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ON A METHOD OF FLEMMING

Wall representation and orthogonal structures

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Abstract. I examine a method of Ulrich Flemming, rooted in graph theory, originally developed for two-dimensional spatial layouts based on mutually orthogonal relations, for extension to the third dimension. Specifically, arrangements of three dimensional rectangles (or *3-rectangles*) are considered. Such arrangements can be classified as *dense* or *loose*, referring to the tightness of packing, and further classified as *unlocked* or *locked*, referring to the interlocking nature of the constituent 3-rectangles. I show that such arrangements can be recursively specified by two kinds of rules: simple insertion of a wall and 'pinwheel' insertion of a cyclical arrangement of walls, mirroring his results in 2D. I further mirror Flemming by establishing a similar structure property for these arrangements, and conclude by establishing the link to a 3D extension of orthogonal structures.

Prologue

In a town near *Koblenz* there was a young man
Who went to *Berlin* with a wonderful *plan*
With a *twist* and a *slice*
He made a *wrep* nice
Das ist *DIS* he exclaimed with élan

He plaited a *wig* with the *space* round the *T*'s
He wore it to *Buffalo* … it was a breeze
Looking *left*, looking *right*
Looking *up*, looking *down*
He *structured* this *rag* and took it to town

Orthogonally speaking, his idea was cool
He *marked* the *stars* by his own *expert rule*
And … broke *loos* a *seed*
Though, my friends, there's lots more to tell
This man from Koblenz has succeeded quite well.

> *Ulrich Flemming grew up in Mayen near Koblenz, in the Eifel region of Germany – he did his PhD at the Technical University of Berlin on generative rules for two dimensional T-plans – developing and extending this work to loose arrangements at SUNY-Buffalo, and Carnegie Mellon – where his ideas took shape in form of the computer systems: DIS, LOOS and SEED*

Ulrich Flemming championed and advanced the cause of spatial layout design as a legitimate area of inquiry bringing rigor to method in regard to schemes of subdivisions of right-angled spaces separated by 'walls' in mutually orthogonal directions. Much of his work focused on arrangements of rectangles in two dimensions, dense and loose, in theory and in practice, on implementation, with application to design and construction; see Flemming (1978, 1979, 1980, 1986, 1988, 1989, 1990; with Fenves et al, 1994; with Chien, 1995; with Woodbury, 1995; with Choi, 1996). Flemming's solo authored papers, cited above, constitute the "method" referred to in the title of this paper.

The basic spatial layout problem he tackled takes the following form: "*For a given set of rectangular spaces, find possible arrangements of the spaces such that no two spaces overlap [and satisfy given constraint]*." Constraints, if any, may take on a variety of forms that relate to geometry, topology, shape, dimension, area, volume, adjacency, material property, costs, and so on (Flemming et al, 1992). Constraints may also apply to collections of spaces, or regions made up of specific spaces. In this paper, I consider unconstrained layouts of rectangular spaces in three dimensions following, in essence, the method of Flemming. Some of the results reported here have been published elsewhere (Krishnamurti and Earl, 1998), although the logic there was mainly number-theoretic – albeit with reference to 3-dimensional maps – rather than graph-theoretic as is the case here.

For any layout of n-dimensional rectangular spaces, $n \geq 2$, termed *n-rectangles*, there is always a smallest n-dimensional rectangular region that encloses the layout; this is referred to as the *bounding* or *enclosing n-rectangle*. An arrangement of non-overlapping n-rectangles in an enclosing n-rectangle is termed an *n-rectangulation*. Each such arrangement falls, naturally, into one of several categories, for example:

- Arrangements in which constituent n-rectangles pack the bounding region. These are 'dense' arrangements.
- Arrangements of n-rectangles that do not completely fill the bounding region. These are 'loose' arrangements.
- Arrangements that satisfy topological or geometrical constraints such as adjacency, tolerances, or minimum and maximum dimensions. These tend to be loose.

3-RECTANGULATIONS

Definition: A *3-rectangulation* is an arrangement of non-overlapping 3-rectangles within a 3-rectangle. The arrangement is *dense* if the 3-rectangles cover the enclosing rectangle, and *loose* otherwise. A loose arrangement can be derived from a dense counterpart by designating certain rectangular spaces as 'holes' or voids. The number of constituent non-void spaces is referred to as its *content*. A 3-rectangulation with content p is denoted as a $(3, p)$-*rectangulation*.

It is not within the purview of this paper to survey the literature pertaining to layouts Although, it is not hard to find practical examples involving elements that occupy volume, the literature on 3D layouts is sparse; in 2D, there are more exemplars. Beyond art and architecture, spatial layout is important in many different disciplines arrived at through a variety of approaches — see, for instance, Preas and van Cleemput (1979), Wong and Liu (1987), Rutenbar (1989) and Galle (1990). Current practice in creating 3D-layouts is, where possible, by extruding 2D-arrangements into the third dimension. For instance, in VLSI design, one, typically, employs optimization techniques especially when the number of spaces is too large – in any practical way – to handle by generative approaches. On the other hand, there are problems that cannot be solved even in this manner, for example, layout of engine body components. Potential solutions are confirmed by 'expensive' simulations (and, perhaps, even prototypes). There is the occasional algorithm to solve constrained packing problems; Szykman and Cagan (1997) provide an example in mechanical engineering design. On the whole, ceteris paribus, it is worthwhile and, perhaps even necessary, to examine properties of general three dimensional arrangements.

T-figurations

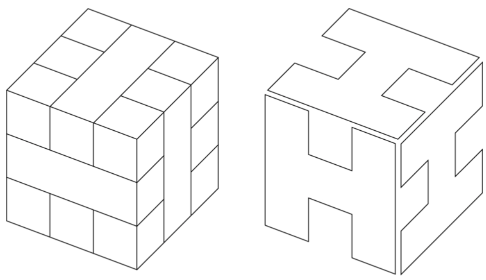


Figure 1: *A 3-rectangulation and interior maximal planes one from each of the three possible orientations. After Earl (1978)*

Maximal lines act as separating walls for 2-rectangulations; in like fashion, maximal planes separate 3-rectangulations. A 3-rectangulation is formed by a set of maximal planes of which six share boundary lines with the bounding 3-rectangle and the rest have boundary lines coincident with other maximal planes but do not share boundary lines (Earl, 1978). In general, the maximal planes are not rectangular. Figure 1 is an example of a symmetrical 3-rectangulation where the interior maximal planes in each of three possible orientations are identical in form.

JOINTS IN 3-RECTANGULATIONS

The intersection of three maximal planes defines a *joint*. For a 3-rectangulation, there are precisely eleven types of interior joints possible. See Figure 2, arranged in rows according to the number of 3-rectangles that have corners at joints.

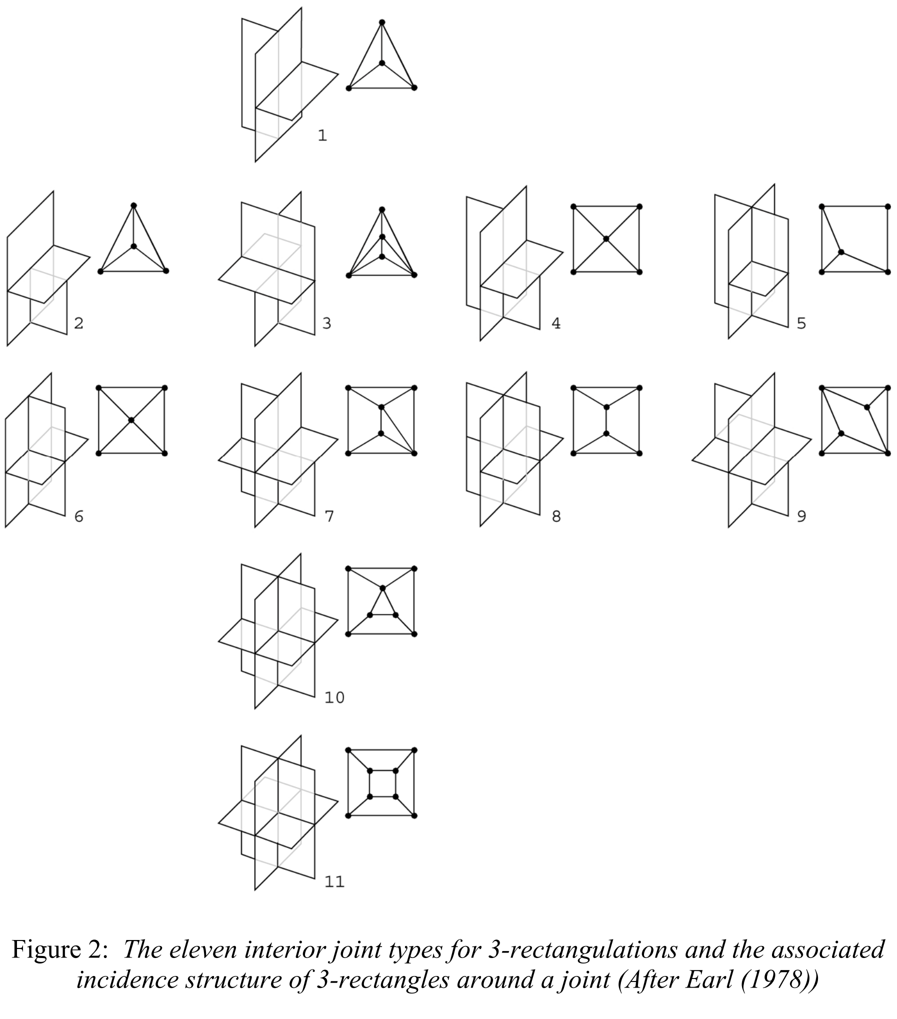


Figure 2: *The eleven interior joint types for 3-rectangulations and the associated incidence structure of 3-rectangles around a joint (After Earl (1978))*

If each constituent 3-rectangle is represented by a vertex, we obtain the incidence structure around a joint, also shown in Figure 2. Note that each graph can be obtained by a sequence of edge contraction (or vertex identification) from the incidence graph for the type 11 joint.

Joints in space are analogous to junctions, the intersection of two lines in the plane, of which there are three types: L-, T-, and +-shaped.

Each cross-section of a 3-rectangulation parallel to a coordinate plane, not coincident with a maximal plane, corresponds to a dense arrangement of rectangles within a rectangle, i.e., a 2-rectangulation. Of interest are those 3-rectangulations the cross-sections of which correspond to trivalent 2-rectangulations. [Flemming (1978) refers to trivalent 2-rectangulations as *T-plans* because the interior maximal lines form T-shaped junctions.]

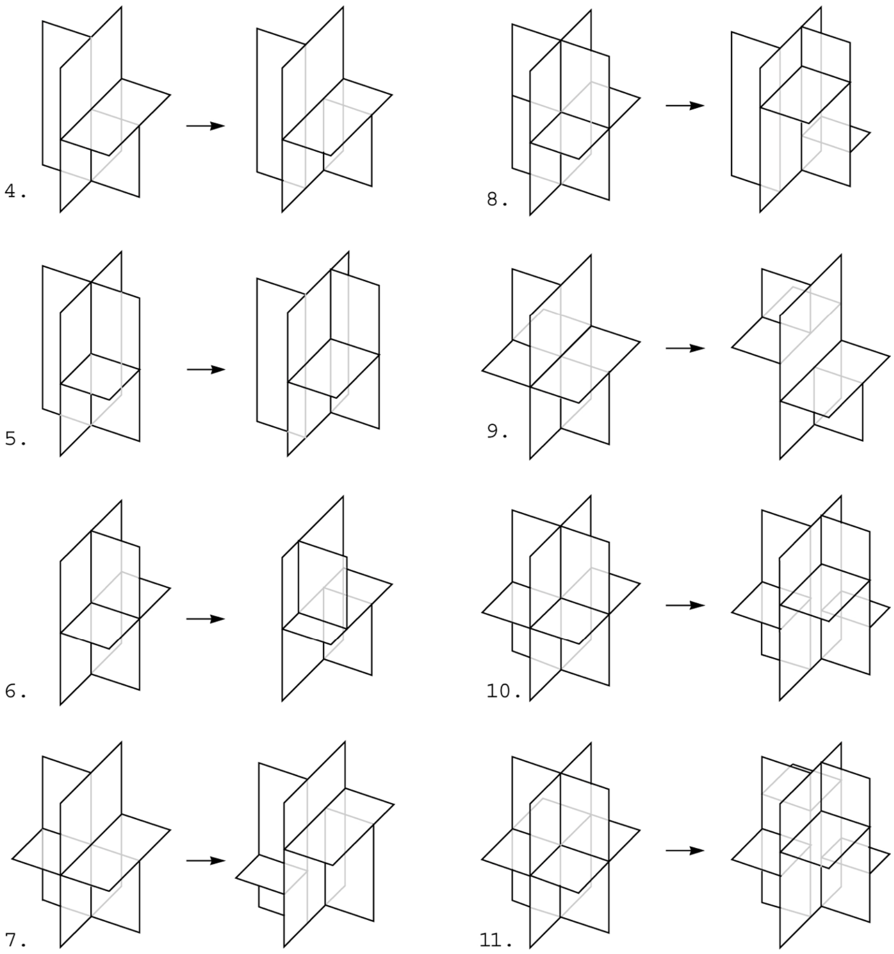


Figure 3: *Rules to convert 3-rectangulations into T-figurations*

Definition: A 3-rectangulation is *trivalent* – that is, each of its cross-sections, in the above sense, is a trivalent 2-rectangulation (or T-plan) – if it contains interior joints of types 1, 2, and 3 and no other. If we apply the rules in Figure 3, repeatedly, to interior joints of type 4 through 11, every 3-rectangulation can be converted to a trivalent 3-rectangulation. These rules are analogous to the rules that Flemming uses to convert 2-rectangulations to T-plans. Observe here that the resulting joints are either of type 1 or type 2. In the manner and spirit of Flemming, a trivalent 3-rectangulation will be referred to as a *T-figuration.*

T-figurations exhibit interesting joint properties..

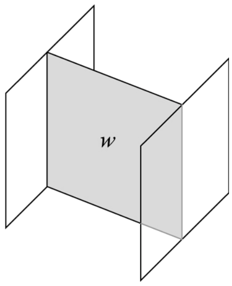
Each corner of a 3-rectangle is determined by three orthogonal planes which meet at the corner; thus, a joint formed by any two maximal planes and a third non-rectangular maximal plane (say, an L-shaped plane) cannot be of type 1 or 2. Hence:

(2.1) *Maximal planes of T-figurations without joints of type 3 are rectangular; conversely, T-figurations with just rectangular maximal planes do not have type 3 joints.*

A T-figuration with type 3 joints is said to be *locked* and *unlocked* otherwise. Figure 1 is an example of a locked T-figuration. Since type 2 joints define corners of 3-rectangles and every face of a 3-rectangle belongs to a maximal plane, it follows that:

(2.2) *Maximal planes of unlocked trivalent T-figurations belong to at least four joints of type 2.*

We can make a stronger observation. Each face of any 3-rectangle in an unlocked T-figuration belongs to a maximal plane, say *w*. Since the neighbouring spaces of the 3-rectangle are either 3-rectangles or the exterior regions, there is a pair of planes that form an I-shape with *w*. This is true in either direction orthogonal to the given plane. Thus:



(2.3) *Each boundary line of any maximal plane of unlocked T-figurations is coincident with but does not share a boundary line with another plane orthogonal to the plane.*

Finally, consider the two pairs of planes that bound a given plane. They must form joints of type 2 at the corners of the given plane. That is:

(2.4) *For each maximal plane of unlocked T-figurations there is a pair of planes orthogonal to it that bound it and form joints of type 2.*

Possible configurations of planes that bound a maximal plane of an unlocked T-figuration are shown in Figure 4. These configurations suggest a set, *bound*(*w*), which specifies for a maximal plane *w* the four maximal planes with which *w* has an incident boundary line. Whence:

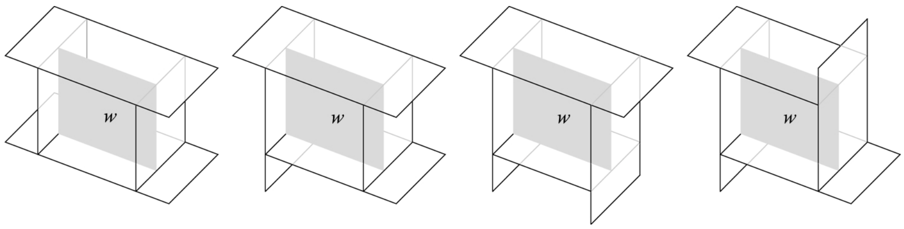
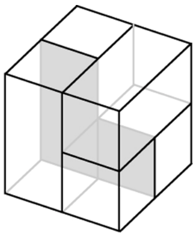


Figure 4*: A maximal plane of an unlocked T-figuration is bound by pairs of parallel planes that form type 2 joints with it*

(2.5) *For each maximal plane of unlocked T-figuration there are exactly four planes to form a configuration equivalent, under identity, rotation or reflection, to one of the configurations shown in Figure 4.*

Lastly, there is a connection between the content of an unlocked T-figuration and the number of its interior maximal planes.

(2.6) *An unlocked trivalent (3, p)-rectangulation has p–1 interior maximal planes (Earl, 1978).*



By (2.6), an unlocked T-figuration with content p+1 can always be constructed from one with content p by adding a new interior maximal plane. This property does not hold for locked T-figurations though. The figuration, shown on the left, is the smallest locked arrangement and has three L-shaped interior planes.

Terminology: A T-figuration with content p is referred to as a *T(p)-figuration*. A T-figuration is said to be *du* if it is dense and unlocked, *dl* if it is dense and locked, *lu* if it is loose and unlocked, and *ll* if it is loose and locked.

REPRESENTING THE MAXIMAL PLANES OF UNLOCKED T-FIGURATIONS

An unlocked T-figuration can be oriented so that the faces of its constituent 3-rectangles are aligned with the coordinate planes. In other words, each maximal plane can be associated with an *orientation* that indicates the coordinate plane to which it is parallel (namely, *xy*, *yz* or *zx*).

The orientation of a maximal plane of an unlocked T-figuration can be used to distinguish it. As a rectangle, a maximal plane can be described by its minimum and maximum coordinates. Depending on orientation, one of the coordinate values is fixed for the descriptors of the maximal plane. For example, an *yz* oriented rectangle shares the same x coordinate value in both its min and max descriptors. Thus, a function *fix* can be specified for maximal planes, taking values from the set $\{x, y, z\}$.

Maximal planes can be arranged so that no two are coplanar. In this case, the T-figuration is said to be in *standard form*. If the number of *xy* oriented planes is n_z, we assign constant z coordinates for the interior xy planes from the set $\{1, \ldots, n_z\text{-}1\}$. The two exterior xy oriented planes are assigned z coordinates of 0 and n_z respectively. The maximal planes oriented in the other two directions are likewise treated. That is, we can define a function, *coord*, which can be used to compare two identically oriented planes for their relative position with respect to each other. The equation of the carrier plane coplanar with the maximal plane w is given by:

$$fix(w) = coord(w).$$

Thus, given the maximal planes and functions *fix*, *coord* and *bound*, we can completely recover an unlocked T-figuration and vice versa. Note, however, that this representation is inadequate for locked T-figurations.

Specifying T-figurations

I now examine three approaches, originally applied to 2-rectangulations, that progressively define larger subsets of 3-rectangulations, culminating in the wall representation (Flemming, 1978), which will be considered in detail.

T-FIGURATIONS AS SLICINGS

Rectangular maximal planes suggest an approach to specifying an unlocked T-figuration by a procedure known as *slicing* (Supowit and Slutz, 1984). A 3-rectangle is sliced into partitions by maximal planes parallel to one of the bounding planes. Each partition is then recursively sliced by maximal planes in an orientation orthogonal to the plane that defines the partition. This process is repeated till each partition is just large enough to house a given 3-rectangular component. The slices form joints of type 1 or 2. A T-figuration that can be specified by slicing is said to be *sliceable*.

A sliceable T(p)-figuration can be constructed from p–1 slices where each slice corresponds to a maximal plane. The slices that meet do so as T-shaped junctions. One of the slices has a boundary line coincident with the other slice. The two slices are respectively termed the *base* and *crosspiece* of the T-shaped junction formed by the two slices. Sliceable T-figurations correspond to a class of acyclic directed graphs.

T-FIGURATIONS AS SUBREGION PARTITIONS

Kundu (1988a) describes a data structure to represent a class of T-plans. A variant of this data structure can be defined for a class of T-figurations. The data structure is an extension of the slicing model, and includes partitioning operators to produce ‘pinwheel’ arrangements.

Quite simply, the data structure is a tree where each node is associated with either a label if it is the root of a subtree, or a name that identifies a 3-rectangle. Leaf nodes correspond to constituent 3-rectangles. The labels of all other nodes identify the partitioning operation that is applied. Each subtree of a node corresponds to a partition of the 3-rectangular space represented by the node. The tree describes a partition of a 3-rectangle into 3-rectangular regions some of which, in turn, are likewise partitioned.

There are three slicing operators, labelled *s*, each oriented parallel to some coordinate plane (i.e., s_x, s_y and s_z), and six pinwheel partitioning operators grouped into three pairs of enantiomorphs, *p* and *q*, each partitioning a 3-rectangular space into five spaces.

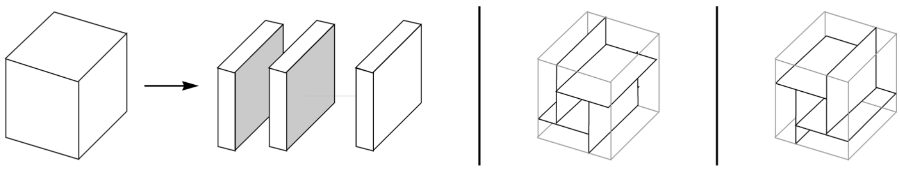
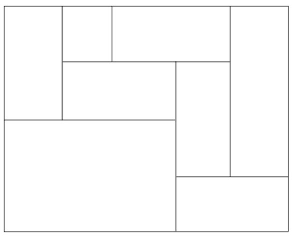


Figure 5: *Slicing s, and pinwheel partitioning operators, p and q.*

Other operators may be defined to generate configurations that cannot be produced by the slicing and pinwheel operations. Kundu (1988b) suggests two for T-plans: the *double spiral* and *weave* operators. Similar operators can be specified for T-figurations.



Partitioning here has the property that any dense T-figuration can be subdivided into 3-rectangular regions, termed *sub-regions*, each the union of one or more subregions. There are, of course, T-figurations that cannot be so described; the T-plan on the left is a cross-section of a T-figuration that cannot be specified by this representation. Such T-figurations can, however, be specified by the 'wall representation,' which is considered next.

WALL REPRESENTATION FOR T-FIGURATIONS

The *wall representation* – *wrep*, for short – is analogous to a combination of the slicing and subregion models for *du* T-figurations (Flemming, 1978). Here, each maximal plane is treated as a wall. The wall representation, W (=$W(R)$), for T-figuration R is given by a set of walls, where each wall is given by the four argument relation:

wall (*index, fix*, $S_\downarrow$, $S_\uparrow$),

where *index* uniquely identifies the wall, *fix* has been defined previously, $S_\downarrow$ is the set of spaces (3-rectangles) that are all directly below, to the front, or to the left of the given wall, and $S_\uparrow$ is the set of spaces that are all directly above, to the back, or to the right of the given wall.

Without introducing further notation, each argument will be considered as a variable as well as a function on walls. In particular, $S_{\downarrow}(wall)$ gives the set of spaces directly to side $\downarrow$ of the wall and $S_{\uparrow}(wall)$ gives the set of spaces directly to the other side $\uparrow$ of the wall. Adding a space to one side of a wall is expressed by assignments of the form:

$$S(wall) \leftarrow S(wall) \cup \{new\text{-}space\}$$

In addition we define truth functions, $\Delta_{\downarrow}$ and $\Delta_{\uparrow}$, to indicate whether a given space is to a specific side of a given wall according to the appropriate orthogonal relation. For a space below a *xy* wall, $\Delta_{\downarrow}(space, wall)$ is true, and false otherwise; for a *xy* oriented wall $\Delta_{\downarrow}$ is defined as, $\Delta_{\downarrow}(space, wall)$: $fix(wall) = z \wedge zmax(space) \leq coord(wall)$, where $\wedge$ denotes conjunction. Likewise, for a space above an *xy* wall, $\Delta_{\uparrow}(space, wall)$ is true, and false otherwise. Δ relations on *yz* or *zx* oriented walls are likewise defined.

Spaces on either side of a wall may be arranged as an ordered list:

- For *xy* oriented walls, spaces are listed front-to-back left-to-right.
- For *yz* oriented walls, spaces are listed below-to-above front-to-back.
- For *zx* oriented walls, spaces are listed left-to-right below-to-above.

In other words, for any *ij* oriented wall, the spaces on either side of the wall are specified in order of increasing *j* and then increasing *i* values.

The wall representation defines an oriented incidence relation between the maximal planes and spaces that make up a T-figuration. The region, exterior to the bounding 3-rectangle, features in this incidence relation. This exterior region is considered to be made up of six distinct spaces, each corresponding to a distinct orthogonal direction. For convenience, these six exterior spaces are identified by symbols E_{above}, E_{below}, E_{front}, E_{back}, E_{left} and E_{right}. The interior spaces are numbered from 1 to p. Likewise, if the six bounding planes of the enclosing 3-rectangle are labelled w_{above}, w_{below}, w_{front}, w_{back}, w_{left} and w_{right}, the interior walls can be numbered from 1 to p–1. We can then specify *bound* for any exterior wall w_e as follows:

$$w \in bound(w_e) \text{ if } w \in \{w_{above}, w_{below}, w_{front}, w_{back}, w_{left}, r_{right}\} \wedge [fix(w) \neq fix(w_e)].$$

Thus, $bound(w_{above}) = \{w_{front}, w_{back}, w_{left}, w_{right}\}$.

The wall representation is independent of the dimensionality of the rectangles, and extends naturally to certain higher dimensional rectangulations. Note that this formulation for the wall representation is independent of the shape of the wall. Initially, we consider T-figurations with rectangular walls.

RULES FOR CONSTRUCTING WALL REPRESENTATIONS

By (2.5), a *du* T(p+1)-figuration can be constructed from a *du* T(p)-figuration by adding a new wall. Let w_1 be a wall in a T-figuration. A new space can be added to the right, to the back or above w_1 – depending on *fix*(w_1) – by inserting a new wall parallel to w_1. Consider the four walls, w_2, w_3, w_4, and w_5 in *bound*(w_1). Suppose w_2 and w_3 are parallel. Then, w_4 and w_5 are parallel. Let *coord*(w_2) < *coord*(w_3) and *coord*(w_4) < *coord*(w_5). Suppose w_1 has bounding lines coincident with walls w_2, w_3, w_4, and w_5 as shown in Figure 6. [Should the walls correspond to one of the other configurations in Figure 4, the relationship between walls w_1, and w_2 through w_5 can be likewise illustrated.

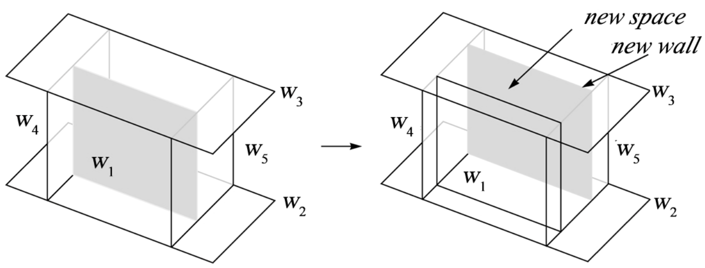


Figure 6: *General rule for creating a new space by inserting a wall*

A new space is created by enclosing it within walls w_1 through w_5 and a new wall parallel to w_1. The new space is bordered by all six walls, and the new plane borders the spaces ‘enclosed’ by the five original walls as illustrated by the right hand side of Figure 6. For the other configurations in Figure 4, the right hand side is accordingly defined.

The following determines the set of spaces ‘enclosed’ by the five walls:

$$B = \{s \in S_\uparrow(w_1) \mid \Delta_\uparrow(s, w_2) \wedge \Delta_\downarrow(s, w_3) \wedge \Delta_\uparrow(s, w_4) \wedge \Delta_\downarrow(s, w_5)\}$$

B is the set of spaces directly to ↑ of w_1 which are simultaneously to ↓ of w_3 and w_5 and to ↑ of w_2 and w_4. After insertion of w_p, these spaces will be directly on side ↑ of w_p with *index p*. [All interior walls are numbered from 1 upwards and thus, are indexed by their number. Exterior bounding walls are numbered from –5 to 0.] The only space directly on side ↓ of the new wall w_p is the new space *p+1*. It is clear that the new space is directly bordered by walls w_2 through w_5 with no changes to the other spaces bordered by these walls. However, the spaces bordered by w_1 and *p* are affected. In other words:

RULE 1 (Simple Insertion)

if *index*(w_1) $\in$ {*right, back, above*}
then *index*(w_p) $\leftarrow$ *index*(w_1)
index(w_1) $\leftarrow$ *p*
else *index*(w_p) $\leftarrow$ *p*

$S_\downarrow(w_p) \leftarrow \{ p{+}1\}$
$S_\uparrow(w_p) \leftarrow B,$
$S_\uparrow(w_1) \leftarrow [S_\uparrow(w_1) - B] \cup \{p{+}1\}$
$S_\uparrow(w_2) \leftarrow S_\uparrow(w_2) \cup \{p{+}1\}$
$S_\downarrow(w_3) \leftarrow S_\downarrow(w_3) \cup \{p{+}1\}$
$S_\uparrow(w_4) \leftarrow S_\uparrow(w_4) \cup \{p{+}1\}$
$S_\downarrow(w_5) \leftarrow S_\downarrow(w_5) \cup \{p{+}1\}$
$bound(w_p) \leftarrow \{w_2, w_3, w_4, w_5\}$

These nine assignments specify a general rule for constructing a wall representation W_{p+1} from a wall representation W_p for certain configurations of walls w_1 through w_5. Note that the index of w_p is p unless w_1 is one of the bounding planes w_{right}, w_{back} or w_{above}, in which case the indices of walls w_p and w_1 are interchanged.

The rule applies to certain other configurations.

Consider any other wall w_j orthogonal to w_1 such that w_j and w_1 intersect at a line. Since walls w_2 through w_5 bound w_1, either (a) w_j is parallel to w_2 and $coord(w_2) < coord(w_j) < coord(w_3)$, or (*b*) w_j is parallel to w_4 and $coord(w_4) < coord(w_j) < coord(w_5)$. Moreover, $w_1 \in bound(w_j)$; otherwise, the 3-rectangulation will not be trivalent.

Consider case (a). That is, suppose w_j is parallel to w_2 (and w_3). Then, there are walls $w_k \in bound(w_j)$ with $w_{k=}w_4$ or $w_{k=}w_5$, or $w_1 \in bound(w_k)$. A similar result holds for case (b) when w_j is parallel to w_4. Therefore:

(3.1) *In any unlocked T-figuration, for each wall w bound by a wall w_b, w is bound by two walls orthogonal to both w and w_b such that each also bounds w_b or is bound by w_b.*

This proposition reinforces the fact that for any T-figuration, the cross-section of the walls that meet w_1 forms a T-plan. See Figure 7.

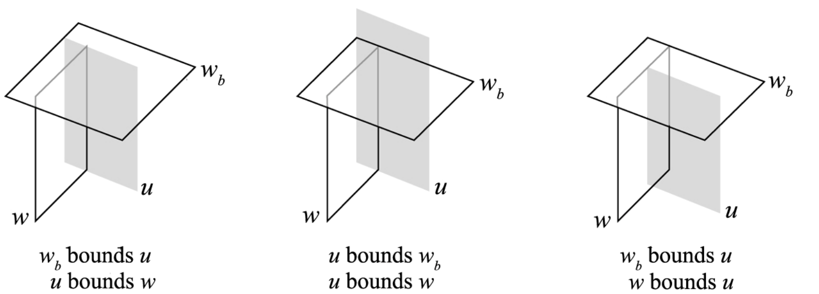


Figure 7: *The spatial situations when a wall w_b bounds wall w*

We introduce the following notation: whenever wall w bounds wall v, we write $w \vdash v$. In other words, $w \vdash v$ implies $w \in bound(v)$ and conversely.

Suppose, in addition, $w_4 \vdash w_j$ and $w_5 \vdash w_j$. Then, the general rule above can be applied in two ways:

- By replacing all occurrences of w_3 in rule 1 by w_j to create the space p+1 bounded by walls w_1, w_2, w_j, w_4, w_5 and w_p.
- By replacing all occurrences of w_2 in rule 1 by w_j to create the space $p+1$ bounded by walls w_1, w_j, w_3, w_4, w_5 and w_p. See Figure 8.

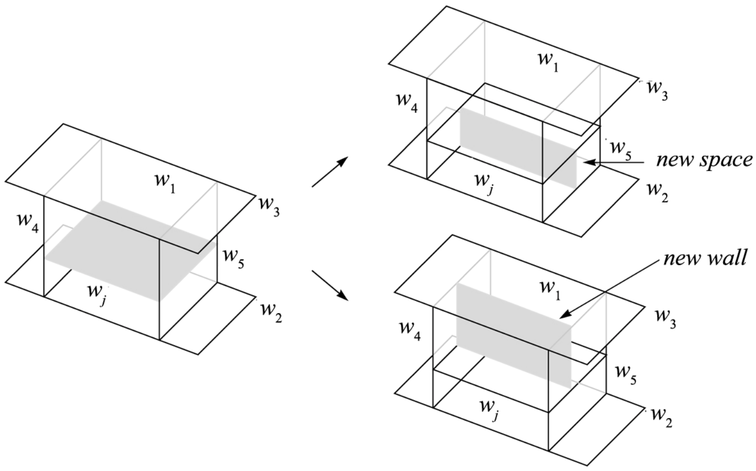


Figure 8: *An instance of the general rule to create a space by inserting a new wall. For illustration, the relative dispositions of w_1 and the new wall have been switched*

In a similar fashion, we can apply the general rule when w_j is parallel to w_4 (and w_5), $w_2 \vdash w_j$ and $w_3 \vdash w_j$. In this case we replace the occurrence of w_4 (or w_5) by w_j in the set expressions.

Suppose there are two walls w_j and w_k both parallel to w_2 and bound by walls w_4 and w_5 that meet w_1 on the same side. Suppose further that $coord(w_j) < coord(w_k)$. Then, the rule can be applied by replacing all occurrences of w_2 and w_3 in the set expressions by w_j and w_k respectively, creating space $p+1$ bounded by walls w_1, w_j, w_k, w_4, w_5 and w_p (see Figure 9).

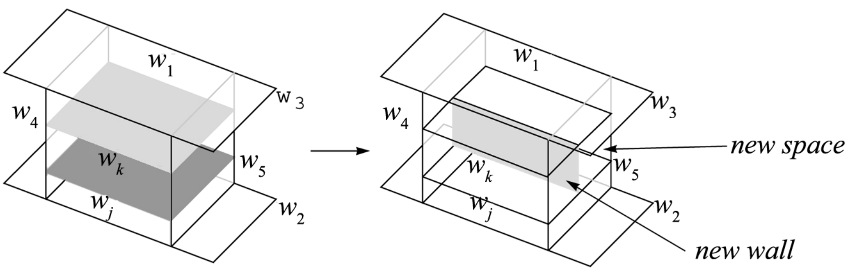


Figure 9: *Another instance of the rule to create a space by inserting a new wall. For illustration, the relative dispositions of w_1 and the new wall have been switched.*

In a similar fashion, we can apply the general rule if w_j and w_k are both parallel to w_4, and bound by walls w_2 and w_3. In this case, occurrences of w_4 and w_5 are replaced by w_j and w_k respectively in the set expressions.

The following case is likewise handled. Suppose there are two walls w_j and w_k such that $w_2 \vdash w_j$, $w_3 \vdash w_j$, $w_j \vdash w_k$, and w_4 (or w_5) $\vdash w_k$. Then, occurrences of w_3 and w_5 (or w_4) are replaced by w_j and w_k respectively. Alternatively, we can treat w_k as the wall to which the rule is applied.

The only case that remains to be considered is when there is a wall w_j that meets w_1 on one side, and a parallel wall w_k that meets w_1 on the other side. In this case a new space that forms a pinwheel arrangement with the spaces that currently border w_1 can be created (see Figure 10). $w_1 \vdash w_j$, $w_1 \vdash w_k$, and both *wj* and *wk* are bound by either w_2 and w_3, or w_4 and w_5. Otherwise, if both w_j and w_k are bound by a common wall other than say, w_2 and w_3, we will have a 3-rectangulation that is not trivalent and thus, a contradiction. Suppose w_j and w_k are both parallel to w_2. Suppose, without loss in generality, that w_j meets w_1 from ↓ (that is, from left, front or below) and w_k from ↑. In this case the rule is described by the following set expressions. Similar expressions can be defined when w_j and w_k are parallel to w_4.

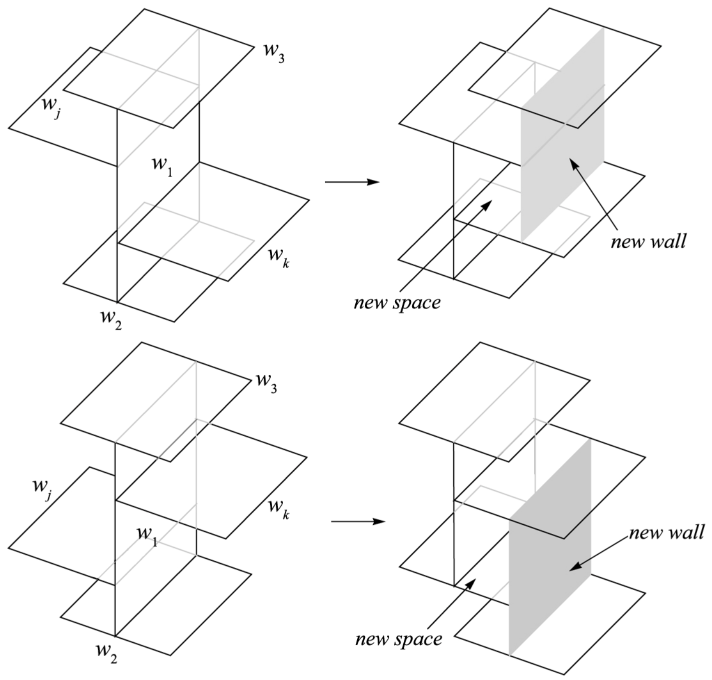


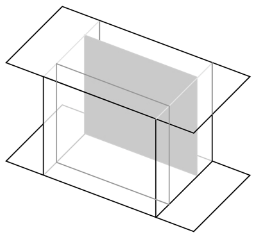
Figure 10: *Examples of the rule for creating a pinwheel arrangement*

RULE 2 (Pinwheel Insertion)

if $coord(w_{\mathrm{j}}) < coord(w_{\mathrm{k}})$
then $B_{\downarrow} \leftarrow \{s \in S_{\downarrow}(w_1) \mid \Delta_{\downarrow}(s, w_j) \wedge \Delta_{\uparrow}(s, w_2) \wedge \Delta_{\uparrow}(s, w_4) \wedge \Delta_{\downarrow}(s, w_5)\}$
$B_{\uparrow} \leftarrow \{s \in S_{\uparrow}(w_1) \mid \Delta_{\downarrow}(s, w_k) \wedge \Delta_{\uparrow}(s, w_2) \wedge \Delta_{\uparrow}(s, w_4) \wedge \Delta_{\downarrow}(s, w_5)\}$
else /* $coord(w_j) > coord(w_k)$ */
$B_{\downarrow} \leftarrow \{s \in S_{\downarrow}(w_1) \mid \Delta_{\uparrow}(s, w_j) \wedge \Delta_{\downarrow}(s, w_3) \wedge \Delta_{\uparrow}(s, w_4) \wedge \Delta_{\downarrow}(s, w_5)\}$
$B_{\uparrow} \leftarrow \{s \in S_{\uparrow}(w_1) \mid \Delta_{\uparrow}(s, w_k) \wedge \Delta_{\downarrow}(s, w_3) \wedge \Delta_{\uparrow}(s, w_4) \wedge \Delta_{\downarrow}(s, w_5)\}$
$index(w_p) \leftarrow p$
$S_{\downarrow}(w_p) \leftarrow B_{\uparrow} \cup \{p+1\}$
$S_{\uparrow}(w_p) \leftarrow B_{\uparrow}$
$S_{\downarrow}(w_1) \leftarrow [S_{\downarrow}(w_1) - B_{\downarrow}]$
$S_{\uparrow}(w_1) \leftarrow [S_{\uparrow}(w_1) - B_{\uparrow}] \cup \{p+1\}$
$S_{\uparrow}(w_2) \leftarrow S_{\uparrow}(w_2) \cup \{p+1\}$
$S_{\downarrow}(w_3) \leftarrow S_{\downarrow}(w_3) \cup \{p+1\}$
$S_{\uparrow}(w_4) \leftarrow S_{\uparrow}(w_4) \cup \{\mathrm{p}+1\}$
$S_{\downarrow}(w_5) \leftarrow S_{\downarrow}(w_5) \cup \{p+1\}$
if $coord(w_{\mathrm{j}}) < coord(w_{\mathrm{k}})$
then $bound(w_p) \leftarrow \{w_2, w_k, w_4, w_5\}$
$bound(w_1) \leftarrow \{w_j, w_3, w_4, w_5\}$
else $bound(w_p) \leftarrow \{w_k, w_3, w_4, w_5\}$
$bound(w_1) \leftarrow \{w_j, w_2, w_4, w_5\}$

The completeness and correctness of the two insertion rules will be established in the next section. However, since each *du* T-figuration has a 3-rectangular space in any one of its corners, a *du* T(p+1)-figuration can always be constructed from a *du* T(p)-figuration by the insertion of a wall, and thus – informally – establishes sufficiency of the simple insertion rule. The pinwheel insertion rule is required when spaces are generated according to some numerical order. In other words, the two rules are sufficient to construct any labelled wall representation from the wall representation of a given T-figuration.

It is convenient to start the generation with a single 3-rectangle bounded by the six exterior planes. By appropriately choosing wall w_1, the rules extend the new space along one of three directions. When the rules are restricted to apply along just two directions, the set expressions correspond to the rules given by Flemming (1978) for generating wall representations in the two-dimensional case. The difference between Flemming's formulation and the one given here is that his directly employs an ordering on the spaces to easily identify the elements of B, the spaces that are shifted when the new wall and space are inserted. The same approach can be adopted here, but would require fairly involved expressions – albeit necessary – which I have avoided for sake of clarity.



Wall incidence graphs for *du* T-figurations

The incidence structure for the walls of a *du* T-figuration is now examined. Let the interior and exterior walls be $\{r_1, r_2, \ldots, r_{p-1}\}$ and $\{r_{above}, r_{below}, r_{left}, r_{right}, r_{back}, r_{front}\}$ respectively. Suppose that the *xy*-oriented enclosing walls, i.e., those with fixed *z* coordinates, bound the other four enclosing walls and the *zx* oriented walls bound the *yz* oriented walls. The six walls divide the region exterior to the enclosed region into six distinct regions.

Definition: A T-figuration R can be represented by a digraph, $WG(R) = (V, E)$, where each vertex $v_j = v(r_j)$ represents wall r_j. Conversely, wall $r_k = r(v_k)$ corresponds to vertex v_k. $V = V_{ext} \cup V_{int}$ where the *exterior* vertices are given by $V_{ext} = \{v_{above}, v_{below}, v_{left}, v_{right}, v_{back}, v_{front}\}$ and *interior* vertices are given by $V_{int} = \{v_k \mid 1 \leq k \leq p{-}1\}$. An edge, $(v_i, v_j) \in E$, is *incident* from v_i to v_j if and only if r_i and r_j form a base cross-piece pair; that is, whenever $r_j \vdash r_i$. Vertices u and v are *adjacent* whenever $(u, v) \in E$, or $(v, u) \in E$. $WG(R)$ is the *wall incidence graph* – *wig*, for short – of T-figuration R.

$WG(R)$ has the following properties: (1) Each interior wall is bound by four walls; that is, for $v \in V_{int}$, outdegree$(v) = 4$. (2) No interior wall bounds an exterior wall; for $v_e \in V_{ext}$ *and* $v_i \in V_{int}$, $(v_e, v_i) \notin E$. (3) The subgraph induced by the exterior vertices, $G_{ext} = (V_{ext}, (V_{ext} \times V_{ext}) \cap E)$, is shown in Figure 11.

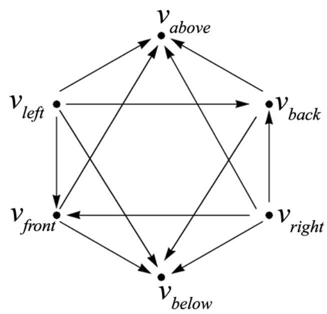


Figure 11: *Graph induced by the exterior vertices*

(4) Each T(p)-figuration has p–1 interior walls and 6 exterior walls. The outdegrees of the exterior vertices are 4, 4, 2, 2, 0 and 0. Thus, $WG(R)$ has p+5 vertices and 4p+8 edges. (5) The vertices can be coloured according to the orientation of their corresponding walls such that no two adjacent

vertices are coloured the same. That is, each vertex $v(r)$ can be assigned the colour $fix(r)$; or, $colour(v(r)) = fix(r)$. No edge connects vertices of the same colour. Of the four edges incident from an interior vertex, two are to vertices of one colour and the other two to vertices of another colour. That is, $WG(R)$ is vertex 3-colourable. (6) Each interior vertex v is associated with 4 vertices that belong to $bound(v)$. These vertices form a weakly connected closed path consisting of vertices of just two colours, which are termed as a *weak 4-cycle*. v is the center of an outwardly weakly connected wheel, denoted by W_v, formed by the weak 4-cycle of its bounding vertices. See Figure 12.

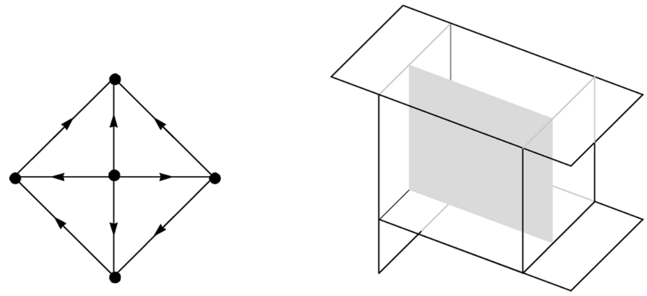


Figure 12: An *outwardly connected wheel of order 4 at a vertex.*

(7) For $v \in V_{int}$, W_v is unique. There are sixteen possible configurations for W_v, which fall into four equivalence classes under rotation and reflection. (8) A weak cycle is *strong* if it is a directed cycle. There are no strong cycles of odd length in $WG(R)$. (9) There are no two strong 4-cycles in $WG(R)$ that share exactly three vertices.

In general, strong cycles in a wall incidence graph can share vertices as illustrated in Figure 13, where two strong cycles share three walls, highlighted by bold black lines.

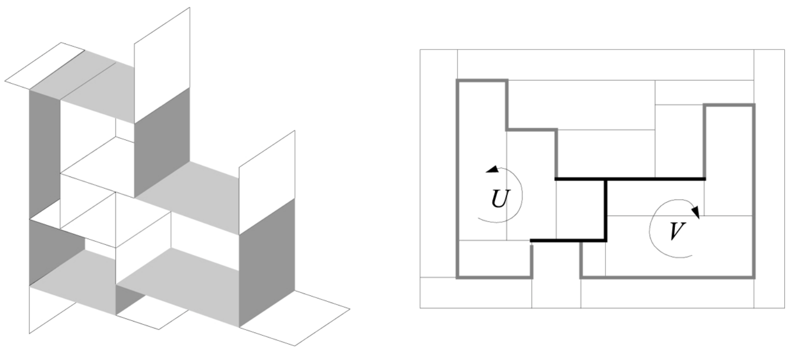


Figure 13: *Two strong cycles that share three walls*

Figure 14 illustrates two strong 4-cycles that share two walls. Note: strong cycles that share vertices are always oriented in opposite directions.

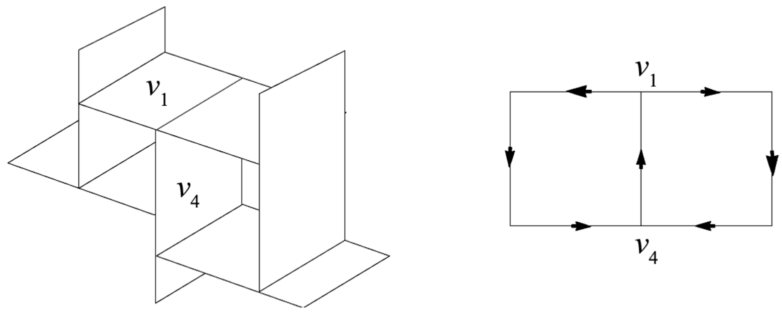


Figure 14: *Strong 4-cycles that share two common walls (v_1 and v_4)*

(4.1) *Every du T-figuration can be produced by simple insertion and pinwheel insertion rules.*

Outline of proof: The proposition is established by induction on the number of vertices. The proposition is clearly true for $G(1)$. Suppose it is true for all $G(q)$, $q \leq p$. Consider any $G(p+1)$.

There are two cases to consider.

(i) *The graph is acyclic.*

The graph has a vertex v with degree = 4. Moreover, the vertices adjacent to v have indegree > 0. Removal of v and all its incident edges (which number exactly 4) will result in a graph with $p+5$ vertices and $4p+8$ edges. The resulting graph – by the induction assumption – satisfies the properties of a wall incidence graph. In other words, it is a $G(p)$. Or, $G(p+1)$ is produced from $G(p)$ by simple insertion.

(ii) *The graph has a strong cycle.*

The graph has a strong 4-cycle on two colours. Moreover, there is a strong 4-cycle with vertices, in cyclic order, v_1, v_2, v_3, and v_4 such that there are two vertices v_5 and v_6 of the third colour to which all four vertices are incident. Let u_j denote the fourth edge incident from v_j, $1 \leq j \leq 4$.

$v_1, \ldots, v_6$ partition the remaining vertices into two classes, say V_1 and V_2, where the vertices in V_1 are outside the 4-cycle and those of V_2 are inside.

In principle, by removing V_2 and associated inside edges we will have a $G(p')$, $p'<p+1$, which by induction can be constructed by the simple and pinwheel insertion rules. There are two cases to consider.

(a) See Figure 15. If V_2 is empty, then by identifying, say v_1 and v_3, into a single vertex, removing the edges (v_1,v_2), (v_3,v_4), (v_3,v_5), (v_3,v_6), and replacing edges (v_2,v_3) and (v_3,u_3) by (v_2,v_1) and (v_1,u_3) respectively, and combining adjacency lists of v_1 and v_3, we produce the required graph $G(p')$.

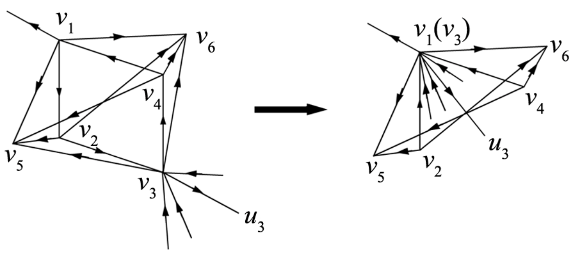


Figure 15: *Illustrating step (a) in the induction step in the proof of proposition (4.1) (Inside of the closure of the strong 4-cycle is empty)*

(b) See Figure 16. If V_2 is not empty, its vertices and associated edges are removed from the original graph and we can then apply case (a) above to the resulting graph. Furthermore, the vertices in V_2 and its associated edges together with six new vertices, designated as exterior vertices, arranged according to Figure 11 and labelled as v_{above}, v_{left}, … according to the original vertex colours of v_1, v_2, …, v_6, define an $G(p'')$, $p''<p+1$, which by induction can be generated by the simple and pinwheel insertion rules. ❑

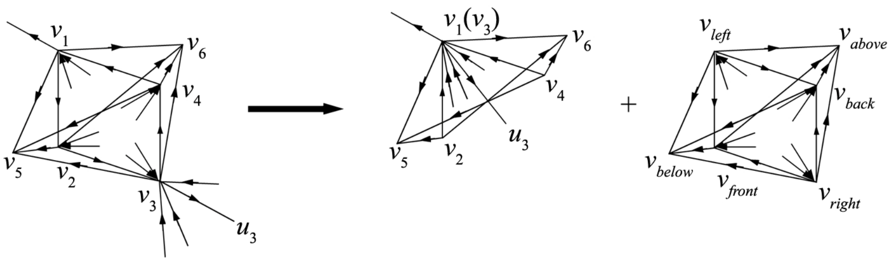


Figure 16: *Illustrating step (b) in the induction proof of proposition (4.1) (Inside of the closure of the strong 4-cycle is not empty)*

ALGORITHM TO REALISE A T-FIGURATION IN STANDARD FORM

The proof for (4.1) can be used to realise $G(p)$ as a T(p)-figuration.

Step 1: Partition the vertices according to their spatial orientation by a depth-first graph search. In the description below, $\veebar$, $\wedge$ and $\neg$ denote disjunction, conjunction, and negation respectively. Start the search at a distinguished vertex and initialize its colour to $\{x\}$. For each vertex w adjacent to v, the current vertex, one of two possibilities can occur:

- w has not been previously visited: w cannot have the same colour as v. Set $colour(w) \leftarrow \neg\, colour(v)$[1] and recursively search from w.
- Otherwise, w has been previously visited: v cannot have the same colour as w. In this case, set $colour(v) \leftarrow colour(v) \wedge \neg\, colour(w)$.

The procedure assigns two possible colours to a vertex when it is first visited. Subsequent visits either leave its colour unchanged or colours it uniquely. Since $G(p)$ is vertex 3-colourable and each vertex is adjacent to vertices of two distinct colours, an instantiation of vertex colouring is always possible. Since the graph is connected and the procedure is a depth-first search of the undirected version, each edge is traversed exactly twice. In other words, the procedure produces a valid colouring of the vertices. It can be readily modified to count the number of x, y, and z vertices, denoted by n_x, n_y, and n_z respectively.

Step 2: Assign descriptors to the vertices each coplanar with a grid plane. Each vertex v can be uniquely identified by an equation of the form $colour(v) = coord(v)$. The enclosing walls have equations given by $x = 0$, $x = n_x-1$, $y = 0$, $y = n_y-1$, $z = 0$ and $z = n_z-1$. In addition, each vertex corresponds to a rectangular wall, the min and max coordinates of which are specified by the equations of the planes that bound it. Initially, set: $min_x \leftarrow min_y \leftarrow min_z \leftarrow 0$, $max_x \leftarrow n_x-1$, $max_y \leftarrow n_y-1$ and $max_z \leftarrow n_z-1$. This step depends on the nature of the graph.

(*a*) *Acyclic graphs.*

For an acyclic $G(p)$ there is at least one interior vertex bounded by exterior vertices; that is, there is a wall w bound by four exterior walls. For ease of argument, suppose w to be an x-wall. Split the graph into two graphs such that w is external in both. Let the graph with external x-wall with coordinate min_x have c_x x-walls. Set $coord(w) = min_x + c_x - 1$. Repeat the procedure for each graph, in one setting max_x to equal the coordinate of w and in the other setting min_x to equal it, until there are no unassigned interior walls. A y- or z- coloured slicing wall is similarly treated. The T-figuration thus realised has no two vertices corresponding to coplanar walls.

(*b*) *Graphs with strong cycles.*

Each strong 4-cycle C in splits the graph into two graphs: one is the current graph replaced by three 'tagged' vertices (explained below), and the other is formed by $c(C)$ and all vertices inside it. Suppose the 4-cycle C is given by v_1, v_2, v_3,, and v_4. Then, either (i) v_1 and v_3 are identified as v_{13}^C and renaming v_2 and v_4 as v_2^C and v_4^C respectively, or (ii) v_2 and v_4 are identified as v_{24}^C and renaming v_1 and v_3 as v_1^C and v_3^C respectively. Suppose (i). In this case, we replace edge (v_1, v_2) by (v_2^C, v_{13}^C) and (v_3, v_4) by (v_4^C, v_{13}^C). All edges

[1] Vertex colour is assumed to be represented by a 3-bit number, and its arithmetic by bitwise-and and -or. Thus, an x vertex is represented by 001, a y vertex by 010, a z vertex by 100, $(x \vee y \vee z)$ by 111, the colour $\neg\, x$ by 110 and so on.

incident on v_1 and v_3 are now incident on $v_{13}{}^C$. All interior vertices of $c(C)$ are removed from the original graph. $c(C)$ and all vertices inside it form the second graph.

Each tagged vertex is associated with the index of the cycle that is found. A vertex may be tagged by the indices of more than one cycle. Eventually all the graphs will be acyclic. We can then assign coordinates to the vertices using procedure (a).

We now consider the strong cycles in reverse order of their detection. For each cycle C, we reassign the coordinates of the vertices of the graphs determined by $c(C)$. This is equivalent to adding some constant to the currently assigned vertex coordinates. Given the coordinates of $c(C)$ we can add a constant to the coordinates of vertices with currently assigned coordinates higher than the tagged vertices in the original graph. Suppose $coord(v_2{}^C) < coord(v_4{}^C)$. Let the difference $coord(v_4{}^C) - coord(v_2{}^C) = k$. That is, there are k–1 vertices with *fix* value equal to $fix(v_2{}^C) = f$. We may assume that these are on the same side of $v_{13}{}^C$ as $v_4{}^C$. If not, we can renumber the vertices on the same side as $v_2{}^C$, which have higher *coord* values than $v_4{}^C$. Suppose there are m vertices in $c(C)$ with *fix* value f. We add m-2 to the *coord* values of vertices v with $fix(v) = f$ and $coord(v) \geq coord(v_4{}^C)$. We also add $coord(v_2{}^C) + k - 2$ to interior vertex in $c(C)$ with *fix* value equal to f. This is illustrated in Figure 17.

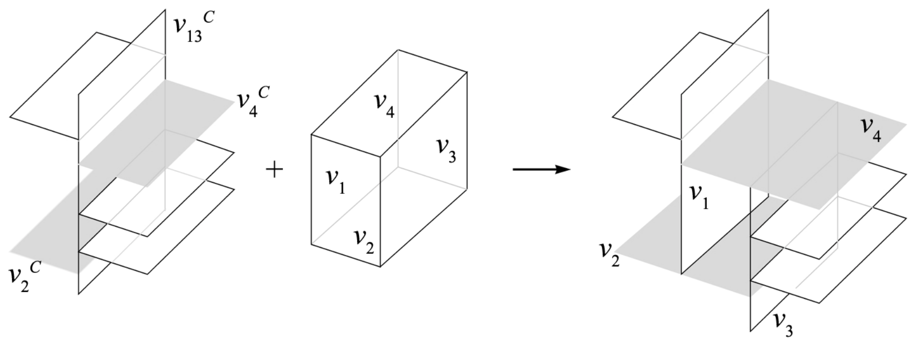


Figure 17: *Replacing three tagged vertices and a T-figuration by a 4-cycle in the process of realizing a T-figuration*

Vertices in the other orientations are similarly treated. Each tagged vertex is untagged by the index of C. Eventually, when all the strong cycles have been considered, a finite process, we have an assignment of vertex coordinates that again corresponds to a T-figuration in standard form. ❑

From dense to loose configurations

We have just seen that a *du* T-figuration can be assigned coordinates in which each wall is aligned with one of the coordinate planes. Let the min and max wall coordinates be denoted by $(x_{min}, y_{min}, z_{min})$ and $(x_{max}, y_{max}, z_{max})$ respectively.

ORTHOGONAL SPATIAL RELATIONS ON 3-RECTANGLES

There are six basic *orthogonal relations* which can be defined between pairs of 3-rectangles, *a* and *b*, not all that hold simultaneously.

$above(a, b) \Leftrightarrow zmin(a) \leq zmax(b)$

$below(a, b) \Leftrightarrow above(b, a) \Leftrightarrow zmax(a) \leq zmin(b)$

Likewise in the *y*- and *x*-directions, the relations *back*, *front* and *left*, *right* respectively taken in pairs can be defined. These relations are transitive; that is, $above(a, b) \wedge above(b, c) \Rightarrow above(a, c)$.

$directly\text{-}above(a, b) \Leftrightarrow$
$above(a, b) \wedge [\forall r \neq a, b \neg (above(a, r) \wedge above(r, b))]$

The relations *directly-below*, *directly-back*, *directly-front*, *directly-left*, and *directly-right* are similarly defined. The relation *above* is the transitive closure of the *directly-above* relation. Likewise, the other orthogonal relations are transitive closures of their respective *directly-* counterpart. These relations can be employed to define relations that partition the region exterior to a given 3-rectangle in other ways. For example,

$strictly\text{-}above(a, b) \Leftrightarrow$
$above(a, b) \wedge \neg [right(a, b) \vee left(a, b) \vee back(a, b) \vee front(a, b)],$

specifies a region exterior to a 3-rectangle in which only the above orthogonal relation holds and no other. $\neg$ denotes negation.

NOTATION

D denotes the set of the six *directly*-orthogonal relations, and D^*, the set of powers of elements of *D*. That is: $\delta^* \in D^*$ if and only if $\exists\ \delta \in D$, an integer $k > 0$ and $\delta^* = \delta^k$. Let $\Re$ be the set of the six orthogonal relations. For each $\delta \in D$ there corresponds a general orthogonal relation $\Delta \in \Re$, and for each $\Delta \in \Re$, there is a corresponding *directly-* relation $\delta \in D$. That is, for any two 3-rectangles u, v and relation δ, $\delta^*(u, v) \Rightarrow \exists\ \Delta \in \Re,\ \Delta(a, b)$. Conversely, $\Delta(a, b) \Rightarrow \exists\ \delta^* \in D^*,\ \delta^*(a, b)$. Thus, for any $\delta \in D$ acting on a set of 3-rectangles, $\bigcup \delta^* = \Delta$.

The *inverse relation* Δ^{-1} of a relation Δ is defined by the following table:

Δ	left	right	front	back	below	above
Δ^{-1}	right	left	back	front	above	below

The inverse notation applies to δ. For ease, the following short-hand is employed: $\delta_r \equiv \delta_{right} \equiv$ *directly-right*, $\delta_b \equiv \delta_{back} \equiv$ *directly-back,* and $\delta_a \equiv \delta_{above} \equiv$ *directly-above.* The other three relations are denoted as inverses of these. The subscript notation extends to Δ.

Orthogonal relations are equivalent to elements of the cyclic group c_6:

σ	front	back	below	above	right	left

Then, $\sigma^3\Delta = \Delta^{-1}$. With respect to a wall, $\exists\ \kappa, 0 \le \kappa \le 5$, $\varphi = \sigma^k\Delta$ represents the *right* relation. Then, $\{\varphi, \sigma\,\varphi, \sigma^2\,\varphi\}$ represent the $\uparrow$ side of the walls, and, with $\varphi^{-1} = \sigma^3\varphi$, the relations $\{\varphi^{-1}, \sigma\,\varphi^{-1}, \sigma^2\,\varphi^{-1}\}$ the $\downarrow$ side.

REGION ADJACENCY GRAPH OF A T-FIGURATION

Definition: For any T(p)-rectangulation, R_p, a labelled digraph on p vertices, $G_p = G(R_p) = (V_p, E_p)$, can be defined as follows: each vertex represents a 3-rectangle and edges satisfy the condition: $(a, b) \in E_p \Leftrightarrow [\delta_a(b, a) \vee \delta_b(b, a) \vee \delta_r(v, u)]$. Equivalently, $(u, v) \in E_p \Leftrightarrow [\delta_a^{-1}(a, b) \vee \delta_b^{-1}(a, b) \vee \delta_r^{-1}(a, b)]$. Here, a and b denote both the vertices and 3-rectangles that these represent. G_p is the *region adjacency graph – rag*, for short – of R_p.

Let λ denote an edge labelling function. Then, $\lambda(a, b) \equiv$ *right* $\Leftrightarrow \delta_r(b, a)$, $\lambda(a, b) \equiv$ *back* $\Leftrightarrow \delta_b(a, b)$ and $\lambda(a, b) \equiv$ *above* $\Leftrightarrow \delta_a(b, a)$. An edge (a, b) in G_p indicates the presence of a wall w between a and b in R_p. That is, if (a, b) is an edge in G_p, then either $a \in S_\downarrow(w)$ and $b \in S_\uparrow(w)$, or $b \in S_\downarrow(w)$ and $a \in S_\uparrow(w)$. The edge label can be used to identify the orientation of the wall.

Consider the neighbourhood of 3-rectangles around joints of types 1, 2 and 3 shown in Figure 18.

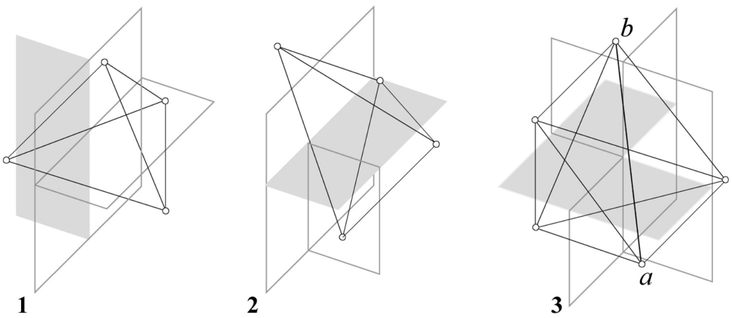


Figure 18: *Neighbourhood of 3-rectangles around a joint*

For unlocked T-figurations – that is, those with joints of types 1 and 2 – there is at most one directed edge between any two interior vertices. For locked T-figurations, there are at least two vertices with three labelled edges between them. For the orientation shown, $\delta_r(b, a)$, $\delta_b(a, b)$ and $\delta_a(a, b)$ hold. In all cases, the graph is weakly cyclic.

Consider the subgraph around a type 3 joint, involving the five vertices. There are four directed edges in each of the three labels. Of the vertices corresponding to interlocking 3-rectangles, there is always one which has two edges directed towards it and two directed away from it, of the other two, one has four (three) edges directed away from it, and the other has the same number of edges directed towards it. *a* and *b* are each incident to six edges, *a* has two (or three) edges directed towards it, and *b* has four (or three) edges directed towards it. This pattern occurs in all possible orientations of a type 3 joint.

REACHABILITY PROPERTIES FOR T-FIGURATIONS

In this section I enunciate a property that relates the orthogonal relations to the constituent 3-rectangular spaces of a T-figuration. Let *a* be an arbitrary 3-rectangle within R_p. For each *a*, define the sets:

$$S_\Delta(a) = \{b \mid \Delta(b, a)\},\ \delta \in D.$$

S_Δ specifies a reachability function in the direction of Δ. That is, if $b \in S_\Delta(a)$, then there exists a trail from *a* to *b* along Δ.

Since each internal 3-rectangle is bounded by six walls, it follows that there are at least six distinct neighbouring spaces directly related to it. [By assigning a vertex to the region exterior to the T-figuration, we can deal with those 3-rectangles that neighbour onto the exterior.]

Secondly, suppose *a* and *b* are two internal 3-rectangles such that the relation $\Delta(b, a)$ holds. Suppose *c* is a space and $\Delta(c, b)$ holds. Since Δ is transitive, it follows that $\Delta(c, a)$ holds. In other words, $S_\Delta(b) \subseteq S_\Delta(a)$. That is, if there is a trail from *a* to *b*, then there is trail from *a* to all vertices to which there is a trail from *b*. Thus,

(5.1) *For each $\delta \in D$, for each constituent 3-rectangle $a \in R_p$,*
$b \in S_\Delta(a) \Rightarrow S_\Delta(b) \subseteq S_\Delta(a)$.

(5.2) *For any $\delta \in D$, for each constituent 3-rectangle $a \in R_p$,*
$S_{\Delta^{-1}}(a) = \{b \mid a \in S_\Delta(b)\}$

The following proposition, stated without proof, shows that from any interior vertex in the graph of a *du* T-figuration, every other vertex is reachable by a trail or a reverse trail.

(5.3) *Suppose R_p is dense. Then, for each internal 3-rectangle $a \in R_p$,*
$$\bigcup_{\delta \in D} S_\Delta(a) = R_p - \{a\}.$$
Moreover, if R_p is unlocked, the sets $S_\Delta(a)$ are pairwise disjoint.

TAGGED RELATIONS AND TRAILS

The proposition (5.3) for T-figurations is similar to the *structure property* that Flemming (1980: Theorem 3) enunciates for T-plans. While (6.3) is tight for *du* T-figurations, it is unsatisfactory for *dl* T-figurations, owing to the non-disjointedness of the structural sets. This is due to the fact that there are the multiple relations between the non-interlocking rectangles around a type 3 joint. We can get round this situation by considering the adjacency relationship in the following manner.

Consider a type 3 joint where a and b are the non-interlocking 3-rectangles. Without loss in generality, we may suppose that $\delta_r(b, a)$ holds. We express this by a tagged edge (a, b, *tag*) where *tag* is a 3-digit binary number $r\ b\ a$ where $\kappa = 1$ if $\delta_\kappa(b, a)$ holds and 0 otherwise. Thus, the region adjacency graph can always be described by at most one directed edge, tagged or otherwise, between any pair of vertices. For ease, for a tagged edge has no edge label. The region adjacency graph so specified is weakly cyclic. Note that for each tag value, there are two possible configurations of the interlocking 3-rectangles.

If we now consider structural sets in terms of just 'untagged' relationships, the structural sets, once again, are pairwise disjoint.

ORTHOGONAL STRUCTURES

The region adjacency graph – where the edges are tagged or untagged – does not take into consideration the exterior of the T-figuration. This can be rectified by associating a vertex E with the exterior space. All other vertices are then deemed as *interior.* Suppose E is specified by its 'inverse' coordinate in the sense that its minimum coordinates are given by $(x_{max}, y_{max}, z_{max})(E)$ and its maximum coordinates by $(x_{min}, y_{min}, z_{min})(E)$. We can then specify edges (E, a) with label $\lambda \in \{right,\ back,\ above\}$ if $\delta_\lambda(E, a)$ holds and edges (a, E) if $\delta_\lambda(a, E)$ holds. In this way, all 3-rectangles on the boundary of the enclosing 3-rectangle are joined by edges to the exterior space.

Definition: A sequence $v_1, v_2, \ldots, v_k$, $k \geq 2$, of distinct vertices defines a *trail* if and only if each successive pair of vertices (v_i, v_{i+1}), $0 < i < k$, in the sequence is an edge in the graph and each edge of the trail is labelled the same. A sequence of distinct vertices defines a *reverse trail* if the reverse of the sequence defines a trail. A trail *exists* between a pair of vertices if one is the first vertex and the other is the last vertex of the trail. A trail is *simple* if there is a directed edge between the first and last vertex in the sequence, and *nonsimple* otherwise. Two trails are *parallel* if they have the same label, and if they are between the same pair of vertices and they differ in at least one vertex in their sequences. Two trails are *perpendicular* if they have the distinct labels, and if they are between the same pair of vertices and they differ in at least one vertex in their sequences. For any trail $v_1, v_2, \ldots, v_k$, $k \geq 2$ defined by $\delta \in D$, for each pair of vertices v_i, v_j, $0 < i < j \leq k$, the relationships $\Delta(v_j, v_i)$ and $\Delta^{-1}(v_i, v_j)$ hold.

The augmented adjacency graph, $AG(R_p)$, with $p+1$ vertices has the following properties. Let R_p denote the set of internal and external spaces of a T(p)-figuration. Then,

(i) $\forall a \in R_p, \forall \delta \in D, E \in S_\Delta(a)$

In other words, the exterior is reachable by a trail (or reverse trail) from any vertex in each of the six directions.

(ii) $\forall a \in R_p, \forall \delta, \delta' \in D, \delta \neq \delta', \bigcup S_\Delta(a) = R_p - \{a\}$

If R_p is *du*, $S\Delta(a) \cap S\Delta'(a) = \{E\}$. That is, every other vertex is reachable by a trail (or reverse trail) from any vertex. Moreover, if R_p is unlocked, all trails between two interior vertices must be parallel.

Definition: A graph is an *orthogonal structure* if and only if

(i) for each pair of interior vertices, there is at least one trail between them and all trails between them are either parallel or perpendicular;
(ii) for every interior vertex and the exterior vertex, there is at least one trail in each of the three labels between them; and
(iii) all parallel trails have at least three vertices.

Condition (i) states that: if a and b are two distinct interior vertices, $\exists \delta \in D$, $\Delta(a, b)$, and $\forall \delta' \in D$, $\delta \neq \delta'$, either $\neg \Delta'(a, b)$ or $[\Delta'(a, b)$ or $\Delta'^{-1}(a, b)]$ Condition (ii) states that: for any interior vertex a, $\forall \delta \in D$, either $\Delta(a, E)$ or $\Delta^{-1}(a, E)$ must hold. Condition (iii) states that: any two distinct relationships of the same kind, between the same pair of 3-rectangles, differ transitively by at least one 3-rectangle.

(5.4) *Every loose T-figuration can be represented by an orthogonal structure and conversely.*

Proof: We follow Flemming's proof for T-plans (Flemming 1981: Theorem 1). Every loose T-figuration can be made dense by suitably filling the holes by 3-rectangles in such a way that the 3-rectangulation is trivalent and unlocked. The graph of this dense 3-rectangulation can be defined by directed edges between 3-rectangles that share a maximal plane according to the *directly*- relations (δ_r, δ_b or δ_a). By the structure property and the definition of the minimum and maximum coordinates of the exterior vertex, it follows that the graph satisfies conditions (i), (iii) and (iii) above.

For every trail <*u, v, w*> such that *v* represents a filled hole, it can be replaced by a directed edge (*u,w)*. Repeating this process and removing all multiple edges between the same pair of vertices results in a graph that satisfies the conditions for an orthogonal structure which represents the loose arrangement.

The converse is established by assigning to each rectangle *v*, its maximum coordinates $(x_{max}, y_{max}, z_{max})(v)$, given by 1+ the length of the longest trails from E_{left} to *v*, E_{front} to *v,* and E_{below} to *v* respectively. By condition (i) the rectangles do not overlap. By condition (ii), there is a trail of each colour between each interior vertex and the exterior vertex *E.* [Note that it does not alter the argument if *E* is considered as split into six exterior vertices E_{left}, E_{front}, E_{below}, E_{right}, E_{back} and E_{above}.] . ❑

Since dense T-figurations are a special case of loose T- figurations:

(5.5) *Every dense T-figuration can be represented by an orthogonal structure and conversely.*

An orthogonal structure does not represent all possible orthogonal spatial relations that can exist simultaneously between a pair of 3-rectangles. For example, a constituent 3-rectangle may be simultaneously above and to the right of another. Nonetheless, these structures have interesting properties, and the reasons, given by Flemming (1986; 1988; 1989 [in regard to the LOOS system]) for considering orthogonal structures in two dimensions, I believe, apply equally to the three dimensional case.

Epilogue

There is still work on 3-dimensional arrangements left unfinished, but for a variety of reasons it has not been possible for me to carry out further examination at the present time. For example, one could look at properties of orthogonal structures and compare rules for generating *lu* arrangements with those for *du* arrangements. Locked T-figurations, on the other hand, perhaps, merit an article on their own. Nonetheless, if one were to examine the incidence structure of the 3-rectangles around a joint, we would observe that a type 3 joint can be obtained by edge contraction from a type 7 joint (see Figure 19). It is interesting to note that these are the only joints that occur as enantiomorphs.

A type 7 joint can be replaced by an arrangement of walls that form two joints of type 2 and one joint of type 1. There are four possible ways that a type 7 joint can be so converted (see Figure 20). In examining the corresponding incidence structures of 3-rectangles arranged around the joint, if the edges shown dotted are removed and emboldened edge contracted, we obtain a type 3 joint, and hence a locked T-figuration.

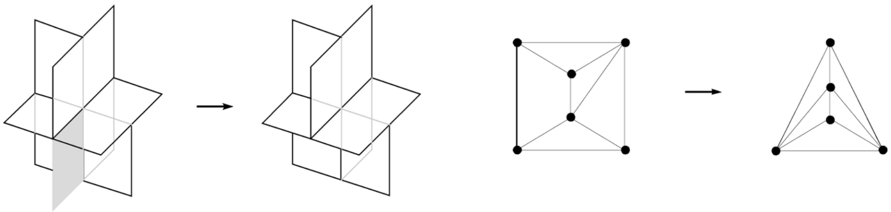


Figure 19: *Transforming a type 7 joint to a type 3 joint by edge contraction*

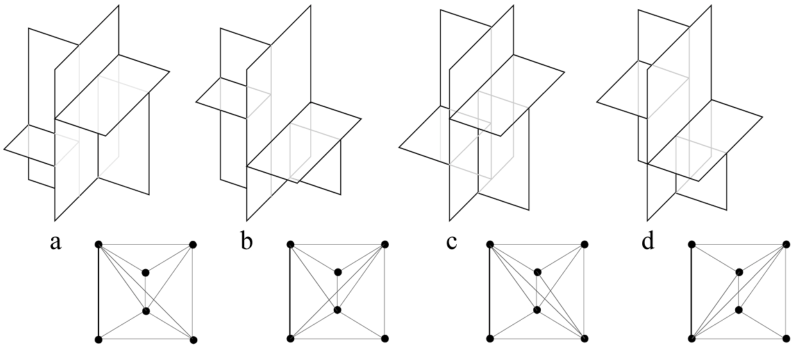
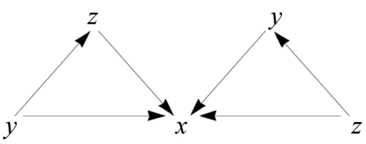


Figure 20: *Conversions of type 7 joint and corresponding incidence structure*

Now consider the wall adjacencies around a type 7 joint replaced by types 1 and 2 joints by the rules in Figure 3. We obtain the subgraph shown to the left. To obtain a type 3 joint, we would have to identify the vertices labelled *y* and the vertices labelled *z* to obtain a graph structure. The difficulty here is that the vertices now represent walls that are not rectangular; consequently, the resulting graph structure will not have the properties of a normal wall incidence graph. To develop a wall incidence structure for locked T-figurations would require a different kind of representation for the geometry of the walls.



IN APPRECIATION – UJF

It is fitting to end where I started – with due mention and acknowledgement of Ulrich Flemming and a body of his work. His contribution to the area of spatial layouts has been tremendous and his influence, considerable. In applying his method, I hope to have conveyed a measure of this accomplishment. All told, a lot has been achieved, some still remains to be done, and much has he inspired.

Acknowledgements

The subject matter of this paper has been of interest to me in many variations and over many years, during which time, the material and with it, my understanding, has undergone change, revision, and refinement.

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References

Choi B, and U Flemming (1996) Adaptation of a layout design system to a new domain: construction site layouts *in* J Vanegas, P Chinowsky (eds) *Proceedings of the 3Rd Congress on Computing in Civil Engineering* (III-CCCE), ASCE, 711-717.

Earl CF (1978) Joints in two- and three-dimensional rectangular dissections, *Environment and Planning B: Planning and Design* **5**, 179-187.

Fenves S, U Flemming, C Hendrickson, ML Maher, R Quadrel, M Terk and R Woodbury (1994) *Concurrent Computer-Integrated Building Design*, Prentice Hall, New Jersey.

Flemming U (1978) Wall representations of rectangular dissections and their use in automated space allocation, *Environment and Planning B,* **5**, 215-232.

Flemming U (1979) Representing an infinite set of solutions through a finite set of principal options *in* A Seidel, S Danford (eds) *Proceedings of the 10th Conference of the Environmental Design Research Association*, Buffalo, NY, 190-197.

Flemming U (1980) Wall representations of rectangular dissections: additional results, *Environment and Planning B*, **7**, 247-251.

Flemming U (1986) On the representation and generation of loosely packed arrangements of rectangles, *Environment and Planning B,* **13**, 189-205.

Flemming U (1988) Rule-based systems in computer-aided architectural design *in* MD Rychener (ed) *Expert Systems for Engineering,* Design Academic Press, New York, 93-112.

Flemming U (1989) More on the representation and generation of loosely packed arrangements of rectangles, *Environment and Planning B,* **16**, 327-359.

Flemming U (1990) Knowledge Representation and Representation in the LOOS System *Building and Environment*, **25**(3), 209-219.

Flemming U, C A Baykan, RF Coyne and MS Fox (1992) Hierarchical generate -and-test vs constraint-directed search: a comparison in the context of layout synthesis *in* J Gero (ed) *Artificial Intelligence in Design'92,* Kluwer, Dordrecht, 817-838.

Flemming U, and R Woodbury (1995) Software Environment to Support Early Phases in Building Design (SEED): Overview, *Journal of Architectural Engineering,* **1**, 147-152.

Galle P (1990) A language for abstract floor plans, *Environment and Planning B: Planning and Desi*gn, **17**, 173-204.

Krishnamurti R, and CF Earl (1998) Dense rectangulations, *Environment and Planning B: Planning and Design* **25**, 773-787.

Kundu S (1988a) The Equivalence of the Subregion Representation and the Wall Representation for a Certain Class of Rectangular Dissections, *Communications of the ACM*, **31** (6), 752-763.

Kundu S (1988b) An optimal O(n) algorithm for determining the subregion representation of a rectangular dissection from its wall representation, Tech. Report 88-029, Department of Computer Science, Louisiana State University, Baton Rouge.

Preas BT, and WM van Cleemput (1979) Placement algorithms for arbitrary shaped blocks, *Proceedings of the 16th Design Automation Conference*, San Diego, ACM, IEEE, SIGDA, 474-480.

Rutenbar R (1989) Simulated Annealing Algorithms: An Overview, *IEEE Circuits and Devices Magazine*, 19-26.

Supowit KJ, and EA Slutz (1984) Placement algorithms for VLSI, *Computer-Aided Design*, **16** (1), 45-50.

Szykman S, and J Cagan (1997) Constrained three-dimensional component layout using simulated annealing, *ASME Journal of Mechanical Design*, **119** (1), 28-35.

Wong DF, and CL Liu (1987) Floorplan Design for Rectangular and L-shaped Modules, *Proceedings of the IEEE International Conference on Computer-Aided Design*, 520-523.