of orthogonal orientations in 3 -space. These planes and 3 -sets provide lattices for modular distributions.

By similar inspection of each solid, we find the polyhedra contribute a total of 17 distinct 3 -sets of orthogonal orientations: the cube, 4 , the octahedron, 3 , the dodecahedron, 5 , and the icosahedron, 5 . Moreover, while the 3 -sets are bound to the spatial orientations of their respective solid, varying the orientation of the solid within the modeling space will engender an even greater range of spatial lattice orientations. The purpose of analysing these polyhedra was to hone in on specific characteristics inherent in their formal structure that might be used as a structural basis in form generation. The 173 -sets found within the Platonic solids serve that purpose.

The symmetry of any symmetric polyhedra belongs to one of five families of groups: I, O, T, cyclic and dihedral (Cromwell 1997). This places the five regular solids somewhat at the heart of symmetry explorations. A polyhedron with $\mathbf{I}, \mathbf{O}$, or $\mathbf{T}$ symmetry may replace a regular solid without adversely affecting any structural characteristics predicated on symmetry. For this reason, the complexity and compositional richness in formal models may be heightened by substitutions between regular and more complex polyhedra. While interesting to speculate, such investigations are not included here. It is sufficient to show that a manageable subset of polyhedra under algorithmic manipulation can produce the intended type of formal construct.

## Algorithms of Construction

Three Sketcher functions - vertex, edge, and face - are based in algorithms not associated with the structural characteristics of the constituent modular form: these are generalised adjacency functions that form aggregations of uniformly sized modules. The three functions do however follow 3-space lattice generations that are based on the generalised structure of
the cube. Spacing in the lattice varies as the modular dimensions vary. With equal dimensions, the lattice is cubic; otherwise, the lattice is rectangular in each of the three planes.

For a cubic module, dimensional change increases the complexity and proportional implications suggested by a composition. Far from being an ordinary structure, the cube offers a surprisingly rich set of formal constructs through simply varying its dimensions. The cube metaphor generalises to a place holder function for other polyhedra. Data transformations that ensure correct positional relationships among modules rely on the sanctity of the orthogonal 3 -space defined by the module. Therefore, no matter what kind of formal distribution an algorithm might call for, there is a consistent underlying method of functional distribution.

Dimensional variation of a module also implies a method of implementing scale. A cube is scaled whenever any of its dimensions change relative to another. It is not necessary for scale ratios to proceed equally along the three orthogonal axes. Varying the dimensions of a module, relative to the dimensions of the modeling space that contains the module, is also a scale.

The three adjacency functions distribute modules independent of scale. Each function is dependent upon a transformation that establishes a dimensional relationship between module centroids and module size. Vertex adjacency places each succeeding module at a location that is at one of the eight vertex positions adjacent to the previous module. Placement is done recursively, restricted by the requirement that no centroid location can be occupied more than once. Vertex selection is specified by a stochastic that responds to the available vertices remaining and the sequence in the program execution where the selection was invoked. The process is not entirely random and, in some cases, tends to maintain a historical trend. Occasionally, all candidate vertices are occupied at the time of vertex selection, in which case, stochastic search algorithms then respond by looking for an empty vertex adjacent to an existing module.

Aggregations are least dense under vertex distribution and tend towards an even apportioning between solid and void over an occupied region of the modeling space. Aggregation by edges places module centroids at locations that establish edge-to-edge relationships. Regions inhabited by edge aggregations see an increased solid to void ratio over the vertex adjacency, yet maintain incremental voids between modules. Face-to-face adjacencies are similarly determined. Face adjacency tends to increase density. Voids are eliminated between modules and the aggregation assumes greater solid massing. Modular mixing between solid and void is replaced by mass mixing of solid and void.

Figure 2 (plates I-IV) illustrates aggregations of octohedra. Vertex adjacency is used to layout I-II. Spacing density looks thin, particularly as the octahedron only half fills the volume of a cubic module. Plan (I) and elevation (II) suggest various ideas for compositional interpretation. c4 symmetry is maintained in plan except for a few modules removed at the right; elevation indicates bilateral symmetry. The choice of $c 4$ symmetry follows from a symmetry variable provided to enhance the consistency of construction on a polyhedral form. Note that by removing a few modules the sense of mirror image remains while the slight symmetry breaking begins to add complexity to the scene.

Images implied by the octahedral module vary in plan and elevation. In plan the octahedron is square, suggesting a square grid. However, its quatrefoil shape implies a $45^{\circ}$ axiality that might exist as well. In elevation the shape takes on a pyramidal image, which, in this composition, strongly suggests an angular construct. Without the octahedral image such schemes would not be obvious candidates for ideation and exploration of alternatives.


Figure 2. Aggregations of octahedra
A different sort of octahedron appears in plates III-IV The octahedron is scaled up so that edges in the $x y$ plane are coincident with the boundaries of the cubic module. Scale effectively increases the mass of each module and induces an overlap between octahedra in the vertical axis. Edge adjacency is used to aggregate the composition. Compared to the composition in plates I-II, both organisation and continuity are better defined without losing any of the complexity introduced by the octahedral form. The elevation, taken from a slight off- axis axonometric viewpoint, shows a consistent massing while still projecting a sense of mirror plane image. $c 4$ symmetry appears here as well. Any of the four corners of the original aggregation would suffice as a point of symmetry. Each corner in turn defines a c4 symmetry composition of a different spatial character. The four elevations assume various interesting interpretations as the viewpoint moves sequentially between the normals to each elevation. Since the elevations are not mirrors in 3-space, symmetry breaking begins to appear also in the sequential movement to off axis positions.

The quatrefoil form of the octahedral shape implies numerous possibilities for organisational schemes. In an architectural context the plan is particularly intriguing. It suggests an outside space enclosed by inside spaces (solid forms) which are enclosed by outside spaces, which are once again partially enclosed by solid inside spaces. All sorts of vertical structure may also be inferred from the elevation.

Figure 3, showing two compositions, introduces several new features. Both are face adjacent aggregations. Face adjacency sets a stronger sense of density and evenness in the modular distribution, characteristics that are most evident in plan and elevation. Both are also aggregations of 30 modules of the same octahedron, composed under the same dimensional variables, and reflect $c 4$ symmetry in plan. The two compositions differ only in
the stochastic influences on the generating algorithm. An even greater exploitation of the axial regularity of the octahedron is attainable using the face adjacency algorithm.


Figure 3. Aggregations of face-adjacent octahedra

Scale, depicted in Figure 4 (plates IX-X), is inherent in the specification and variation of modular dimensions. Plate IX illustrates the effects of scaling an octahedron to a 1:2:6 ratio, X, the reverse 6:2:1 ratio. Scale is susceptible to infinite variation and thus enhances ideation by enabling a capacious set of formal constructs that interweave numerous dimensional variations.


Figure 4. Illustrating independent scaling in aggregations

Scaling may also introduce differences in modular orientation, which in this case appears either 'vertical' (IX) or 'horizontal' (X). Modular orientation introduces another level of complexity available to enrich and enhance ideation, particularly when dynamically specified at the origination point of the aggregation. Orientation algorithms prove useful in combination with algorithms that select and aggregate modules in axial distributions.

It was noted that among the characteristics of the five solids each contained some combination of axial, planar, and 3-set planar orientations that could serve as references for orientating spatial lattices. Using one or more such spatial references requires that the aggregations of polyhedra be both directed and bound by the lattices. Such binding introduces formal constructs assuming adjacency relationships that may differ fundamentally from those of vertex, edge, or face adjacency. Moreover, lattice orientation within the modeling space introduces another level of complexity that breaks the strict orthogonality relationship between the axes of the aggregation and those of the modeling space. Variations in orientation schemes such as these add useful complexity to ideation and concept exploration.

Sketcher provides algorithms for selecting a particular axis, or 2-point axial orientation, of propagation for any modular form. The module is instantiated in the modeling space at the origination point for the aggregation. The algorithm transforms a specified axis designated in 2-point fashion into a linear aggregation lattice. It is possible to select more than one axis, which then forms a propagation structure in which the axes have the same angular relationship as their structure in the solid. Consequently, the axes may or may not hold orthogonal relationships either mutually or with the axes of the modeling space. See Figure 5 (plates XI-XII).


Figure 5. Axial distribution with tetrahedra and dodecahedra
Plate XI illustrates one such axial structure taken on two axes of the tetrahedron. One axis of the structure is normal to the $z$-axis while the other extends in a direction that is not orthogonal to any of the three modelling space axes. The structure reflects the essential angularity of the tetrahedron. As demonstrated in the compositions in Figure 3, modular orientation is an algorithmic variable available to the designer. This means that the structure of the construct could assume numerous different directional orientations in modeling space while maintaining its internal topography.

Other details are notable. One axis holds six modules, the other seven. Module count is a parameter useful for varying axis length. In plan the composition reveals $c 3$ symmetry which reflects the triangular topography of the tetrahedron. While the centroid-to-centroid
spacing in this example equals the side length, it is possible to specify an arbitrary spacing along the axis to control, for example, composition density. Further, the module scaling ratio may vary from the $1: 1: 1$ shown. Scale is also a variable in this algorithm. Lastly, the two large triangles were added to the composition after construction as one interpretation of connection points implied by the structure.

Another example of axial distribution is shown in plate XII. The composition derives from a two axes aggregation in $c 5$ symmetry. One axis follows an edge and holds seven modules. The other is an edge-to-edge axis that passes through the dodecahedron centroid. It holds six modules. Note that module spacing along the edge axis is compressed so that modules overlap adjacent modules. Conversely, spacing along the edge-to-edge axis is such that the distance between modules is exactly one module and adjacent modules occupy the same edge line.

There is a certain concept of density associated with the spacing function that is unique to the implied linearity of an axis. Where constant module spacing is inherent to the vertex, edge, and face algorithms as a necessary requirement for internal lattice consistency, it is less restricted along a line. Variations in spacing do not destroy linearity but rather serve to articulate the line and define its place as an axis of construction.

Spacing differences not only increase complexity and interest in a composition but also suggest certain characteristics inherent to the modular form. For example, the density in the module overlap along the edge axis seems to imply that 'stacks' of dodecahedra might hold together simply on the strength of their internal structure. However, such cannot be the case since the structure of a dodecahedron is inherently unstable. A dichotomy of this sort, between a visual interpretation and a physical interpretation, is a useful one in the quest for conceptual ideas.

Where axial propagation projects strong directional properties, planar arrangements seem not so decisive. Figure 6 illustrates a composition for comparative purposes, based on the dodecahedron, which lacks the strength of directionality and density. Yet, such planar compositions remain both useful and interesting. The aggregation is taken along a plane formed by two edge-to-edge axes normal to each other, passing through the centroid of the dodecahedron. In plan, the composition is both regular on the dodecahedron, and planar. Module adjacency across the plane is edge to edge. Note that edge and face adjacencies work for the plane, while the vertex scheme fails by violating a sense of planarity (single module thickness).


Figure 6. Planar arrangements of dodecahedra

The planar generation algorithm provides for selection of either a face or a set of orthogonal axes to establish the relationship of the planar orientation to the modeling space volume. In plate XIV the plane is at an angle. Several orientations are possible for the dodecahedron, in particular one for each of the 12 faces, and one for each of the 15 planes associated with the 15 edge-to-edge axes. Since the plane selected is predicated on the orientation of the first solid placed in the modeling space, it is conceivable that additional planar orientations might occur by arbitrary orientations of the initial solid. This is valuable because the more interesting configurations of planes might be those of multiples that define compositions in 3 -space.

Figure 7 illustrates one such algorithm for a 3 -space composition. In plan, one sees a planar aggregation in a $c 3$ configuration. The icosahedron provides the modular definition and as a consequence of its triangular topography, an 'interior' elevation taken normal to the plane of propagation looks identical from all three viewpoints. Two of the internal axes of the icosahedron define the plane of propagation. Aggregation is straightforward in the plane with edge adjacency and spacing equal to the length of one module. Complexity in the composition arises from both $c 3$ symmetry distribution and the segmented arrangement of icosahedra in the plane. The selected orientation is biased vertically while the modular propagation grows in a way that suggests stacking along an axis of the icosahedron. These features juxtaposed in the three spatially related planes are sufficiently speculative to be useful as progenitors of ideas in an architectural context.


Figure 7. A 3-space composition of icosohedra
Lastly, it is worth considering an example illustrating the interaction of several of the previous algorithms (Figure 8). The three images of the composition appear thoroughly speculative. The composition is steeped in ambiguity, a characteristic of human cognition long thought to be essential in the search for meaning (Minsky 1988; Stiny 1989).

Several characteristics are notable. As the side view in plate XVI shows, the local modeling space is reoriented relative to global space, following a 3 -set of orthogonal axes of the icosahedron. Note that scaling the icosahedron, while modifying the dimensional relations between its vertices, does not change its basic topographical structure. As a consequence, the internal relationships between its sets of orthogonal axes remain unchanged.

The basic aggregation is composed of 15 icosahedra in edge adjacency, each scaled in 1:2:3 ratio. Modeling space dimensions are the same in the three directions such that the smallest module dimension will fit ten times along an axis. The aggregation is placed in a $c 3$ symmetry plan and mirrored about the model space plane parallel to the global $y$-axis.

Finally, the construction is mirrored once again about the model space plane parallel to the global $x$-axis to give the final composition.


Figure 8. Three distinct images of an aggregation of icosohedra
Plate XVI illustrates the first mirror isometry, XVII illustrates the second. In plate XVIII, which shows an image taken from a viewpoint off the normal to any plane, one sees that the mirror symmetries seem to disappear while the 12 composite aggregates of 15 modules still maintain their identity. This composition is quite complex, a sense of which is apparent in the images. By transiting the endless viewpoints in global space that surround the composition, the more tantalising and curious formal assets of the composition are revealed. For this composition the numerous views provide enough spatial material to spark ideas about spatial relationships, solid-void contrasts, axial organisations and modular constructions based in the images of the icosahedron. In some cases, the complexity of construction might seem unmanageable; it is, in fact, quite stimulating.

## Conclusion

Investigations of the sort presented here are relatively rare in their consideration of design as firstly an analysis of form. Yet formal constructions are central to architectural design and comprise an essential domain of inquiry. Sketcher's purpose, expressed by its algorithms, is to generate ideas through formal models that otherwise might not be found in traditional design episodes. The expectation is that the architect would encounter a more robust sketching experience and, in turn, respond more aggressively to the ideas presented. The examples presented are for the purpose of demonstrating two formal issues: the computability of formal constructs, and the influence the modular image, used as a nucleus, might have on the formal content of the construct.

The presentation proceeded in two parts. Firstly, a discussion considering the salient characteristics of modular form that might propagate throughout a construction. For this the five Platonic solids served a high purpose. Secondly, a description of the various algorithms, all basic but founded in mathematical certainty, which proved sufficiently robust to use the formal features of the solids to advantage.

We found that the combination of an instantiation of algorithmic procedures in a modeling environment and the use of a richly endowed seed form could produce compositions with significant potential for exploitation in design. Numerous examples of such compositions proved to be both gratifyingly speculative and competent progenitors of architectural ideation.

Many interesting and useful issues were raised in the course of this investigation. Two were quite significant. The first is that, in general, computing environments are useful for
formal investigations in conceptual design. The second shows that the use of an idea such as that implied by polyhedra - which is closely aligned with aesthetics, the intent of art and of the senses (and by extension, architecture), and also possessing an inherent sense of structure - may be useful in transcribing an image that is both a focused organisational paradigm and a sensitive human event.

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