Probability Fact Sheets

Combinatorics

The number of ways of choosing k objects from n, with repetitions, where the order counts, is n^k.

The number of ways of choosing k objects from n, without repetitions, where the order counts is

$$\frac{n!}{(n-k)!} = n(n-1)\cdots n-(k-1).$$

This is called the number of *permutations* of k objects from n.

The number of ways of choosing k objects from n, without repetititions, where the order doesn't count, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots n-(k-1)}{k(k-1)\cdots 1}$$

This is called the number of *combinations* of k objects from n, and happens to be the kth entry of the nth row of Pascal's triangle (assuming you start counting with 0).

The binomial theorem:

$$\begin{aligned}(x+y)^n &= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \\ &= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots \binom{n}{n-1} x y^{n-1} + y^n\end{aligned}$$

Probability Measures

Let X be the set of possible outcomes of an experiment. Subsets of X are called *events*. If A and B are events, then

- $A \cup B$ is the union of the two events (that is, the outcomes that are in either A or B)
- $A \cap B$ is the intersection of the two events (that is, the outcomes that are in both A and B)

- $\bar{A}$ is the complement of A (that is, the outcomes that are not in A)

Events A and B are *incompatible* (or "mutually exclusive," or "disjoint") if $A \cap B = \emptyset$, that is, there is no outcome common to both events.

Definition: A (finitely additive) *probability measure* on X is a function P that assigns a real number between 0 and 1 to subsets of X (that is, elements of some field of subsets of X), satisfying the following:

- $P(X) = 1$
- If A and B are incompatible events, then $P(A \cup B) = P(A) + P(B)$

These facts follow from the definition:

- $P(\emptyset) = 0$
- If $A \subseteq B$ then $P(A) \leq P(B)$

Suppose X is finite, and each of the outcomes in X is equally likely. Then for any event A,

$$P(A) = \frac{|A|}{|X|},$$

where $|A|$ is the number of elements in A, and $|X|$ is the number of elements in X.

Bernoulli's Theorem

Two events A and B are said to be *independent* if

$$P(A \cap B) = P(A)P(B).$$

Suppose one runs n independent trials of an experiment in which the probability of success in each trial is p and the probability of failure is $q = 1 - p$. Then the probability that there are exactly k successes in these n trials is

$$\binom{n}{k} p^k q^{n-k}.$$

Bernoulli's Theorem: Suppose one repeatedly runs independent trials of an experiment in which the probability of success in each trial is p. Let A and

B be small positive numbers. Then there is a value of n large enough so that the probability that the ratio of successes in n trials is not within A of p is less than B.

In other words: if you run the experiment long enough, the fraction of successes is very likely to be close to the "correct" probability. See the discussion on page 203 of TTT.

Bayes' Theorem

Let X be a (finite) probability space, and let A and B be events. The *probability of A given B* is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplying through, we see that

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

Let A and B be any two events. Since $A = (A \cap B) \cup (A \cap \bar{B})$, and $(A \cap B)$ and $(A \cap \bar{B})$ are incompatible, we have

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap \bar{B}) \\ &= P(A|B)P(B) + P(A|\bar{B})P(\bar{B}). \end{aligned}$$

Now let H and E be events. Putting all the information together, we have Bayes' theorem:

$$\begin{aligned} P(H|E) &= \frac{P(H \cap E)}{P(E)} \\ &= \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|\bar{H})P(\bar{H})} \end{aligned}$$

Interpret this as follows:

- Let H be some scientific hypothesis you want to test.
- Let $P(H)$ be your *a priori* estimate of the probability that the hypothesis is correct (before running an experiment).
- Then $P(\bar{H})$ your estimate of the probability that the hypothesis is false.

- After running an experiment, you find some evidence, E.
- Then $P(E|H)$ is the probability that the experiment would turn out this way, assuming your hypothesis is correct.
- And $P(E|\bar{H})$ is the probability that the experiment would turn out this way, if your hypothesis is incorrect.

Bayes' theorem then allows you to calculate the probability that the hypothesis is correct, given the evidence you have gathered.

In the discussion on pages 206-210 of TTT, the "hypothesis" is that the probability of success in each trial of the experiment is somewhere between a and b, and the "evidence" is the fraction of successes in some large number of trials. The analysis shows that if the experiment returns a large number of sucesses and no failures, then the only likely hypothesis is that the probability of success is very close to 1.