Computer Organization: Boolean Logic

Representing and Manipulating Data

Last Unit

- How to represent data as a sequence of bits
- How to interpret bit representations
- Use of levels of abstraction in representing more complex information (music, pictures) using simpler building blocks (numbers)

This Unit

- How sequences of bits are implemented using electrical signals, and manipulated by circuits
- Use of levels of abstraction in designing more complex computer components from simpler components

Foundations

Boolean logic is the logic of digital circuits

Implementing Bits

- Computers compute by manipulating electricity according to specific rules.

- We associate electrical signals inside the machine with bits. Any electrical device with two distinct states (e.g. on/off switch, two distinct voltage or current levels) could implement our bits.

- The rules are implemented by electrical circuits.

Conceptualizing bits and circuits

- **ON** or **1**: **true**
- **OFF** or **0**: **false**
- circuit behavior: expressed in *Boolean logic* or *Boolean algebra*

Boolean Logic (Algebra)

- Computer circuitry works based on Boolean Logic (Boolean Algebra) : operations on True (1) and False (0) values.

A	B	A ∧ B (A AND B) (conjunction)	A ∨ B (A OR B) (disjunction)
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

A	¬ A (NOT A) (negation)
0	1
1	0

- A and B in the table are Boolean variables, AND and OR are operations (also called functions).

Foundations of Digital Computing

- Boolean Algebra was invented by George Boole in 1854 (before digital computers)
  - Variables and functions take on only one of two possible values: True (1) or False (0).

- The correspondence between Boolean Logic and circuits was not discovered until 1930s
  - Shannon's thesis: A Symbolic Analysis of Relay and Switching Circuits argued that electrical applications of Boolean Algebra could construct any logical, numerical relationship.
  - We forget about the *logical* (truth and falsehood) aspect of Boolean logic and just manipulate symbols.

Boolean Logic & Truth Tables

- Example: You can think of A ∧ B below as *15110 is fun and 15110 is useful* where A stands for the statement *15110 is fun*, B stands for the statement *15110 is useful.*

A	**B**	**A Λ B (A AND B) (conjunction)**	**A ∨ B (A OR B) (disjunction)**
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

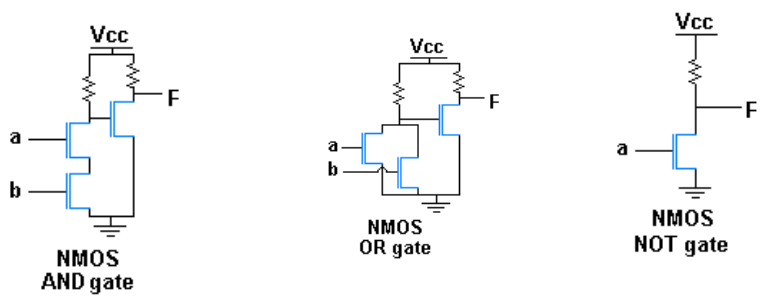
A	**¬¬A (NOT A) (negation)**
0	1
1	0

Logic gates

the basic elements of digital circuits

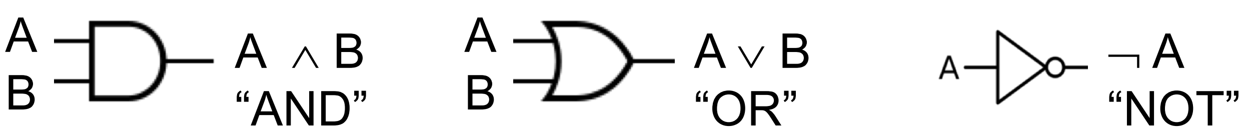
Logic Gates

- A gate is a physical device that implements a Boolean operator by performing basic operations on electrical signals.
- Nowadays, gates are built from transistors.



physical picture of gates

Physical behavior of circuits is beyond the scope of our course.

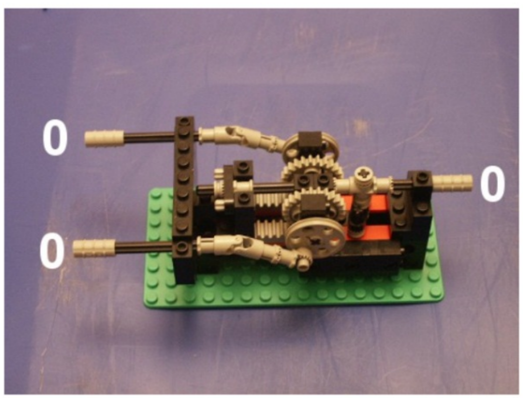


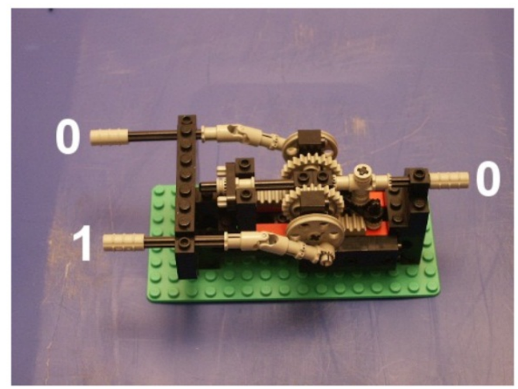
logical picture of gates

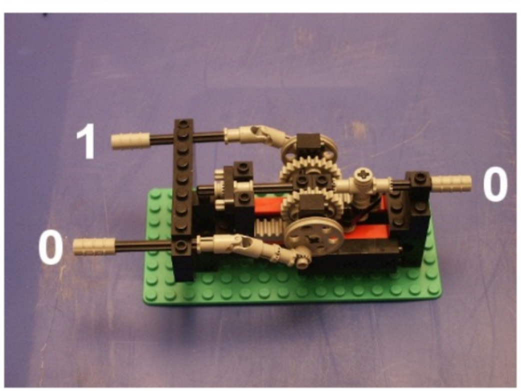
A Mechanical Implementation

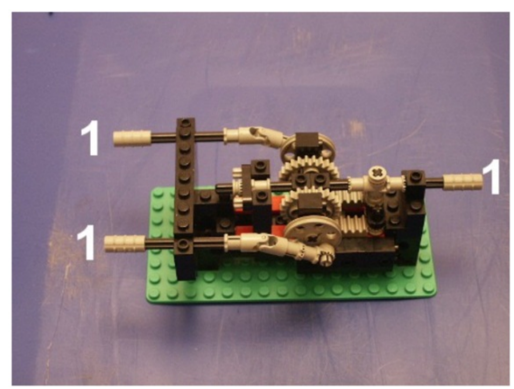
Push-pull logic AND gate

- For an input pushed-in lever represents 1
- For an output pushed-in lever represents 0









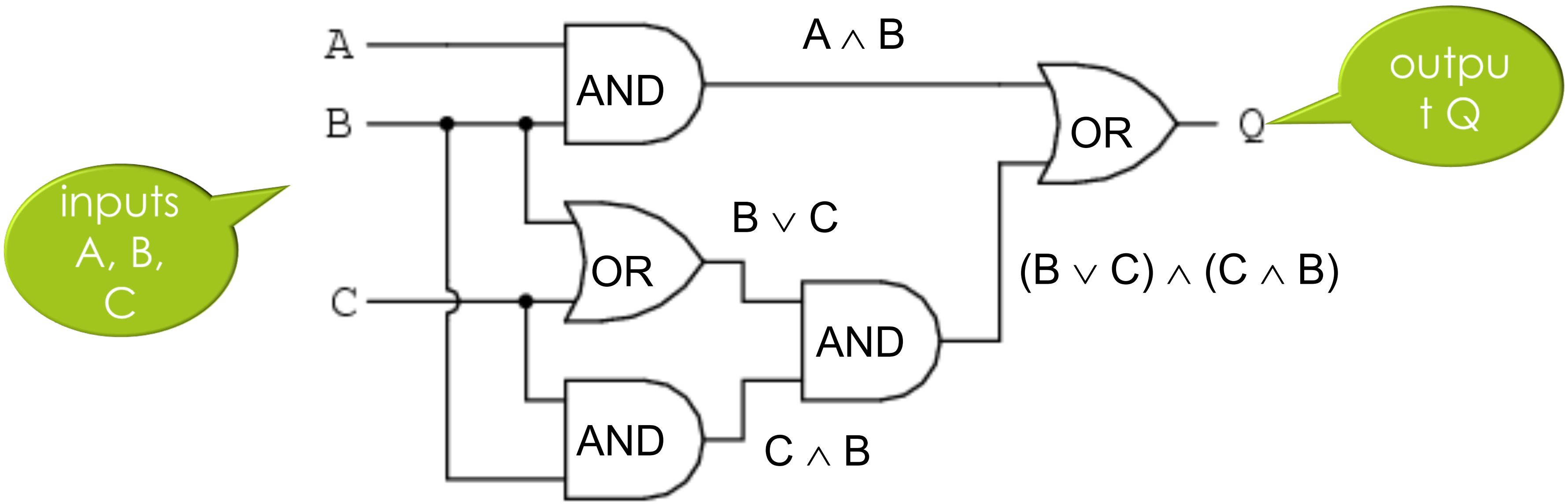
Source:
randomwraith.com
by Martin Howard

Combinational circuits

combinations of logic gates

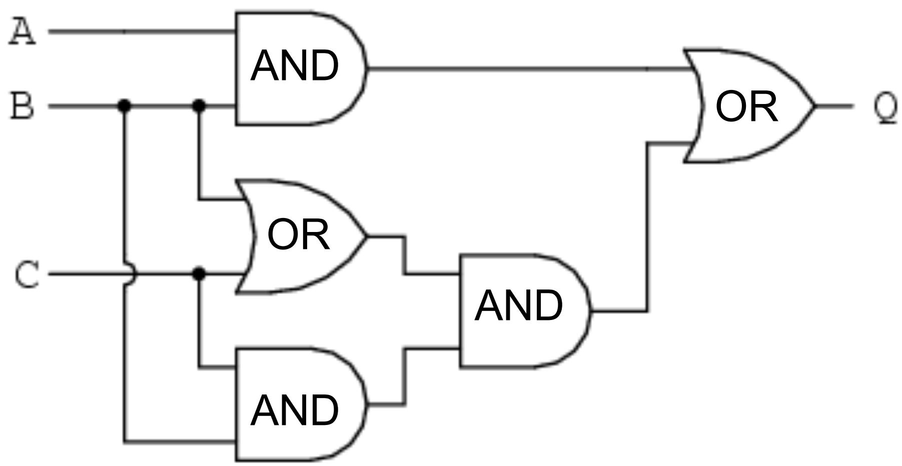
Combinational Circuits

The logic states of inputs at any given time determine the state of the outputs.



What is Q? $(A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$

Truth Table of a Circuit



$$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$$

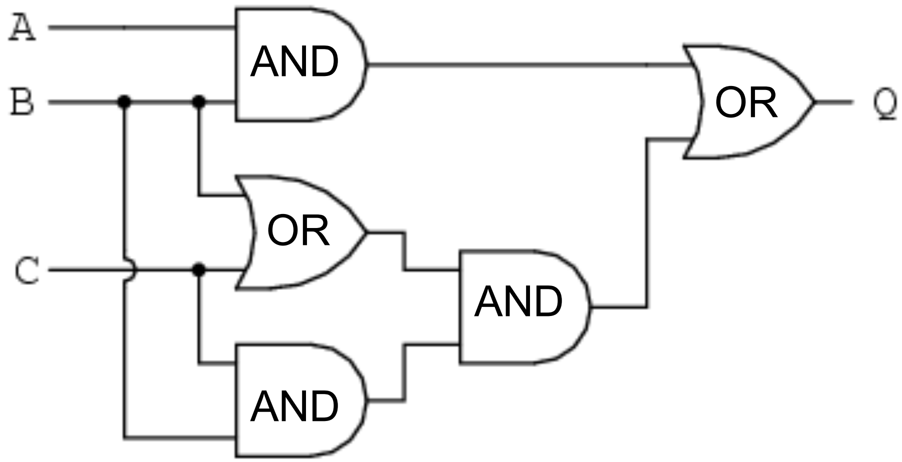
A	B	C	Q
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

How do I know that there should be 8 rows in the truth table?

Describes the relationship between inputs and outputs of a device

http://www.allaboutcircuits.com/vol_4/chpt_7/6.html

Truth Table of a Circuit



$$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$$

A	B	C	Q
0	0	0	0
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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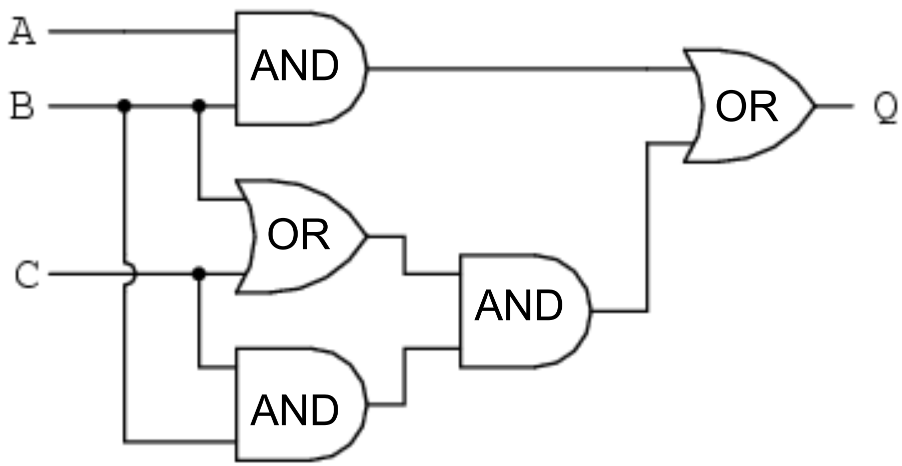
Truth Table of a Circuit



$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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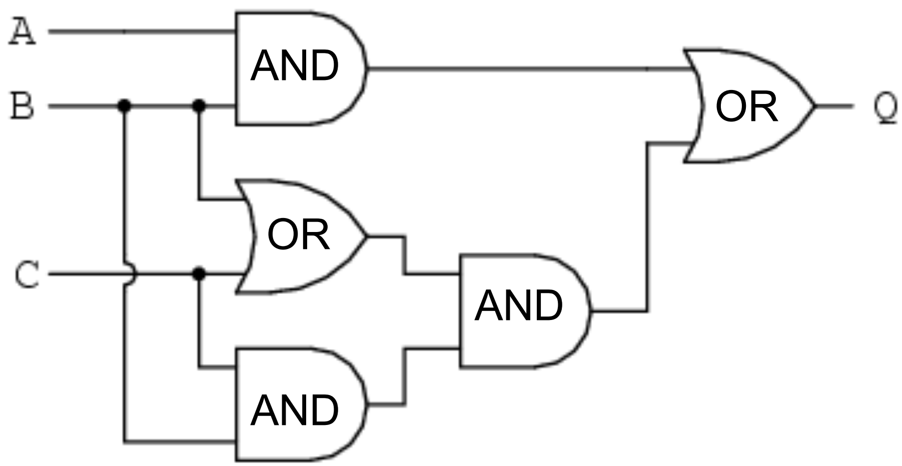

$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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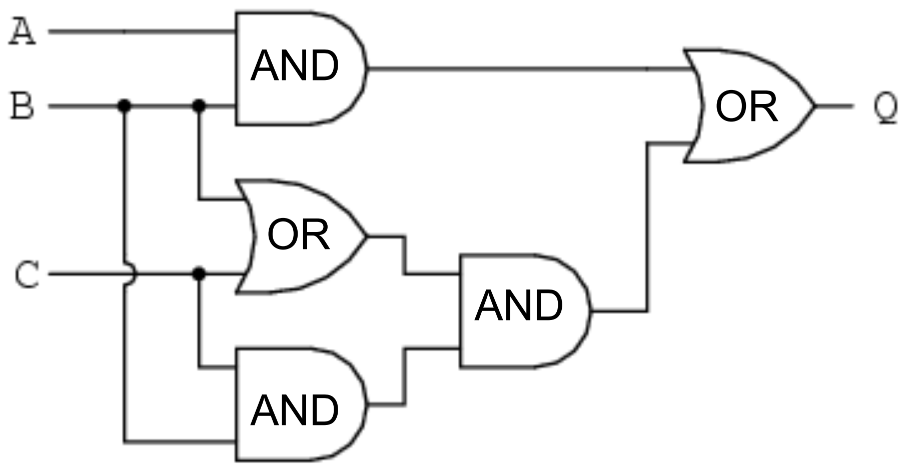

Q = (A ∧ B) ∨ ((B ∨ C) ∧ (C ∧ B))

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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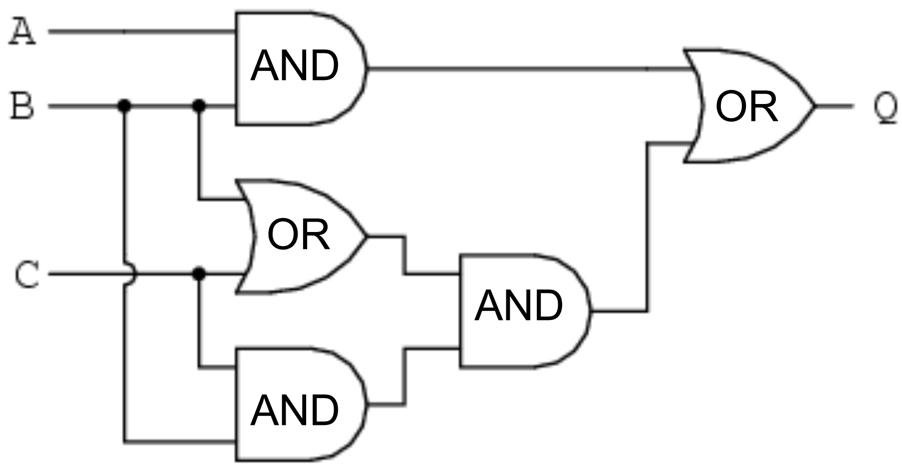

Q = (A ∧ B) ∨ ((B ∨ C) ∧ (C ∧ B))

A	B	C	Q
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0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	
1	1	0	
1	1	1	

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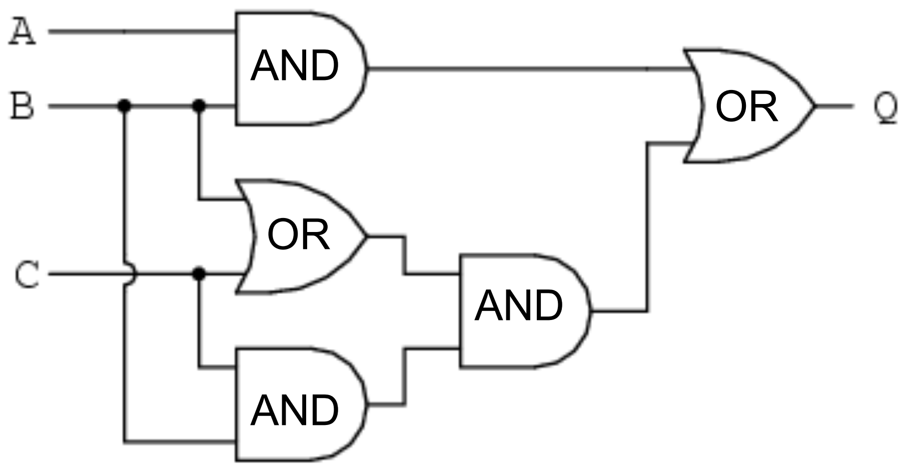

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A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
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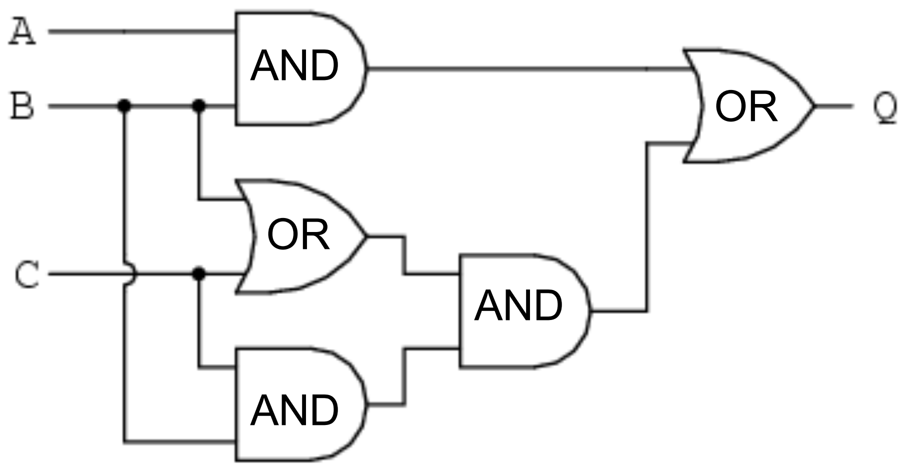

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0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	

Describes the relationship between inputs and outputs of a device

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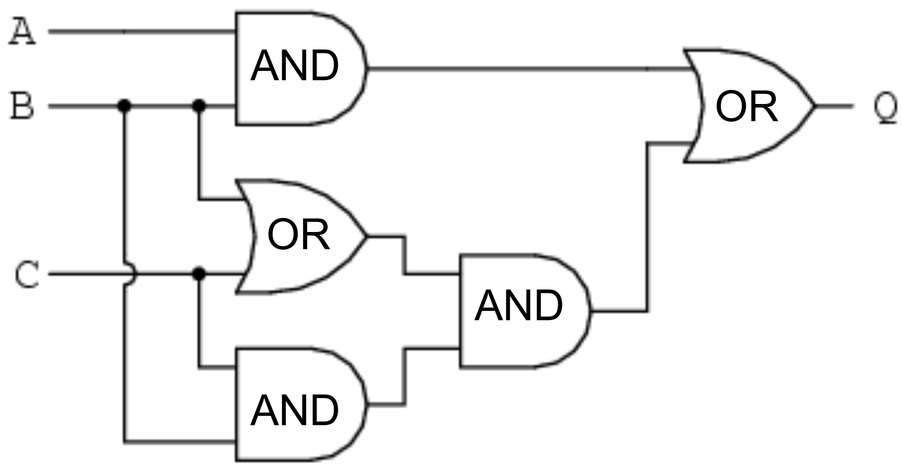
Truth Table of a Circuit



$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Describes the relationship between inputs and outputs of a device

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Describing Behavior of Circuits

- Boolean expressions
- Circuit diagrams
- Truth tables

Equivalent notations

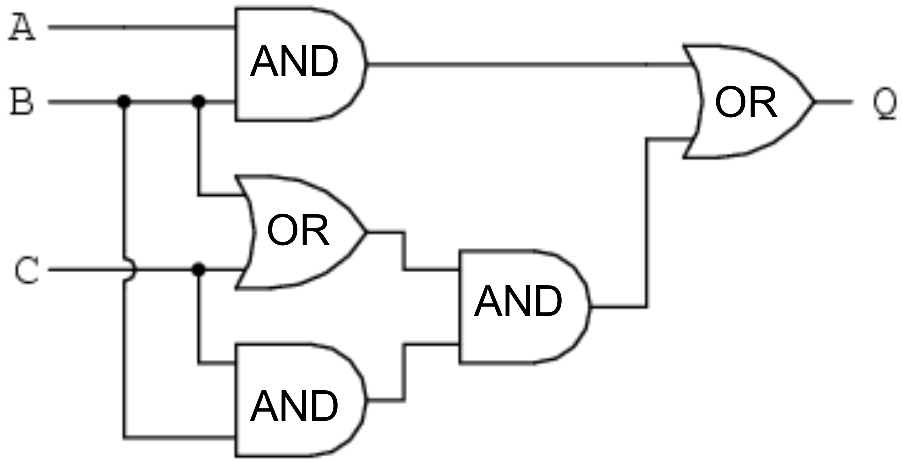
Manipulating circuits

Boolean algebra and logical equivalence

Why manipulate circuits?

- The design process
  - simplify a complex design for easier manufacturing, faster or cooler operation, …
- Boolean algebra helps us find another design guaranteed to have same behavior

Logical Equivalence

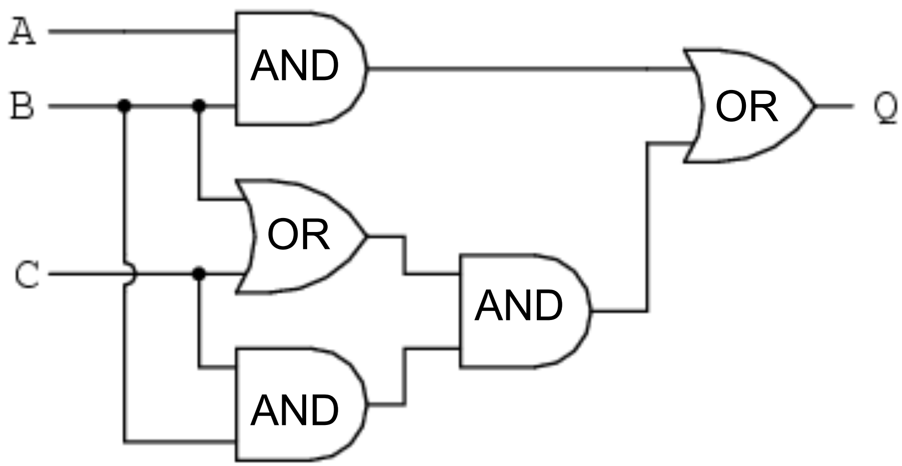


$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

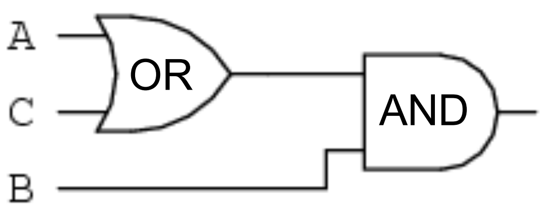
Can we come up with a simpler circuit implementing the same truth table? Simpler circuits are typically cheaper to produce, consume less energy etc.

Logical Equivalence



$Q = (A \wedge B) \vee ((B \vee C) \wedge (C \wedge B))$

A	B	C	Q
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1



$Q = B \wedge (A \vee C)$

This smaller circuit is logically equivalent to the one above: they have the same truth table. By using laws of Boolean Algebra we convert a circuit to another equivalent circuit.

Laws for the Logical Operators $\wedge$ and $\vee$ (Similar to $\times$ and $+$)

- Commutative: $A \wedge B = B \wedge A$ $\quad$ $A \vee B = B \vee A$
- Associative: $A \wedge B \wedge C = (A \wedge B) \wedge C = A \wedge (B \wedge C)$
  $A \vee B \vee C = (A \vee B) \vee C = A \vee (B \vee C)$
- Distributive: $A \wedge (B \vee B) = (A \wedge B) \vee (A \wedge C)$
  $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
- Identity: $A \wedge 1 = A$ $\quad$ $A \vee 0 = A$
- Dominance: $A \wedge 0 = 0$ $\quad$ $A \vee 1 = 1$
- Idempotence: $A \wedge A = A$ $\quad$ $A \vee A = A$
- Complementation: $A \wedge \neg A = 0$ $\quad$ $A \vee \neg A = 1$
- Double Negation: $\neg \neg A = A$

Laws for the Logical Operators ∧ and ∨ (Similar to × and +)

- Commutative: $A \wedge B = B \wedge A$ $A \vee B = B \vee A$
- Associative: $A \wedge B \wedge C = (A \wedge B) \wedge C = A \wedge (B \wedge C)$
  $A \vee B \vee C = (A \vee B) \vee C = A \vee (B \vee C)$
- Distributive: $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
  $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ ← Not true for + and ×
- Identity: $A \wedge 1 = A$ $A \vee 0 = A$

The A's and B's here are schematic variables! You can instantiate them with any expression that has a Boolean value:

$(x \vee y) \wedge z = z \wedge (x \vee y)$ (by commutativity)

$A \wedge B = B \wedge A$

Applying Properties for $\wedge$ and $\vee$

Showing **$(x \wedge y) \vee ((y \vee z) \wedge (z \wedge y)) = y \wedge (x$ or $z)$**

Commutativity $A \wedge B = B \wedge A$

$(x \wedge y) \vee ((z \wedge y) \wedge (y \vee z))$

Distributivity $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

$(x \wedge y) \vee (z \wedge y \wedge y) \vee (z \wedge y \wedge z)$

Associativity, Commutativity, Idempotence

$(x \wedge y) \vee ((z \wedge y) \vee (y \wedge z))$

Commutativity, idempotence $A \wedge A = A$

$(y \wedge x) \vee (y \wedge z)$

Distributivity (backwards) $(A \wedge B) \vee (A \wedge C) = A \wedge (B \vee C)$

$y \wedge (x \vee z)$

Conclusion:

$(x \wedge y) \vee ((y \vee z) \wedge (z \wedge y)) = y \wedge (x \vee z)$

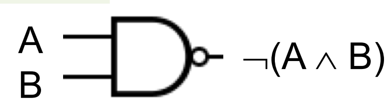
Extending the system

more gates and DeMorgan’s laws

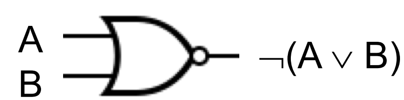
More gates (NAND, NOR, XOR)

A	B	A nand B	A nor B	A xor B
0	0	1	1	0
0	1	1	0	1
1	0	1	0	1
1	1	0	0	0

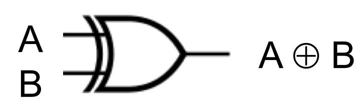
- nand ("not and"): A nand B = not (A and B)



- nor ("not or"): A nor B = not (A or B)



- xor ("exclusive or"):
  A xor B = (A and not B) or (B and not A)



A curious fact

- Functional Completeness of NAND and NOR
  - Any logical circuit can be implemented using NAND gates only
- Same applies to NOR

DeMorgan’s Law

Nand: $\neg(A \wedge B) = \neg A \vee \neg B$

Nor: $\neg(A \vee B) = \neg A \wedge \neg B$

DeMorgan's Law

Nand: $\neg(A \wedge B) = \neg A \vee \neg B$

```
if not (x > 15 and x < 110):  ...
```

is logically equivalent to

```
if (not x > 15) or (not x < 110): ...
```

Nor: $\neg(A \vee B) = \neg A \wedge \neg B$

```
if not (x < 15 or x > 110): ...
```

is logically equivalent to

```
if (not x < 15) and (not x > 110): ...
```

A circuit for parity checking

Boolean expressions and circuits

A Boolean expression that checks parity

- 3-bit odd parity checker F: an expression that should be true when the count of 1 bits is odd: when 1 or 3 of the bits are 1s.

$$P = (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C) \vee (A \wedge B \wedge C)$$

A	B	C	P
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

There are specific methods for obtaining canonical Boolean expressions from a truth table, such as writing it as a disjunction of conjunctions or as a conjunction of disjunctions.

Note we have four subexpressions above each of them corresponding to exactly one row of the truth table where P is 1.

The circuit

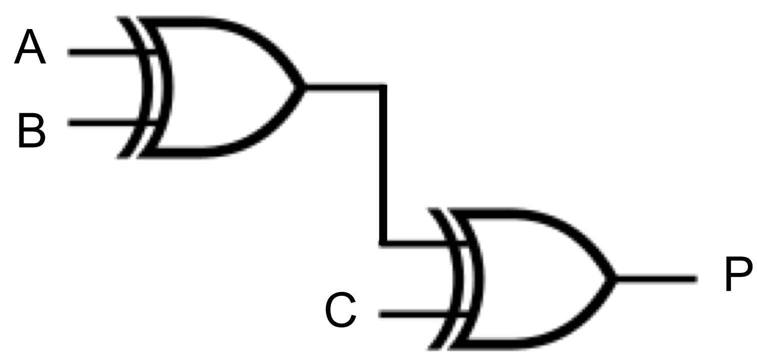
3-bit odd parity checker

$P = (\neg A \land \neg B \land C) \lor (\neg A \land B \land \neg C) \lor (A \land \neg B \land \neg C) \lor (A \land B \land C)$

logically equivalent

$P = (A \oplus B) \oplus C$

A	B	C	P
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1



Summary

You should be able to:

- Identify basic gates
- Describe the behavior of a gate or circuit using Boolean expressions, truth tables, and logic diagrams
- Transform one Boolean expression into another given the laws of Boolean algebra

Next Time

- How circuits are combined to form a computer
  - Von Neumann architecture revisited
  - Fetch – Decode - Execute Cycle