Dependent Types and Program Equivalence

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Abstract

The definition of type equivalence is one of the most important design issues for any typed language. In dependently-typed languages, because terms appear in types, this definition must rely on a definition of term equivalence. In that case, decidability of type checking requires decidability for the term equivalence relation.

Almost all dependently-typed languages require this relation to be decidable. Some, such as Coq, Epigram or Agda, do so by employing analyses to force all programs to terminate. Conversely, others, such as DML, ATS, Ωmega, or Haskell, allow nonterminating computation, but do not allow those terms to appear in types. Instead, they identify a terminating index language and use singleton types to connect indices to computation. In both cases, decidable type checking comes at a cost, in terms of complexity and expressiveness.

Conversely, the benefits to be gained by decidable type checking are modest. Termination analyses allow dependently typed programs to verify total correctness properties. However, decidable type checking is not a prerequisite for type safety—and, in this context, type safety implies partial correctness. Furthermore, decidability does not imply tractability. A decidable approximation of program equivalence may not be useful in practice.

Therefore, we take a different approach: instead of a fixed notion for term equivalence, we parameterize our type system with an abstract relation that is not necessarily decidable. We then design a novel set of typing rules that require only weak properties of this abstract relation in the proof of the preservation and progress lemmas. This design provides flexibility: we compare valid instantiations of term equivalence which range from beta-equivalence, to contextual equivalence, to some exotic equivalences.

1. Introduction

Dependent type systems promise the smooth integration of lightweight invariant checking with full program verification. In languages with dependent types, the types of a program may express and statically verify rich properties about its behavior.

Central in the design of a dependently-typed language is the notion of type equivalence. Because types include programs, type checking requires a definition of term equivalence. Therefore, the decidability of type checking requires that the term equivalence relation be decidable.

Previous work has almost uniformly insisted on decidable type checking, and hence decidable term equivalence. Some *full-spectrum* languages, such as Coq [Coq Development Team 2009], Epigram [McBride and McKinna 2004] or Agda [Norell 2007], do so by employing analysis that force all programs to terminate. This strong requirement has the benefit that type checking implies total correctness. If a function has type $\tau \to \Sigma y{:}\tau'.\ P\ y$ then one can be assured that it will terminate and produce a value satisfying property P.

Other, *phase-sensitive* languages, such as Dependent ML [Xi and Pfenning 1998], ATS [Xi 2004], Ωmega [Sheard 2006] and Haskell (with GADTs [Peyton Jones et al. 2006]), allow nonterminating computation and sacrifice total correctness. They retain decidable type checking by not allowing terms to appear in types. Instead, they identify a terminating index language (the type language in the case of Haskell) and use singleton types to connect indices to computation.

Finally, some projects such as Ynot [Nanevski et al. 2008], GURU [Stump et al. 2009] and PIE [Vytiniotis and Weirich 2007] extend dependent types to reason about languages with effects and states. These languages use termination and effect analyses to only allow pure expressions to appear in dependent types.

In each of these cases, decidable type checking comes at a cost, in terms of both complexity and expressiveness. Requiring all programs to terminate severely limits the generality of a programming language. Furthermore, the complexity of the termination analysis can make it difficult for programmers to understand why their code does not type check. In phase-sensitive languages, singleton types lead to code duplication, as programs must often be written twice, once in the computation language, and again in the index language. More troublesome, there is no restriction that the semantics of the index language match that of the computation language: only their first-order values are required to agree.

At the same time, the benefits to be gained by decidable type checking are modest. Although termination analyses provide stronger correctness guarantees, they do not need to be integrated into the type system. Partial correctness guarantees that are naturally implied by type safety could be separately extended to total correctness where necessary by an external termination analysis. Furthermore, decidability does not imply practicality or tractability. Why rule out undecidable specifications a priori, when they could behave well in practice?

Therefore, we design a full-spectrum, dependently-typed language $\lambda^{\cong}$, pronounced "lambda-eek", that does not presuppose decidable program equivalence. This language is both simple and expressive: not only does it include general recursive function definitions and dependent products, but it also supports dependent datatypes (also called inductive families) with elimination forms to both terms (case expressions) and types (large eliminations).

It is a folklore belief that undecidable type checking is compatible with type safety for languages similar to $\lambda^{\cong}$ [Augustsson 1998]. As a demonstration of the simplicity of our design, $\lambda^{\cong}$ supports a straightforward proof of type safety based on standard preservation and progress lemmas. We have formalized this proof in the Coq proof assistant.

An important aspect of $\lambda^{\cong}$ is that it is actually a *family of languages* because its type system is parameterized by an abstract relation that specifies program equivalence. This three-place relation, written $\mathbf{isEq}\,(\,\Delta\,,\,e_1\,,\,e_2\,)$, asserts when terms e_1 and e_2 are equivalent in some context Δ of assumptions about the equivalence of terms. This specification of program equivalence is isolated from typing, and the type safety proof depends on properties of program equivalence that make no reference to the type system. This separation simplifies the type safety proof.

For generality, we would like weak requirements for **isEq**. In particular, we would like to admit call-by-value respecting equivalences, since the operational semantics of $\lambda^{\cong}$ is call-by-value. Surprisingly, we revised our design several times before we found one that would admit such relations.

Although it is impossible to claim that we have the weakest possible set of requirements, our design permits many different relations: from standard beta-equivalence, to contextual equivalence, to some exotic equivalences. The finest equivalence makes our system admit no more terms than the simply-typed lambda calculus. More surprisingly, equivalences based on call-by-name evaluation are also valid, as well as some exotic equivalences that identify certain terminating and nonterminating expressions.

We also found that the requirements of the preservation proof force all valid instantiations of **isEq** to be undecidable. However, preservation is not a necessary requirement for type safety. Any language that type checks strictly fewer programs than a type-safe language is itself type safe. Therefore, any *decidable, conservative approximation* of a particular notion of program equivalence also defines a type-safe language. Consequently, $\lambda^{\cong}$ can be used as a template for languages with both decidable and undecidable definitions of program equivalence.

The organization of this paper is as follows. In Section 2 we introduce the syntax and call-by-value operational semantics of $\lambda^{\cong}$. We then describe its type system, parameterized by the abstract predicate **isEq** in Section 3. Working through a standard proof of preservation and progress leads to requirements on **isEq**—we describe those properties in Section 4. In Section 5 we give several definitions of **isEq** that satisfy our requirements. Variations of our type system lead to stronger requirements on **isEq**, which we discuss in Section 6. We discussion extensions to this system and other issues in Section 7. Finally, in Sections 8 and 9 we discuss related work and conclude.

$$
\begin{array}{lrcl}
\text{Terms} & e,\ u & ::= & x \mid \mathbf{unit} \mid \mathbf{fun}\,f(x) = e \mid e_1\,e_2 \\
 & & \mid & \langle\, e_1\,,\, e_2 \,\rangle \mid e.1 \mid e.2 \\
 & & \mid & C\ e \mid \mathbf{case}\,e\,\mathbf{of}\,\{\,\overline{C_i\,x_i \Rightarrow e_i}^{\,i}\,\} \\
\text{Values} & v & ::= & \mathbf{unit} \mid \mathbf{fun}\,f(x) = e \mid \langle\, v_1\,,\, v_2 \,\rangle \mid C\ v
\end{array}
$$

Figure 1. Syntax

All Coq proof developments [1] for this paper are available online at `http://www.seas.upenn.edu/liminjia/research/lambdaEq/lambdaEqCoq.tgz`.

2. A call-by-value language

Figure 1 presents the syntax of terms and values of $\lambda^{\cong}$. Importantly, terms do not contain typing annotations. We use untyped terms to isolate the specification of **isEq** from the type system of $\lambda^{\cong}$ and simplify its metatheory. A worry is that **isEq** might distinguish between terms with syntactically different but semantically equivalent type annotations. To trivially rule this possibility out, terms do not contain types, and $\lambda^{\cong}$ uses a Curry-style type system, presented in Section 3.

The term language includes only standard features of programming languages: variables, unit, (recursive) functions, applications, binary products, projections, data constructors and case analysis. We use the metavariables e and u to denote terms and v to denote values. In a recursive function $\mathbf{fun}\,f(x) = e$, the variables f and x are bound in the body of recursive functions. If f does not appear in the body of the function, then we write it as $\lambda x.e$. In a case expression $\mathbf{case}\,e\,\mathbf{of}\,\{\,\overline{C_i\,x_i \Rightarrow e_i}^{\,i}\,\}$, the variables x_i are bound within each of the branches e_i. We follow the Barendregt convention for bound variables [Barendregt 1981].

For simplicity, every data constructor must be of arity one and must always be applied to its argument. This limitation does not affect expressiveness—nullary and multiargument data constructors can be encoded. Throughout the paper, we assume a standard Peano encoding of natural numbers, with, for example, 0 represented as C_{zero} **unit** and 1 represented by C_{succ} (C_{zero} **unit**). The boolean values `true` and `false` can be similarly encoded.

The small-step, call-by-value (CBV) evaluation rules for $\lambda^{\cong}$ appear in Figure 2. This semantics is completely standard. Importantly, applications of recursive functions only step when their arguments are values.

3. A parameterized type system

We now define a Curry-style type system for $\lambda^{\cong}$. Figure 3 defines the necessary additions to the syntax. The judgment forms of the type system are summarized in Figure 4. The rules of the type system itself appear in Figures 5, 6 and 7.

The types of $\lambda^{\cong}$ are divided into *proper types* of kind $*$ that classify terms directly; and *indexed types* of kind $(x{:}\tau) \Rightarrow *$ that must first be applied to a single term (of type τ).

Proper types include **Unit**, the type of the **unit** term, function types $(x{:}\tau) \to \tau'$ and product types $\Sigma x{:}\tau.\,\tau'$. In the latter two types, the variable x may appear in τ'. The result type of a function may depend on the argument value, and the type of the second component of a product may depend on the first component.

[1] Because we are dealing with potentially undecidable relations, our developments use Coq's classical logic library.

$$\frac{}{(\,\mathbf{fun}\, f(x) \,=\, e_1\,)\; v_2 \;\longrightarrow\; e_1\{v_2/x\}\{\mathbf{fun}\, f(x) \,=\, e_1/f\}}$$

$$\frac{e_1 \longrightarrow e_1'}{e_1\; e_2 \longrightarrow e_1'\; e_2} \qquad \frac{e_2 \longrightarrow e_2'}{v_1\; e_2 \longrightarrow v_1\; e_2'}$$

$$\frac{e_1 \longrightarrow e_1'}{\langle\, e_1 \,,\, e_2 \,\rangle \longrightarrow \langle\, e_1' \,,\, e_2 \,\rangle} \qquad \frac{e_2 \longrightarrow e_2'}{\langle\, v_1 \,,\, e_2 \,\rangle \longrightarrow \langle\, v_1 \,,\, e_2' \,\rangle}$$

$$\frac{}{\langle\, v_1 \,,\, v_2 \,\rangle.\,1 \longrightarrow v_1} \qquad \frac{}{\langle\, v_1 \,,\, v_2 \,\rangle.\,2 \longrightarrow v_2}$$

$$\frac{e \longrightarrow e'}{e.\,1 \longrightarrow e'.\,1} \qquad \frac{e \longrightarrow e'}{e.\,2 \longrightarrow e'.\,2} \qquad \frac{e \longrightarrow e'}{C\; e \longrightarrow C\; e'}$$

$$\frac{C_j \in \overline{C_i}^{\,i \in 1..n}}{\mathbf{case}\; C_j\; v\; \mathbf{of}\; \{\, \overline{C_i\; x_i \Rightarrow e_i}^{\,i \in 1..n} \,\} \longrightarrow e_j\{v/x_j\}}$$

$$\frac{e \longrightarrow e'}{\mathbf{case}\; e\; \mathbf{of}\; \{\, \overline{C_i\; x_i \Rightarrow e_i}^{\,i} \,\} \longrightarrow \mathbf{case}\; e'\; \mathbf{of}\; \{\, \overline{C_i\; x_i \Rightarrow e_i}^{\,i} \,\}}$$

Figure 2. Operational Semantics

$$\begin{array}{lll}
\text{Kinds} & \kappa & ::= * \mid (x{:}\tau) \Rightarrow * \\
\text{Types} & \tau, \sigma & ::= \mathbf{Unit} \mid (x{:}\tau) \to \tau' \mid \Sigma x{:}\tau.\, \tau' \mid T \\
& & \quad \mid \tau\; e \mid \mathbf{case}\; e \,\langle\, T\; u \,\rangle\, \mathbf{of}\; \{\, \overline{C_i\; x_i \Rightarrow \tau_i}^{\,i} \,\} \\
\text{Signatures} & \Sigma & ::= \cdot \mid \Sigma, C : (x{:}\tau) \to T\; e \\
& & \quad \mid \Sigma, T : (x{:}\tau) \Rightarrow * \\
\text{Contexts} & \Gamma & ::= \cdot \mid \Gamma, x : \tau \mid \Gamma,\, e \cong e' \\
\text{Eq ctxs} & \Delta & ::= \cdot \mid \Delta,\, e \cong e' \\
\text{Pure terms} & w & ::= x \mid \mathbf{unit} \mid \mathbf{fun}\, f(x) = e \\
& & \quad \mid \langle\, w_1 \,,\, w_2 \,\rangle \mid w.\,1 \mid w.\,2 \mid C\; w
\end{array}$$

Figure 3. Types and Contexts

Data constructors are typed by datatype constants, T, which are indexed types. The kinds of datatype constants and the types of data constructors are recorded by a signature Σ. We assume that there is one fixed, well-formed signature Σ_0 for an entire program, so we leave it implicit. We also assume that all data constructors and datatype constants are in the domain of Σ_0.

For simplicity, we require that all datatype constants be of kind $(x{:}\tau) \Rightarrow *$. Standard data types use the uninformative index **unit**. For example, the notation **Nat** abbreviates the type T_{Nat} **unit**, where the constant T_{Nat} has kind $(x{:}\mathbf{Unit}) \Rightarrow *$. We use a similar definition for the type **Bool**.

Often, however, the index is informative. For example, suppose the constant T_{List} is indexed by its length, a natural number. The data constructor C_{nil} creates a list of type $T_{List}\; 0$. When type checking a case analysis where the scrutinee has type $T_{List}\; x$, the type checker can assume that x is equal to 0 in the C_{nil} branch.

The type language also includes a strong elimination form: case analysis of terms to produce types. In a type pattern match, $\mathbf{case}\; e \,\langle\, T\; u \,\rangle\, \mathbf{of}\; \{\, \overline{C_i\; x_i \Rightarrow \tau_i}^{\,i} \,\}$, a finite number of types τ_i are indexed by a term e that is expected to be of type $T\; u$. (We discuss the need for this annotation in Section 3.3.) This mechanism provides the technique of "Universes" in dependently-typed languages. For example, in a context containing the assumption $x : \mathbf{Bool}$, the term

$\mathbf{case}\; x\; \mathbf{of}\; \{\, \texttt{true} \Rightarrow 1 \,;\, \texttt{false} \Rightarrow \texttt{false} \,\}$

can be assigned the type

Formation Judgments		Equivalence Judgments	
$\vdash \Sigma$	Signature	$\vdash \Gamma \equiv \Gamma'$	Context
$\vdash \Gamma$	Context	$\Delta \vdash \tau \equiv \tau'$	Types
$\Gamma \vdash \kappa$	Kinds	$\Delta \vdash \kappa \equiv \kappa'$	Kinds
$\Gamma \vdash \tau : \kappa$	Types		
$\Gamma \vdash e : \tau$	Terms		

Figure 4. Type System Judgment Forms

$\mathbf{case}\; x \,\langle\, \mathbf{Bool} \,\rangle\, \mathbf{of}\; \{\, \texttt{true} \Rightarrow \mathbf{Nat} \,;\, \texttt{false} \Rightarrow \mathbf{Bool} \,\}$

The type system is defined in terms of a number of assumption lists. Besides signatures Σ, there are *contexts* Γ and *equivalence contexts* Δ. Contexts are ordered lists of variable type assumptions and term equivalence assumptions. The domain of a context is the set of variables for which there are type assumptions. Equivalence contexts Δ contain term equivalence assumptions only. We denote context concatenation with $\Gamma\,,\,\Gamma'$ (and $\Delta\,,\,\Delta'$). We use $\Gamma^\star$ to produce the equivalence context containing the equivalence assumptions in Γ.

Some places in the specification of the type system require the definition of *pure terms*. We use the metavariable w to range over a simple set of terms that are known to terminate.

3.1 Parameterized equivalence: isEq

As mentioned above, the type system of $\lambda^{\cong}$ is parameterized by the predicate $\mathbf{isEq}\,(\,\Delta\,,\, e\,,\, e'\,)$. This predicate decides whether the terms e and e' are equivalent under the set of equivalence assumptions in Δ.

We use **isEq** to define two auxiliary relations used for type checking. First, the predicate $\mathbf{incon}\,(\,\Delta\,)$ determines if there exists a contradiction in the equivalence assumptions of Δ. An equivalence context Δ is inconsistent when **isEq** equates two pure terms headed by different constructors.

DEFINITION 3.1 (Inconsistency).
Define $\mathbf{incon}\,(\,\Delta\,)$ *if there exist terms* $C_i\; w_i$ *and* $C_j\; w_j$ *such that* $\mathbf{isEq}\,(\,\Delta\,,\, C_i\; w_i\,,\, C_j\; w_j\,)$ *and* $C_i \neq C_j$*.*

Furthermore, we also define when two equivalence contexts are equivalent according to **isEq**.

DEFINITION 3.2 (Equivalence Context Equivalence). *We define the judgment* $\Delta \equiv_{\mathbf{ctx}} \Delta'$ *from the following rules:*

$$\frac{}{\cdot \equiv_{\mathbf{ctx}} \cdot}$$

$$\frac{\Delta \equiv_{\mathbf{ctx}} \Delta' \qquad \mathbf{isEq}\,(\,\Delta\,,\, e_1\,,\, e_1'\,) \qquad \mathbf{isEq}\,(\,\Delta\,,\, e_2\,,\, e_2'\,)}{\Delta\,,\, e_1 \cong e_2 \equiv_{\mathbf{ctx}} \Delta'\,,\, e_1' \cong e_2'}$$

Most existing dependently typed languages use β−equivalence or $\beta\eta$-equivalence to decide term equivalence. In our language, we leave **isEq** abstract. However, to ensure that our system enjoys standard properties (such as preservation and progress) **isEq** must itself satisfy a number of properties that we describe in Section 4.

The equivalence assumptions in Δ are equations between arbitrary terms. These terms do not need to be well-typed or even have the same type (though our rules only add such assumptions to the equivalence context). Furthermore, these equations do not need to be consistent, though when they are not, all terms are typeable with all types.

3.2 Typing

The type system of $\lambda^{\cong}$ is defined by two main categories of judgments (see Figure 4). One set determines when syntac-

$\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \cdot}\ \text{C\_E} \qquad \frac{\vdash \Gamma \quad x \notin dom(\Gamma) \quad \Gamma \vdash \tau : *}{\vdash \Gamma, x : \tau}\ \text{C\_TERM}$$

$$\frac{\vdash \Gamma \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\vdash \Gamma,\, e_1 \cong e_2}\ \text{C\_EQ}$$

$\boxed{\Gamma \vdash \kappa}$

$$\frac{\vdash \Gamma}{\Gamma \vdash *}\ \text{K\_TYPE} \qquad \frac{\Gamma, x : \tau \vdash \kappa}{\Gamma \vdash (x{:}\tau) \Rightarrow \kappa}\ \text{K\_PI}$$

$\boxed{\Gamma \vdash \tau : \kappa}$

$$\frac{\vdash \Gamma}{\Gamma \vdash \textbf{Unit} : *}\ \text{T\_UNIT} \qquad \frac{\vdash \Gamma \quad T : \kappa \in \Sigma_0}{\Gamma \vdash T : \kappa}\ \text{T\_CST}$$

$$\frac{\Gamma, x : \tau_1 \vdash \tau_2 : *}{\Gamma \vdash (x{:}\tau_1) \to \tau_2 : *}\ \text{T\_PI} \qquad \frac{\Gamma, x : \tau_1 \vdash \tau_2 : *}{\Gamma \vdash \Sigma x{:}\tau_1.\,\tau_2 : *}\ \text{T\_SIGMA}$$

$$\frac{\begin{array}{c}\Gamma \vdash \tau : (x{:}\tau_1) \Rightarrow \kappa_1 \quad \Gamma \vdash e : \tau_1 \\ \Gamma^\star,\, x \cong e \vdash \kappa_1 \equiv \kappa \quad \Gamma \vdash \kappa\end{array}}{\Gamma \vdash \tau\, e : \kappa}\ \text{T\_APP}$$

$$\frac{\begin{array}{c}\Gamma \vdash \kappa \quad \Gamma \vdash e : T\,u \quad \mathsf{CtrOf}(T) = \overline{C_i}^{\,i\in 1..n} \\ \overline{C_i : (x_i{:}\tau_i) \to T\,u_i \in \Sigma_0}^{\,i\in 1..n} \\ \overline{\Gamma, x_i : \tau_i,\, u \cong u_i,\, e \cong C_i\,x_i \vdash \tau_i : \kappa}^{\,i\in 1..n}\end{array}}{\Gamma \vdash \textbf{case}\, e\, \langle\, T\,u \,\rangle\, \textbf{of}\, \{\, \overline{C_i\,x_i \Rightarrow \tau_i}^{\,i\in 1..n} \,\} : \kappa}\ \text{T\_CASE}$$

$$\frac{\vdash \Gamma \quad \textbf{incon}\,(\Gamma^\star)}{\Gamma \vdash \tau : \kappa}\ \text{T\_INCON}$$

$$\frac{\Gamma \vdash \tau : \kappa \quad \Gamma^\star \vdash \kappa \equiv \kappa' \quad \Gamma \vdash \kappa'}{\Gamma \vdash \tau : \kappa'}\ \text{T\_KCONV}$$

Figure 5. Context, kind, and type formation rules

tic elements are well-formed. The other set determines when they are equivalent. The formation rules refer to the equivalence rules, but the equivalence rules are independent. We start our discussion with the formation rules, and discuss the equivalence rules in Section 3.3.

The formation rules appear in Figures 5 and 6. Most rules are straightforward; we focus on the term typing rules. One significant departure from standard rules is that we use equivalence assumptions instead of substitution. For example, a standard rule for application substitutes the operand e_2 for the variable in the result type:

$$\frac{\Gamma \vdash e_1 : (x{:}\tau_1) \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\, e_2 : \tau_2\{e_2/x\}}\ \text{E\_APP}'$$

However, in $\lambda^{\cong}$, instead of substituting the operand e_2 in the result type, rule E_APP checks if τ_2 is equal to some τ under an equivalence context that extends $\Gamma^\star$ with the equation $x \cong e_2$. Furthermore, to ensure that x is not free in τ, the rule checks that τ is well-formed under the context Γ. Similarly, the typing rules for dependent pairs, projections and constructors also extend the context with equivalence assumptions rather than use explicit substitution.

We use equivalence assumptions instead of substitution because substituting e into a type leads to stronger requirements on the substitution property of **isEq**. Intuitively, requiring that **isEq** be closed under substituting an arbitrary e limits our term equivalence relations to those based on call-by-name evaluation. However, our system is call-by-value, leading to an undesirable mismatch. We discuss this issue further in Section 6.1.

The typing rule for pattern-match E_CASE also uses equivalence assumptions, but for a different purpose. This rule first checks that the branch is exhaustive, with the premise $\mathsf{CtrOf}(T) = \overline{C_i}^{\,i\in 1..n}$. During execution, if the i^{th} branch is taken, the scrutinee must match the pattern $C_i\,x_i$, and the index u of the scrutinee's type must match with the index u_i in the signature. Therefore, this rule checks each branch under a context that extends Γ with equivalence assumptions that the indices are the same ($u \cong u_i$) and that the scrutinee is equal to the pattern ($e \cong C_i\,x_i$).

The fact that $\lambda^{\cong}$ uses equivalences to represent the information gained via case analysis is powerful. In particular, $\lambda^{\cong}$ can take advantage of information such as $f\ x \ \cong\ \texttt{true}$ in a way that languages, such as Coq and Agda, which use substitution and unification to specify pattern matching, cannot. For example, suppose we have a datatype T indexed by booleans with constructors $C_1 : T\,\texttt{true}$ and $C_2 : T\,\texttt{false}$.

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{\vdash \Gamma \quad x : \tau \in \Gamma}{\Gamma \vdash x : \tau}\ \text{E\_VAR} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \textbf{unit} : \textbf{Unit}}\ \text{E\_UNIT}$$

$$\frac{\Gamma, x : \tau_1, f : (x{:}\tau_1) \to \tau_2 \vdash e : \tau_2}{\Gamma \vdash \textbf{fun}\, f(x) = e : (x{:}\tau_1) \to \tau_2}\ \text{E\_FIX}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : (x{:}\tau_1) \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1 \\ \Gamma^\star,\, x \cong e_2 \vdash \tau_2 \equiv \tau \quad \Gamma \vdash \tau : *\end{array}}{\Gamma \vdash e_1\, e_2 : \tau}\ \text{E\_APP}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2' \\ \Gamma^\star,\, x \cong e_1 \vdash \tau_2' \equiv \tau_2 \quad \Gamma, x : \tau_1 \vdash \tau_2 : *\end{array}}{\Gamma \vdash \langle\, e_1 ,\, e_2 \,\rangle : \Sigma x{:}\tau_1.\,\tau_2}\ \text{E\_SIGMA}$$

$$\frac{\Gamma \vdash e : \Sigma x{:}\tau_1.\,\tau_2}{\Gamma \vdash e.1 : \tau_1}\ \text{E\_PROJ1}$$

$$\frac{\begin{array}{c}\Gamma \vdash e : \Sigma x{:}\tau_1.\,\tau_2 \quad \Gamma \vdash \tau : * \\ \Gamma^\star,\, x \cong e.1 \vdash \tau_2 \equiv \tau\end{array}}{\Gamma \vdash e.2 : \tau}\ \text{E\_PROJ2}$$

$$\frac{\begin{array}{c}C : (x{:}\sigma) \to T\,u \in \Sigma_0 \quad \Gamma \vdash e : \sigma \\ \Gamma^\star,\, x \cong e \vdash T\,u \equiv \tau \quad \Gamma \vdash \tau : *\end{array}}{\Gamma \vdash C\,e : \tau}\ \text{E\_CTR}$$

$$\frac{\begin{array}{c}\Gamma \vdash e : T\,u \quad \mathsf{CtrOf}(T) = \overline{C_i}^{\,i\in 1..n} \\ \Gamma \vdash \tau : * \quad \overline{C_i : (x_i{:}\tau_i) \to T\,u_i \in \Sigma_0}^{\,i\in 1..n} \\ \overline{\Gamma, x_i : \tau_i,\, u \cong u_i,\, e \cong C_i\,x_i \vdash e_i : \tau}^{\,i\in 1..n}\end{array}}{\Gamma \vdash \textbf{case}\, e\, \textbf{of}\, \{\, \overline{C_i\,x_i \Rightarrow e_i}^{\,i\in 1..n} \,\} : \tau}\ \text{E\_CASE}$$

$$\frac{\vdash \Gamma \quad \textbf{incon}\,(\Gamma^\star)}{\Gamma \vdash e : \tau}\ \text{E\_INCON}$$

$$\frac{\Gamma \vdash e : \tau \quad \Gamma^\star \vdash \tau \equiv \tau' \quad \Gamma \vdash \tau' : *}{\Gamma \vdash e : \tau'}\ \text{E\_TCONV}$$

Figure 6. Term formation rules (typing)

Then, in the following context

$$f : \mathbf{Nat} \to \mathbf{Bool}, x : \mathbf{Nat}, h : T\,(f\,x\,) \to \mathbf{Bool}$$

there are instantiations of **isEq** such that the following term typechecks

$$\mathbf{case}\, f\, x\, \mathbf{of}\, \{\mathtt{true} \Rightarrow h\, C_1; \mathtt{false} \Rightarrow \mathit{false}\}$$

To typecheck the application of h, the type checker must show the equivalence of $T\,(f\,x\,)$ and $T\,\mathtt{true}$ in the first branch, when the equation $f\,x \cong \mathtt{true}$ is available. Systems based on unification cannot make this information available via a substitution, so require the result of $f\,x$ to be named.[2]

Note that in rule E_CASE, the order in which the equivalence assumptions are added to the context is important for maintaining the well-formedness of the context. The type of e is $T\,u$, and the type of $C_i\,x_i$ is $T\,u_i$. For the extended context to be well-formed, we need to insert the assumption $u \cong u_i$ before $e \cong C_i\,x_i$, so that $u \cong u_i$ is available for checking that e and $C_i\,x_i$ have the same type.

The equivalence assumptions in Γ could become inconsistent, for example, while checking the `false` branch when the scrutinee is `true`. In that case, the assumption $\mathtt{true} \cong \mathtt{false}$ is added to the context. However, this branch is inaccessible at runtime, so there is no need to type check it. Therefore, rule E_INCON assigns an arbitrary type τ to e when the equivalence assumptions in Γ are contradictory.

The last typing rule is a conversion rule. If e can be assigned type τ, then E_TCONV allows e to be given any well-kinded type that is equivalent to τ.

3.3 Equivalence

Several typing rules require determining when two types are equivalent. One type formation rule requires kind equivalence. We present these two equivalence judgments for $\lambda^{\cong}$ in Figure 7. These judgments do not check well-formedness. Instead, the formation rules only use the equivalence judgments on well-formed constructs. For instance, in E_TCONV rule, both τ and τ' must be well-kinded. This framework simplifies the metatheory of $\lambda^{\cong}$ because the properties of equivalence may be proven independently of those for formation.

Most of the rules are straightforward. Below, we focus on the type equivalence rules. The type equivalence judgment has the form $\Delta \vdash \tau_1 \equiv \tau_2$, where Δ is the equivalence context under which τ_1 and τ_2 are considered.

The first rule, TQ_INCON, states that when Δ is inconsistent, any two types are equivalent. The next few rules are congruence rules stating that two types are equivalent if the corresponding sub-terms are equivalent. Rule TQ_APP uses **isEq** to check the equivalence of the two embedded terms. The congruence rule for case types TQ_CASE checks that the corresponding branches are equivalent with added assumptions that the actual index is equal to the stated index of the constructor and that the scrutinee is equal to pattern for that branch. This rule must check not only the equivalence of the scrutinees, but also that the indices in the scrutinees' types are equal. Because our equivalence rules do not depend on well-formedness rules, the only way to find out the type of the scrutinee is to annotate the case type with $\langle T\,u\rangle$.

The last two rules consider the situation when a case type could reduce along one of the branches. The rule TQ_RED2 is symmetric to TQ_RED1. The first premise of TQ_RED1

$$\boxed{\Delta \vdash \kappa \equiv \kappa'}$$

$$\frac{}{\Delta \vdash * \equiv *}\ \textsc{KQ\_Refl}$$

$$\frac{\Delta \vdash \tau \equiv \tau' \quad \Delta \vdash \kappa \equiv \kappa'}{\Delta \vdash (x{:}\tau) \Rightarrow \kappa \equiv (x{:}\tau') \Rightarrow \kappa'}\ \textsc{KQ\_Pi}$$

$$\boxed{\Delta \vdash \tau \equiv \tau'}$$

$$\frac{\mathbf{incon}\,(\,\Delta\,)}{\Delta \vdash \tau \equiv \tau'}\ \textsc{TQ\_InCon}$$

$$\frac{}{\Delta \vdash \mathbf{Unit} \equiv \mathbf{Unit}}\ \textsc{TQ\_URefl}$$

$$\frac{T : \kappa \in \Sigma_0}{\Delta \vdash T \equiv T}\ \textsc{TQ\_TRefl}$$

$$\frac{\Delta \vdash \tau_1 \equiv \tau_1' \quad \Delta \vdash \tau_2 \equiv \tau_2'}{\Delta \vdash (x{:}\tau_1) \to \tau_2 \equiv (x{:}\tau_1') \to \tau_2'}\ \textsc{TQ\_Pi}$$

$$\frac{\Delta \vdash \tau_1 \equiv \tau_1' \quad \Delta \vdash \tau_2 \equiv \tau_2'}{\Delta \vdash \Sigma x{:}\tau_1.\, \tau_2 \equiv \Sigma x{:}\tau_1'.\, \tau_2'}\ \textsc{TQ\_Sigma}$$

$$\frac{\Delta \vdash \tau \equiv \tau' \quad \mathbf{isEq}\,(\,\Delta\,,\, e\,,\, e'\,)}{\Delta \vdash \tau\, e \equiv \tau'\, e'}\ \textsc{TQ\_App}$$

$$\frac{\begin{array}{l}\mathbf{isEq}\,(\,\Delta\,,\, e\,,\, e'\,) \quad \mathbf{isEq}\,(\,\Delta\,,\, u\,,\, u'\,)\\ \overline{C_i : (x_i{:}\sigma_i) \to T\,u_i \in \Sigma_0}^{\,i\in 1..n}\\ \overline{\Delta\,,\, u \cong u_i\,,\, e \cong C_i\,x_i \vdash \tau_i \equiv \tau_i'}^{\,i\in 1..n}\end{array}}{\begin{array}{c}\Delta \vdash \mathbf{case}\, e\,\langle\, T\,u\,\rangle\, \mathbf{of}\, \{\, \overline{C_i\,x_i \Rightarrow \tau_i}^{\,i\in 1..n}\,\} \equiv \\ \mathbf{case}\, e'\,\langle\, T\,u'\,\rangle\, \mathbf{of}\, \{\, \overline{C_i\,x_i \Rightarrow \tau_i'}^{\,i\in 1..n}\,\}\end{array}}\ \textsc{TQ\_Case}$$

$$\frac{\begin{array}{l}\mathbf{isEq}\,(\,\Delta\,,\, e\,,\, C_j\,w\,) \quad C_j \in \overline{C_i}^{\,i\in 1..n}\\ C_j : (x_j{:}\sigma_j) \to T\,u_j \in \Sigma_0\\ \mathbf{isEq}\,(\,(\,\Delta\,,\, w \cong x_j\,)\,,\, u\,,\, u_j\,)\\ \Delta\,,\, w \cong x_j\,,\, e \cong C_j\,x_j \vdash \tau_j \equiv \tau\end{array}}{\Delta \vdash \mathbf{case}\, e\,\langle\, T\,u\,\rangle\, \mathbf{of}\, \{\, \overline{C_i\,x_i \Rightarrow \tau_i}^{\,i\in 1..n}\,\} \equiv \tau}\ \textsc{TQ\_Red1}$$

$$\frac{\begin{array}{l}\mathbf{isEq}\,(\,\Delta\,,\, e\,,\, C_j\,w\,) \quad C_j \in \overline{C_i}^{\,i\in 1..n}\\ C_j : (x_j{:}\sigma_j) \to T\,u_j \in \Sigma_0\\ \mathbf{isEq}\,(\,(\,\Delta\,,\, w \cong x_j\,)\,,\, u\,,\, u_j\,)\\ \Delta\,,\, w \cong x_j\,,\, e \cong C_j\,x_j \vdash \tau \equiv \tau_j\end{array}}{\Delta \vdash \tau \equiv \mathbf{case}\, e\,\langle\, T\,u\,\rangle\, \mathbf{of}\, \{\, \overline{C_i\,x_i \Rightarrow \tau_i}^{\,i\in 1..n}\,\}}\ \textsc{TQ\_Red2}$$

Figure 7. Kind and Type Equivalence

checks if the scrutinee e is equal to some pure term $C_j\,w$, where C_j heads one of the patterns. The rule also checks that the index u in e's type is equal to u_j, which is the index of $C_j\,x_j$'s type. If the j^{th} branch τ_j is equivalent to a type τ (which does not contain x_j by the variable convention), then we can conclude that the case type is equivalent to τ.

Like E_CASE, TQ_RED1 extends Δ with the equation $w \cong x_j$ rather than using explicit substitution. Notice that when checking if τ_j is equal to τ, both $w \cong x_j$ and $e \cong C_j\,x_j$ are in the context. Although the latter assumption is semantically redundant, not including this assumption leads to stronger requirements for **isEq**. Another design choice is why we require a pure term $C_j\,w$ in the first premise, instead of $C_j\,v$ or $C_j\,e$. We address this decision in Section 6.1.

[2] Agda includes some ad hoc machinery that typechecks this particular example, but breaks down on small variations of it.

PROPERTY 4.1 (IsEq Weakening).
If $\mathbf{isEq}\,((\Delta\,,\,\Delta'')\,,\,e_1\,,\,e_2\,)$,
then $\mathbf{isEq}\,((\Delta\,,\,\Delta'\,,\,\Delta'')\,,\,e_1\,,\,e_2\,)$.

PROPERTY 4.2 (IsEq Substitution). *If* $\mathbf{isEq}\,(\Delta\,,\,e_1\,,\,e_2\,)$*, then* $\mathbf{isEq}\,(\Delta\{w/x\}\,,\,e_1\{w/x\}\,,\,e_2\{w/x\}\,)$.

PROPERTY 4.3 (IsEq Cut).
If $\mathbf{isEq}\,((\Delta\,,\,u_1 \cong u_2\,,\,\Delta')\,,\,e_1\,,\,e_2\,)$*, and* $\mathbf{isEq}\,(\Delta\,,\,u_1\,,\,u_2\,)$*,*
then $\mathbf{isEq}\,((\Delta\,,\,\Delta')\,,\,e_1\,,\,e_2\,)$.

PROPERTY 4.4 (IsEq Context Conversion).
If $\mathbf{isEq}\,(\Delta\,,\,e_1\,,\,e_2\,)$*, and* $\Delta \equiv_{\mathbf{ctx}} \Delta'$*, then* $\mathbf{isEq}\,(\Delta'\,,\,e_1\,,\,e_2\,)$.

PROPERTY 4.5 (IsEq Reflexivity). $\mathbf{isEq}\,(\Delta\,,\,e\,,\,e\,)$.

PROPERTY 4.6 (IsEq Symmetry). *If* $\mathbf{isEq}\,(\Delta\,,\,e_1\,,\,e_2\,)$*, then* $\mathbf{isEq}\,(\Delta\,,\,e_2\,,\,e_1\,)$.

PROPERTY 4.7 (IsEq Transitivity). *If* $\mathbf{isEq}\,(\Delta\,,\,e_1\,,\,e_2\,)$*, and* $\mathbf{isEq}\,(\Delta\,,\,e_2\,,\,e_3\,)$*, then* $\mathbf{isEq}\,(\Delta\,,\,e_1\,,\,e_3\,)$.

PROPERTY 4.8 (IsEq Injectivity).
If $\mathbf{isEq}\,(\Delta\,,\,C\,w_1\,,\,C\,w_2\,)$*, then* $\mathbf{isEq}\,(\Delta\,,\,w_1\,,\,w_2\,)$.

PROPERTY 4.9 (IsEq Beta). *If* $e \longrightarrow e'$*, then* $\mathbf{isEq}\,(\cdot\,,\,e\,,\,e'\,)$.

PROPERTY 4.10 (IsEq Empty).
If $C_i \neq C_j$*, then* $\neg\mathbf{isEq}\,(\cdot\,,\,C_i\,w_i\,,\,C_j\,w_j\,)$.

Figure 8. The **isEq** Properties

Our type equivalence rules are defined to be easily invertible. For example, by examining the rules, we can conclude that there does not exist a derivation for $\Delta \vdash T\,e \equiv (x{:}\tau_1) \rightarrow \tau_2$ when Δ is consistent, an important property for the progress and preservation lemmas.

4. Properties of the type system

The type system of $\lambda^{\cong}$ depends on the relation $\mathbf{isEq}\,(\Delta\,,\,e_1\,,\,e_2\,)$. Consequently, the type safety property of $\lambda^{\cong}$ depends on properties of this relation. In this Section, we investigate the properties shown in Figure 8 that we use in the proof of the progress and preservation lemmas. Although these proofs are straightforward, we include details here to motivate each of the properties listed in Figure 8.

Note that these properties are independent of the type system. We make no requirements that the arguments to **isEq** have the same type, or even have a type, or that the assumptions in the equivalence context are well-formed in any way. Thus our parameterization is simple and well-defined.

4.1 Basic lemmas

We start with four basic properties (weakening, substitution, cut, and context conversion) that should hold for every judgment. Because our judgments include **isEq** as a hypothesis, these properties are required for **isEq** (see the first four properties in Figure 8).

Weakening states that if a judgment holds under context Γ (or Δ), then it also holds under a larger context.

LEMMA 4.1 (Weakening).

1. *If* $\Delta_1\,,\,\Delta_3 \vdash J$*, then* $\Delta_1\,,\,\Delta_2\,,\,\Delta_3 \vdash J$.
2. *If* $\Gamma_1\,,\,\Gamma_3 \vdash J$*, and* $\vdash \Gamma_1\,,\,\Gamma_2\,,\,\Gamma_3$*, then* $\Gamma_1\,,\,\Gamma_2\,,\,\Gamma_3 \vdash J$.

The Substitution Lemma states that equivalence judgments are closed under the substitution of *pure* terms and that the formation judgments are closed under the substitution of values.

LEMMA 4.2 (Substitution).

1. *If* $\Delta \vdash J$ *then* $\Delta\{w/x\} \vdash J\{w/x\}$.
2. *If* $\Gamma, x : \tau_1\,,\,\Gamma' \vdash J$ *and* $\Gamma \vdash v : \tau_1$
   then $\Gamma\,,\,\Gamma'\{v/x\} \vdash J\{v/x\}$.

Because our language has a call-by-value semantics, we do not need this property to be true for arbitrary terms, only pure terms and values respectively. As a result, **isEq** need only be closed over the substitution of pure terms. Property 4.2 is a particularly weak requirement. We discuss variations of it in more detail in Section 6.

The Cut Lemma removes redundant equivalence assumptions from the context.

LEMMA 4.3 (Cut).

1. *If* $\Delta\,,\,e \cong e'\,,\,\Delta' \vdash J$ *and* $\mathbf{isEq}\,(\Delta\,,\,e\,,\,e'\,)$
   then $\Delta\,,\,\Delta' \vdash J$.
2. *If* $\Gamma\,,\,e \cong e'\,,\,\Gamma' \vdash J$ *and* $\mathbf{isEq}\,(\Gamma^\star\,,\,e\,,\,e'\,)$
   then $\Gamma\,,\,\Gamma' \vdash J$.

Finally, both the equivalence judgments and the formation judgments are closed under equivalent contexts. First, define context equivalence as follows:

DEFINITION 4.1 (Context equivalence).

$$\frac{}{\vdash \cdot \equiv \cdot}\ \text{CQ\_EMPTY}$$

$$\frac{\vdash \Gamma \equiv \Gamma' \quad \Gamma^\star \vdash \tau \equiv \tau'}{\vdash \Gamma, x : \tau \equiv \Gamma', x : \tau'}\ \text{CQ\_TERM}$$

$$\frac{\vdash \Gamma \equiv \Gamma' \quad \mathbf{isEq}\,(\Gamma^\star\,,\,e_1\,,\,e_1'\,) \quad \mathbf{isEq}\,(\Gamma^\star\,,\,e_2\,,\,e_2'\,)}{\vdash (\Gamma\,,\,e_1 \cong e_2\,) \equiv (\Gamma'\,,\,e_1' \cong e_2'\,)}\ \text{CQ\_EQ}$$

Then we can show that all formation judgments are stable under this equivalence, and that all equivalence judgments are stable under the equivalence context equivalence.

LEMMA 4.4 (Context Conversion).

1. *If* $\Delta \vdash J$ *and* $\Delta \equiv_{\mathbf{ctx}} \Delta'$ *then* $\Delta' \vdash J$.
2. *If* $\Gamma \vdash J$ *and* $\vdash \Gamma \equiv \Gamma'$ *and* $\vdash \Gamma'$ *then* $\Gamma' \vdash J$.

4.2 Properties of type equivalence

The type equivalence rules shown in Figure 7 do not contain rules for reflexivity, symmetry, or transitivity, permitting simple inversion. Instead, we prove the following lemmas about the equivalence judgments to show that these rules are admissible. Again, to show these properties, they also must be true of **isEq** (see Properties 4.5-4.7 in Figure 8).

LEMMA 4.5 (Refl). $\Delta \vdash \tau \equiv \tau$.

LEMMA 4.6 (Symm). *If* $\Delta \vdash \tau \equiv \tau'$ *then* $\Delta \vdash \tau' \equiv \tau$.

LEMMA 4.7 (Transitivity). *If* $\Delta \vdash \tau \equiv \tau'$ *and* $\Delta \vdash \tau' \equiv \tau''$ *then* $\Delta \vdash \tau \equiv \tau''$.

The proofs of reflexivity and symmetry are straightforward, but transitivity is less so, so we show one case of the proof below. This proof motivates the Cut and Injection Properties as well as our definition of **incon**. It also explains why the Substitution Lemma requires a pure term w. To show transitivity, we must first generalize the statement of the lemma so that the contexts of the two type equivalence derivations are not the same, but are equivalent.

LEMMA 4.8 (Transitivity'). *If $\Delta \vdash \tau \equiv \tau'$ and $\Delta' \vdash \tau' \equiv \tau''$ and $\Delta \equiv_{\mathbf{ctx}} \Delta'$ then $\Delta \vdash \tau \equiv \tau''$.*

The proof is by a double induction on the structure of the pair of assumed judgments; call the first one $\mathcal{D}$ and the second second one $\mathcal{E}$. Consider the case where the last rule used in $\mathcal{D}$ is TQ_RED2 and the last rule of $\mathcal{E}$ is TQ_RED1. Then, these derivations are of the form:

$$\frac{\begin{array}{ll} C_j \in \overline{C_i}^{i\in 1..n} & C_j : (x_j{:}\sigma_j) \to T\ u_j \in \Sigma_0 \\ \mathbf{isEq}\,(\,\Delta\,,\ e\,,\ C_j\ w\,) & \mathbf{isEq}\,(\,(\,\Delta\,,\ w \cong x_j\,)\,,\ u\,,\ u_j\,) \\ \multicolumn{2}{l}{\Delta\,,\ w \cong x_j\,,\ e \cong C_j\ x_j \vdash \sigma \equiv \tau_j} \end{array}}{\Delta \vdash \sigma \equiv \mathbf{case}\ e\,\langle\, T\ u\,\rangle\ \mathbf{of}\ \{\,\overline{C_i\ x_i \Rightarrow \tau_i}^{i\in 1..n}\,\}}$$

and

$$\frac{\begin{array}{ll} C_n \in \overline{C_i}^{i\in 1..n} & C_n : (x_n{:}\sigma_n) \to T\ u_n \in \Sigma_0 \\ \mathbf{isEq}\,(\,\Delta'\,,\ e\,,\ C_n\ w'\,) & \mathbf{isEq}\,(\,(\,\Delta'\,,\ w' \cong x_n\,)\,,\ u\,,\ u_n\,) \\ \multicolumn{2}{l}{\Delta'\,,\ w' \cong x_n\,,\ e \cong C_n\ x_n \vdash \tau_n \equiv \sigma'} \end{array}}{\Delta' \vdash \mathbf{case}\ e\,\langle\, T\ u\,\rangle\ \mathbf{of}\ \{\,\overline{C_i\ x_i \Rightarrow \tau_i}^{i\in 1..n}\,\} \equiv \sigma'}$$

We need to show that $\Delta \vdash \sigma \equiv \sigma'$. To use the induction hypothesis, we need to know that both $\mathcal{E}$ and $\mathcal{D}$ reduce using the same branch. In other words, $j = n$. We know that $\mathbf{isEq}\,(\,\Delta\,,\ e\,,\ C_j\ w\,)$ and $\mathbf{isEq}\,(\,\Delta'\,,\ e\,,\ C_n\ w'\,)$. By the Symmetry, Transitivity, and Context Conversion Properties of **isEq**, we conclude that $\mathbf{isEq}\,(\,\Delta\,,\ C_j\ w\,,\ C_n\ w'\,)$. To continue the proof, we must conclude either that $C_j = C_n$ or that Δ is inconsistent, hence our definition of $\mathbf{incon}\,(\,\Delta\,)$.

Now suppose that $j = n$. To apply the induction hypothesis, we must show

$$(\,\Delta\,,\ w \cong x_j\,,\ e \cong C_j\ x_j\,) \equiv_{\mathbf{ctx}} (\,\Delta'\,,\ w' \cong x_j\,,\ e \cong C_j\ x_j\,)$$

We have $\Delta \equiv_{\mathbf{ctx}} \Delta'$ by assumption, so for these two contexts to be equivalent, we need only show $\mathbf{isEq}\,(\,\Delta\,,\ w\,,\ w'\,)$. We also have $\mathbf{isEq}\,(\,\Delta\,,\ C_j\ w\,,\ C_j\ w'\,)$, so the Injection Property (4.8) of **isEq** suffices.

By applying the induction hypothesis, we have $\Delta\,,\ w \cong x_j\,,\ e \cong C_j\ x_j \vdash \sigma \equiv \sigma'$. By substituting w for x_j, we conclude that $\Delta\,,\ w \cong w\,,\ e \cong C_j\ w \vdash \sigma \equiv \sigma'$ (because x_j is not free in Δ, e, σ and σ'). To conclude $\Delta \vdash \sigma \equiv \sigma'$, we need only remove $w \cong w$ and $e \cong C_j\ w$ from the context. We already know these facts via reflexivity and assumption, so we use the Cut Lemma (4.3), finishing the case.

In this proof we must substitute a pure term w into the judgment, not a value v. For that reason, our Substitution Lemma (4.2) on equivalence must hold for pure terms.

4.3 Type safety

We prove type safety for our language via standard progress and preservation Lemmas [Wright and Felleisen 1994].

LEMMA 4.9 (Preservation). *If $\Gamma \vdash e : \tau$ and $e \longrightarrow e'$, then $\Gamma \vdash e' : \tau$.*

The proof is by induction on the reduction relation. In some of the cases, the typing of e' depends on a subterm in e' that takes a step. Those cases motivate the IsEq Beta Property (4.9). We use the case when $e = e_1\ e_2$ and $e_2 \longrightarrow e_2'$ as an example. By assumption we know that

$$\frac{\begin{array}{ll} \Gamma \vdash e_1 : (x{:}\tau_1) \to \tau_2 & \Gamma \vdash e_2 : \tau_1 \\ \Gamma^\star\,,\ x \cong e_2 \vdash \tau_2 \equiv \tau & \Gamma \vdash \tau : * \end{array}}{\Gamma \vdash e_1\ e_2 : \tau}\ \text{E\_APP}$$

By the induction hypothesis, we know $\Gamma \vdash e_2' : \tau_1$. We need to show that $\Gamma^\star\,,\ x \cong e_2' \vdash \tau_2 \equiv \tau$. Because Property 4.9 requires **isEq** to identify e_2 and e_2', we know that the context $\Gamma^\star\,,\ x \cong e_2$ is equivalent to the context $\Gamma^\star\,,\ x \cong e_2'$. Therefore, by using context conversion (Lemma 4.4), we can conclude $\Gamma^\star\,,\ x \cong e_2' \vdash \tau_2 \equiv \tau$.

To show progress, we must first prove a canonical forms lemma.

LEMMA 4.10 (Canonical Forms). *Suppose $\neg\mathbf{incon}\,(\,\Gamma^\star\,)$.*

1. *If $\Gamma \vdash v : \mathbf{Unit}$ then v is $\mathbf{unit}$.*
2. *If $\Gamma \vdash v : (x{:}\tau_1) \to \tau_2$ then v is $\mathbf{fun}\,f(x) = e$.*
3. *If $\Gamma \vdash v : \Sigma x{:}\tau_1.\,\tau_2$ then v is $\langle\, v_1\,,\ v_2\,\rangle$.*
4. *If $\Gamma \vdash v : T\ e$ then v is $C\ v'$ and $C : (x{:}\sigma) \to T\ u \in \Sigma_0$.*

To prove the above, we show that the type system does not equate types with different top level forms when the assumptions in the equivalence context are consistent.

DEFINITION 4.2 (Value types). *A type τ is a* value type *if it is of the top level form $\mathbf{Unit}$, $\Sigma x{:}\sigma_1.\,\sigma_2$, $(x{:}\sigma_1) \to \sigma_2$, or $T\ e$.*

LEMMA 4.11 (Value Type Consistency). *If $\neg\mathbf{incon}\,(\,\Delta\,)$ and $\Delta \vdash \tau_1 \equiv \tau_2$, where τ_1 and τ_2 are value types, then τ_1 and τ_2 have the same top-level structure.*

LEMMA 4.12 (Progress). *If $\cdot \vdash e : \tau$, then $\exists e'.\ e \longrightarrow e'$ or e is a value.*

In the proof of this lemma we show that $\neg\mathbf{incon}\,(\,\cdot\,)$, so the Canonical Forms Lemma is available. Because $\mathbf{incon}\,(\,\Delta\,)$ is defined in terms of **isEq**, we require that the empty context be consistent, *i.e.* that $\neg\mathbf{isEq}\,(\,\cdot\,,\ C_i\ w_i\,,\ C_j\ w_j\,)$ if $C_i \neq C_j$ (cf. Prop 4.10).

A straightforward application of preservation and progress gives us the final result: Well-typed $\lambda^{\cong}$ programs do not get stuck.

THEOREM 4.1 (Type Safety). *If $\cdot \vdash e : \tau$, then either there exists a v such that $e \longrightarrow^* v$ or e diverges.*

5. Instantiations

Having identified a set of properties of **isEq** that are strong enough to prove type safety, we now examine definitions of term equivalence that satisfy those properties.

It is not hard to see that any instantiation is undecidable: let **isEqX** be some instantiation and consider the predicate $\phi(e) = \mathbf{isEqX}(\cdot, e, C_1\ \mathbf{unit})$. The properties require this predicate to be nontrivial (since $\phi(C_1\ \mathbf{unit})$ but $\neg\phi(C_2\ \mathbf{unit})$) and respect beta-convertibility, so by a lambda calculus variant of Rice's theorem ([Barendregt 1981] p.144) ϕ is undecidable.

However, we could have a decidable predicate that does not satisfy the **isEq** properties but still allows type safety to hold for $\lambda^{\cong}$. Suppose we have an instantiation **isEqX**, and consider a predicate $\mathbf{isEqX}'$ which is dominated by **isEqX**, that is if $\mathbf{isEqX}'$ returns true then so does **isEqX**. Then any program that typechecks using $\mathbf{isEqX}'$ will also typecheck using **isEqX**, and type safety for **isEqX** tells us that the program will never reach a stuck state.

What we are seeing here is the need to make a distinction between type safety and preservation/progress. Any predicate that is dominated by one that satisfies the properties is sufficiently weak to ensure type safety, so it is safe to use it in a programming language implementation. Such a predicate will not necessarily be strong enough to typecheck all the intermediate states of a computation.

5.1 Beta-equivalence

Many dependently-typed languages use beta-equivalence as the underlying equivalence of the type system. In this section, we show that beta-equivalence is indeed a valid instantiation that satisfies the properties in Figure 8.

Call-by-value evaluation Some dependently typed languages test term equivalence by reducing both inputs to a normal form and then comparing, so one expects this algorithm to be a valid instantiation. Indeed it is, although we must adjust the definition slightly: because of nontermination we cannot reduce to normal form, so instead we say that two terms are **isEq** if they reduce to *some* common term (not necessarily normal). As a result, the predicate is only semidecidable because we do not know how long to evaluate. Thus we define our first instantiation, called $\mathbf{isEq}_{\longrightarrow}$.

DEFINITION 5.1. *Define* $\mathbf{isEq}_{\longrightarrow}(\Delta, e, e')$ *when there exists* u *such that* $e \longrightarrow^* u$ *and* $e' \longrightarrow^* u$.

LEMMA 5.1. $\mathbf{isEq}_{\longrightarrow}$ *satisfies the* **isEq** *properties.*

Note that $\mathbf{isEq}_{\longrightarrow}$ is the finest equivalence satisfying the properties. Because we require that **isEq** be an equivalence relation which includes $\longrightarrow$, any valid instantiation must identify at least as many terms as $\mathbf{isEq}_{\longrightarrow}$.

LEMMA 5.2. *Let* **isEqX** *be a predicate which satisfies the* **isEq** *properties. Then* $\mathbf{isEq}_{\longrightarrow}(\Delta, e, e')$ *implies* $\mathbf{isEqX}(\Delta, e, e')$.

Generalized reduction relations The verification that $\mathbf{isEq}_{\longrightarrow}$ satisfies the properties does not use many specific facts about $\longrightarrow$. Therefore, we can state a more general result about an arbitrary reduction relation $\rightsquigarrow$.

DEFINITION 5.2. *If* $\rightsquigarrow$ *is a binary relation between expressions, then define* $\mathbf{isEq}_{\rightsquigarrow}(\Delta, e_1, e_2)$ *when there exists a* u *such that* $e_1 \rightsquigarrow^* u$ *and* $e_2 \rightsquigarrow^* u$.

LEMMA 5.3. *For a given relation on expressions* $\rightsquigarrow$, *if*

- $\longrightarrow \subseteq \rightsquigarrow$,
- $e \rightsquigarrow e'$ *implies* $e\{w/x\} \rightsquigarrow e'\{w/x\}$,
- $C\ e_0 \rightsquigarrow e'$ *implies that* $e' = C\ e_0'$ *and* $e_0 \rightsquigarrow e_0'$, *and*
- $\rightsquigarrow^*$ *is confluent,*

then $\mathbf{isEq}_{\rightsquigarrow}$ *satisfies the* **isEq** *properties.*

The added generality of the above lemma shows that type safety is insensitive to the evaluation order used by the type checker. In particular, we can use a parallel reduction relation for $\rightsquigarrow$, where terms are nondeterministically reduced throughout, including underneath function definitions and inside case branches. In fact, there are many valid variants of parallel reduction, based on differences in the beta rules.

We identify three variants of parallel reduction below.

$e \Longrightarrow e'$	*Require values in active positions*
$e \Longrightarrow_w e'$	*Require pure terms in active positions*
$e \Longrightarrow_n e'$	*Allow arbitrary reductions*

Surprisingly, all three of these relations are sound, including the last variant which permits $\beta-$reductions for arbitrary expressions. For example, this relation allows the type checker to identify $(\lambda x.y)\,\Omega$ and y—a rather strange fact since these terms are not contextually equivalent under call-by-value evaluation.

However, note that deterministic call-by-name evaluation $\longrightarrow_n$, which never evaluates the argument of an application, is not a valid instantiation. This relation does not contain call-by-value evaluation, so $\mathbf{isEq}_{\longrightarrow_n}$ does not satisfy the Beta property (4.9). Nevertheless, $\mathbf{isEq}_{\longrightarrow_n}$ is strictly dominated by $\mathbf{isEq}_{\Longrightarrow_n}$, which *is* a valid instantiation. This means that even though our language is CBV, it is safe to use CBN evaluation in the type checker.

Expressivity The $\mathbf{isEq}_{\rightsquigarrow}$ instantiations formally satisfy the properties and highlight the similarities between our system and other dependently-typed languages, but they are of minimal use: our type system relies on introducing equations into the context, but $\mathbf{isEq}_{\rightsquigarrow}$ does not even look at them! This is only possible because the properties do not force **isEq** to make use of the context; in particular we do not require the following property:

PROPERTY 5.1 (Assumption).
If $e_1 \cong e_2 \in \Delta$ *then* $\mathbf{isEq}\,(\,\Delta\,,\ e_1\,,\ e_2\,)$.

As we have seen, this property is not necessary for type safety, so we do not require it. However, it *is* interesting when we consider the expressivity of our type system. In fact, the equivalence assumptions provide all the "dependent" features of our type system: if the **isEq** instantiation ignores them, we can type no more terms than in the simply typed lambda calculus.

DEFINITION 5.3. *Define a type erasure function* $(\cdot)^o$, *mapping types* τ *to simple types, as follows:*

$$\begin{array}{lll} (\mathbf{Unit})^o = \mathbf{Unit} & & ((x{:}\tau_1) \to \tau_2)^o = (\tau_1)^o \to (\tau_2)^o \\ (T)^o = T & & (\Sigma x{:}\tau_1.\,\tau_2)^o = (\tau_1)^o \times (\tau_2)^o \\ (\tau\ e)^o = (\tau)^o & & \end{array}$$

$$(\mathbf{case}\ e\ \langle\ T\ u\ \rangle\ \mathbf{of}\ \{\ \overline{C_i\ x_i\ \Rightarrow\ \tau_i}^{\,i}\ \})^o = \begin{cases} (\tau_i)^o & \textit{if}\ \mathbf{isEq}\,(\,\cdot\,,\ e\,,\ C_i\ w\,) \\ \mathbf{Unit} & \textit{otherwise} \end{cases}$$

We write Γ^o *to denote the pointwise lifting of the erase operation applied to* Γ *with all of its equivalence assumptions removed.*

LEMMA 5.4 (Erasure). *Suppose that* $\mathbf{isEq}\,(\,\Delta\,,\ e_1\,,\ e_2\,)$ *iff* $\mathbf{isEq}\,(\,\cdot\,,\ e_1\,,\ e_2\,)$. *Then* $\Gamma \vdash e : \tau$ *implies* $\Gamma^o \vdash_{STLC} e : \tau^o$, *where* $\vdash_{STLC}$ *is the type system for the simply-typed lambda calculus with unit, products and datatypes.*

5.2 Beta-equivalence with assumptions

To extend $\mathbf{isEq}_{\longrightarrow}$ to a relation satisfying the Assumption Property we can give a direct inductive definition and include enough rules to satisfy the properties:

DEFINITION 5.4 (**isEqFiat**).
Define the relation $\mathbf{isEqFiat}\,(\,\Delta\,,\ e_1\,,\ e_2\,)$ *as the least relation satisfying the following rules:*

$$\frac{e_1 \cong e_2 \in \Delta}{\mathbf{isEqFiat}\,(\,\Delta\,,\ e_1\,,\ e_2\,)} \qquad \frac{e_1 \longrightarrow e_2}{\mathbf{isEqFiat}\,(\,\Delta\,,\ e_1\,,\ e_2\,)}$$

$$\frac{\mathbf{isEqFiat}\,(\,\Delta\,,\ C\ w_1\,,\ C\ w_2\,)}{\mathbf{isEqFiat}\,(\,\Delta\,,\ w_1\,,\ w_2\,)}$$

$$\frac{}{\mathbf{isEqFiat}\,(\,\Delta\,,\ e\,,\ e\,)} \qquad \frac{\mathbf{isEqFiat}\,(\,\Delta\,,\ e_1\,,\ e_2\,)}{\mathbf{isEqFiat}\,(\,\Delta\,,\ e_2\,,\ e_1\,)}$$

$$\frac{\mathbf{isEqFiat}\,(\,\Delta\,,\ e_1\,,\ e_2\,) \quad \mathbf{isEqFiat}\,(\,\Delta\,,\ e_2\,,\ e_3\,)}{\mathbf{isEqFiat}\,(\,\Delta\,,\ e_1\,,\ e_3\,)}$$

LEMMA 5.5. **isEqFiat** *satisfies the* **isEq** *properties.*

Properties 4.5–4.9 hold for **isEqFiat** by its definition. The properties about substitution and context operations are proved by easy inductions on **isEqFiat** $(\Delta\,,\ e\,,\ e')$. Finally we get the Empty property for free since when Δ is empty **isEqFiat** coincides with **isEq**$_{\longrightarrow}$.

Just like **isEq**$_{\longrightarrow}$, we can vary the evaluation relation used in the second rule—any relation that works for **isEq**$_{\leadsto}$ also works for **isEqFiat**. We use the notation **isEqFiat**$_{\leadsto}$ for alternate versions of this relation.

Like **isEq**$_{\longrightarrow}$, **isEqFiat**$_{\longrightarrow}$ is semidecidable. However, its definition does not suggest a particularly efficient algorithm to search for derivations. Therefore, **isEqFiat** is a specification of equivalence: the type checker can safely use any algorithm that is dominated by **isEqFiat**.

5.3 Contextual equivalence with assumptions

In the previous subsections we showed that various beta-equivalences are valid instantiations. Our ultimate goal, however, is to find the strongest equivalence we can; then an implementation can use anything weaker than it and be assured of type safety. The natural instantiation to aim for then is contextual equivalence. If we can show that contextual equivalence satisfies the properties, then an implementation will be free to use any known technique from the literature in its equivalence-checking algorithm.

Therefore we must state what it means for two terms to be contextually equivalent in the presence of equivalence assumptions. We take as our starting point the notion of CIU-equivalence, which is one of many equivalent definitions of contextual equivalence [Mason and Talcott 1991]. It says that two terms are equivalent if all Closed Instantiations (substitutions of values for free variables) of them have the same termination behavior when Used (placed in a closed evaluation context).

The one subtlety here is what evaluation relation we should consider the termination behavior under. Recall that the type-equivalence rule for **case** will reduce with an open scrutinee $C\ w$, while the operational semantics will only reduce when the scrutinee is a closed value $C\ v$. The **isEq** predicate is part of typechecking, so it is the former behavior that is relevant and shows up in Empty (Prop 4.9); for instance we must not identify the stuck terms $C_1\ (\,(\,\lambda x.x\,)\,.\,1\,)$ and $C_2\ (\,(\,\lambda x.x\,)\,.\,1\,)$ even though they are contextually equivalent under CBV reduction.

Therefore, we define a "CBW" variant of the evaluation relation, which we write $\longrightarrow_w$. This relation is exactly the same as $\longrightarrow$ except that it replaces all vs with terminal ws. For example, the beta rule reads:

$$\frac{w_2 \not\longrightarrow_w}{(\,\mathbf{fun}\,f(x)\ =\ e_1\,)\ w_2\ \longrightarrow_w\ e_1\{w_2/x\}\{\mathbf{fun}\,f(x)\ =\ e_1/f\}}$$

In the definition of contextual equivalence, we use the $\longrightarrow_w$ relation and let the substitutions range over ws.

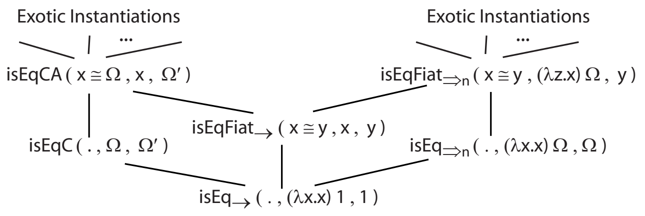
Note that this subtlety is only for stuck terms. For well-typed terms, it does not matter whether we use $\longrightarrow$ or $\longrightarrow_w$, the same terms will be equated. Therefore, we are justified in considering this a "CBV" contextual equivalence.

DEFINITION 5.5. *Define $e \Downarrow$ if there exists u such that $e\ \longrightarrow_w^*\ u$ and not $u\ \longrightarrow_w\ u'$ for any u'.*

Now define evaluation contexts in the standard manner.

DEFINITION 5.6 (Evaluation contexts).

$$\begin{array}{lcl} E & ::= & \Box \mid E\,e \mid v\,E \mid \langle\, E\,,\ e\,\rangle \mid \langle\, v\,,\ E\,\rangle \mid E\,.\,1 \\ & \mid & E\,.\,2 \mid C\,E \mid \mathbf{case}\,E\,\mathbf{of}\,\{\,\overline{C_i\,x_i\ \Rightarrow\ e_i}^{\,i}\,\} \end{array}$$



Exotic Instantiations
...
Exotic Instantiations
...
isEqCA (x ≅ Ω , x , Ω′)
isEqFiat⟹n (x ≅ y , (λz.x) Ω , y)
isEqFiat→ (x ≅ y , x , y)
isEqC (. , Ω , Ω′)
isEq⟹n (. , (λx.x) Ω , Ω)
isEq→ (. , (λx.x) 1 , 1)

Figure 9. Inclusions between the instantiations

DEFINITION 5.7 (CBV Contextual Equivalence). *Define* **isEqC** $(\,e_1\,,\ e_2\,)$ *iff $\forall E$, $\forall\delta$ such that δ maps variables to ws, if $E\,[\,\delta e_1\,]$ and $E\,[\,\delta e_2\,]$ are closed then $E\,[\,\delta e_1\,]\Downarrow$ iff $E\,[\,\delta e_2\,]\Downarrow$.*

As one might expect, **isEqC** satisfies the **isEq** properties. However, it does not make any use of the equivalence context. The key idea to generalize the definition is to restrict what instantiations should be considered. For instance, if the context contains the equivalence $e_1\ \cong\ e_2$, we should only consider substitutions that make e_1 and e_2 equal. We thus introduce a new judgment $\Delta \vdash \delta$ (pronounced "δ respects Δ") as follows.

DEFINITION 5.8 (Equivalence respecting substitution).

$$\frac{}{\cdot \vdash \delta} \qquad \frac{\Delta \vdash \delta \quad \mathbf{isEqC}\,(\,\delta e_1\,,\ \delta e_2\,)}{\Delta, e_1 \cong e_2 \vdash \delta}$$

We define two expressions to be equivalent under an equivalence context Δ if they have the same behavior for all substitutions that respect the context.

DEFINITION 5.9 (CBV Contextual Equiv. with Assumptions). *Define* **isEqCA** $(\,\Delta\,,\ e_1\,,\ e_2\,)$ *iff $\forall E$, $\forall\delta$ such that δ maps variables to ws, if $E\,[\,\delta e_1\,]$ and $E\,[\,\delta e_2\,]$ are closed and $\Delta\ \vdash\ \delta$ then $E\,[\,\delta e_1\,]\Downarrow$ iff $E\,[\,\delta e_2\,]\Downarrow$.*

When Δ is empty, **isEqCA** coincides with **isEqC**.

LEMMA 5.6. *The relation* **isEqCA** *satisfies the* **isEq** *properties.*

LEMMA 5.7. *If $e_1 \cong e_2 \in \Delta$, then* **isEqCA** $(\,\Delta\,,\ e_1\,,\ e_2\,)$.

5.4 Exotic Instantiations

The relation **isEqCA** is a strong instantiation, strictly coarser than **isEqFiat**. But it is not the limit—we have already seen that **isEq**$_{\Longrightarrow_n}$ can safely identify terms that are not contextually equivalent. In fact, the **isEq** properties place very weak restrictions on what terms may be identified, the only negative statements are Empty and Injectivity, and they only apply when both terms are of the form $C\ w$.

Therefore, given a valid **isEq** instantiation, we can create another coarser one by merging two of its equivalence classes, as long as the two classes do not contain pure terms headed by different constructors; for instance, contextual equivalence considers all diverging terms to be equal. Certainly no diverging expression is a constructor value, so we can create a coarser instantiation by also saying that any non-terminating term is equal to the integer constant 3 (and all additional equivalences forced by transitivity). Of course, we could also make it equal to 4—but we had better not do both, since then transitivity would make 3 and 4 equal.

This example shows that while there is a weakest valid instantiation, **isEq**$_{\longrightarrow}$, there is no strongest one. Figure 9 summarizes the ordering of the various instantiations we have discussed as a Hasse diagram.

6. Variations

Different versions of our typing rules lead to different requirements for **isEq**, which in turn affects what instantiations of **isEq** are valid. In this section, we present variations to $\lambda^{\cong}$'s type system, show how they lead to stronger properties for **isEq**, and discuss what instantiations are no longer available.

6.1 Values, pure terms or terms

A few rules have flexibility about whether some component must be a value, a pure term, or an unrestricted term. Although the last is the most permissive, we have chosen in some cases to restrict to pure terms to weaken the substitution requirement for **isEq**.

For example, consider the type equivalence rule below.

$$\frac{\begin{array}{l}\mathbf{isEq}\,(\,\Delta\,,\ e\,,\ C_j\,w\,) \quad C_j \in \overline{C_i}^{\,i\in 1..n} \\ C_j : (x_j{:}\sigma_j) \rightarrow T\,u_j \in \Sigma_0 \\ \mathbf{isEq}\,(\,(\,\Delta\,,\ w \cong x_j\,)\,,\ u\,,\ u_j\,) \\ \Delta\,,\ w \cong x_j\,,\ e \cong C_j\,x_j \vdash \tau_j \equiv \tau\end{array}}{\Delta \vdash \mathbf{case}\,e\,\langle\,T\,u\,\rangle\,\mathbf{of}\,\{\,\overline{C_i\,x_i \Rightarrow \tau_i}^{\,i\in 1..n}\,\} \equiv \tau}\ \text{TQ\_RED1}$$

The first precondition requires the scrutinee to be equal to a constructor applied to a pure term. Possible alternatives allow the argument to the constructor to be an arbitrary expression, or require it to be a value.

If we had used an arbitrary expression, then the proof of transitivity in Section 4.2 would require the stronger properties shown below:

PROPERTY 6.1 (Impure Substitution). *If* $\mathbf{isEq}\,(\,\Delta\,,\ e_1\,,\ e_2\,)$, *then* $\mathbf{isEq}\,(\,\Delta\{e/x\}\,,\ e_1\{e/x\}\,,\ e_2\{e/x\}\,)$.

PROPERTY 6.2 (Impure Empty). *If* $C_i \neq C_j$, *then* $\neg\mathbf{isEq}\,(\,\cdot\,,\ C_i\,e_i\,,\ C_j\,e_j\,)$.

Unfortunately, instantiations of **isEq** that are based on CBW-evaluation, such as $\mathbf{isEqBeta}_{\Longrightarrow_w}$ or $\mathbf{isEqCA}$ do not satisfy these properties because they are not closed under substitution of arbitrary terms. For example, if Ω is a nonterminating expression, then $(\,\lambda x.z\,)\,y$ is equivalent to z under $\mathbf{isEqBeta}_{\Longrightarrow_w}$ and $\mathbf{isEqCA}$, but $(\,\lambda x.z\,)\,\Omega$ is not. Further, although $\mathbf{isEqBeta}_{\Longrightarrow_w}$ trivially satisfies Impure Empty, $\mathbf{isEqCA}$ does not; all contexts identify $C_i\,\Omega$ and $C_j\,\Omega'$, which breaks 6.2.

Alternatively, if we require the scrutinee to be equivalent to some constructor *value*, such as $C_j\ v$, then we would limit the expressiveness of the type system. For example, the case type

$$\mathbf{case}\ C_1\,y\ \langle T\ u\rangle\ \mathbf{of}\{C_1\,x_1 \Rightarrow \mathbf{Nat} \mid C_2\,x_2 \Rightarrow \mathbf{Bool}\,\}$$

cannot be shown equivalent to $\mathbf{Nat}$.

Finally, the syntactic categorization of pure terms in $\lambda^{\cong}$ can be viewed as a very weak and conservative termination analysis. However, unlike Coq or Agda, complex termination analysis only slightly increase the expressiveness of $\lambda^{\cong}$'s term language, and only in terms of type convertibility. For instance, if $\lambda^{\cong}$ were to use Coq's termination checker, in the above example, after replacing $C_1\,y$ by $C_1\,(\mathit{factorial}\,n)$, the two types would be still equivalent.

6.2 Substitution versus equivalence assumptions

As we discussed in Section 3.2, some of our typing rules diverge from standard practice in that, instead of substitution, they add equivalence assumptions to the context. We have designed our rules in this manner for two reasons. One reason is that we can make the E_CASE rule more expressive by using equations. A second reason is that stating rules with substitution requires a stronger substitution property for **isEq**. With the the alternate E_APP′ rule in Section 3.2, **isEq** would need to be closed under the substitution of related expressions inside related expressions.

PROPERTY 6.3 (Equivalent substitution).
If $\mathbf{isEq}\,(\,\Delta\,,\ e_1\,,\ e_2\,)$, *and* $\mathbf{isEq}\,(\,\Delta\,,\ e\,,\ e'\,)$,
then $\mathbf{isEq}\,(\,\Delta\{e/x\}\,,\ e_1\{e/x\}\,,\ e_2\{e'/x\}\,)$.

The reason for this property is the need to show a stronger substitution property for type equivalence $\Delta \vdash \tau\{e_2/x\} \equiv \tau\{e_2'/x\}$ in the case of the preservation lemma when e is an application $e_1\,e_2$ and $e_2 \longrightarrow e_2'$. Our previous proof required a weaker lemma that substituted the same pure term throughout the judgment.

We could modify the definitions of **isEqFiat** to satisfy the Equivalent Substitution Property. However, Property 6.3 implies Impure Substitution Property (Property 6.1); therefore, neither $\mathbf{isEq}_{\longrightarrow}$ nor $\mathbf{isEqCA}$ satisfies it.

These two examples show two different axes: whether CBW-respecting relations are allowed and whether the equivalence must be stronger than reflexivity for binders, e.g. is $\lambda x.e$ equivalent to $\lambda x.e'$ when e reduces to e'. It is possible to design the type system that interpolates between these two requirements, requiring a "pure equivalent substitution" property, by maintaining the invariant that only ws are ever substituted in terms. Then $\mathbf{isEq}_{\Longrightarrow_w}$ satisfies the pure equivalent substitution, but $\mathbf{isEq}_{\longrightarrow}$ does not since it does not reduce under the binder.

7. Extensions

Our design of $\lambda^{\cong}$ has been simplified in a few ways so that we can emphasize its novel features. Here, we revisit some aspects of $\lambda^{\cong}$ and discuss extensions that would make it more practical.

Polymorphism For simplicity, $\lambda^{\cong}$ is not polymorphic. Adding Haskell-style higher-order polymorphism [Jones 1995] would require straightforward changes to the language. Another simple extension to the specification of $\lambda^{\cong}$ is first class-polymorphism, as in Curry-style System F [Girard 1972]. (Note that type checking for Curry-style System F is also undecidable [Wells 1999].) In both cases, type abstraction and application would be implicit as we do not wish to include types in the syntax of terms.

Adding abstractions to the type language, such as in F_ω, would require more significant changes. In particular, our definition of type equivalence would have to be extended to include beta-equivalence for these abstractions. A kind-directed specification, which retains the easy inversions of our current definition of type equivalence seems possible, but we leave this extension to future work.

Church-style type system For reasons discussing in Section 2, $\lambda^{\cong}$ does not include typing annotations in expressions. As a result, the type system can assign multiple non-equivalent types to the same expressions. Given the difficulty of complete type inference for dependently-typed languages, a practical source language would include annotations to guide type inference and eliminate ambiguity.

An extension to $\lambda^{\cong}$ with type annotations would take the form of an *external language* that elaborates to and is defined by $\lambda^{\cong}$ typing derivations. This external language would be free to use any type inference technology available for elaboration. As long as elaboration produces valid $\lambda^{\cong}$ typing derivations, this external language is type safe.

Type-directed term equivalence Our design decision that the properties of **isEq** should not refer to the type system means that **isEq** cannot receive any typing information from the type checker, such as type annotations embedded in the terms, or the types of the terms, or a typing context. Therefore, certain type-directed equivalence algorithms [Coquand 1991, Stone and Harper 2000], which use type information to provide stronger extensionality properties, cannot be used for **isEq**. However, in a call-by-value language with nontermination, eta-equivalences are restricted: $\lambda x. e\, x$ is not equivalent to e because e could diverge. Instead, this equivalence only holds for pure terms. Therefore, it is not clear how to extend type-directed equivalences to this setting.

Termination analysis Like most type systems, type safety for this language provides a fairly weak result: *if the language terminates, then the resulting value has the expected type.* In a dependent type system, if the expected type is $\Sigma x{:}\tau.\ P\,(\,x\,)$, then this property is a partial correctness proof that the program satisfies property P. The incorporation of a termination analysis, as a separate analysis, would allow reasoning about the total correctness of the program. Users would have more confidence that the program would behave as expected.

Furthermore, termination analysis provide a significant source of program optimizations. In a dependently-typed program, many values are the encodings of irrelevant proofs. These proofs show that the program type checks, but otherwise do not affect the actual result of computation. Some languages [Coq Development Team 2009, Barras and Bernardo 2008, Mishra-Linger and Sheard 2008] distinguish between computational and proof terms, allowing the latter to be erased prior to execution. This erasure leads to significant gains in performance.

Note that in a call-by-value language, computationally irrelevant code can be erased only if it terminates. Even if x is not free in e_2, $\mathbf{let}\ x\ =\ e_1\ \mathbf{in}\ e_2$ is only equivalent to e_2 if e_1 is known to terminate. Optimization must not change the termination behavior of the program, lest an infinite loop, which was preventing the program state reaching a stuck computation from being removed.

8. Related work

The past decade has seen much research in the design of dependently-typed programming languages, including Cayenne [Augustsson 1998], Epigram [McBride and McKinna 2004], Ωmega [Sheard 2006], PIE [Vytiniotis and Weirich 2007], DML [Xi and Pfenning 1998], ATS [Xi 2004], DML reformulated [Licata and Harper 2005], GURU [Stump et al. 2009], ConCoqtion [Fogarty et al. 2007], Delphin [Poswolsky and Schrmann 2008], Ynot [Nanevski et al. 2008] and Liquid Types [Jhala 2008]. A number of proof assistants, such as Agda [Norell 2007] and Coq [Coq Development Team 2009], have also successfully been used as dependently-typed languages [Leroy 2006, Oury and Swierstra 2008]. We do not attempt to survey this vast field here. Instead, we only describe aspects of the most related systems. Of these languages, only Cayenne and DML do not require decidable type checking. As far as we know, no results, such as type safety, have been proven about Cayenne.

Dependent ML Like, $\lambda^{\cong}$, Dependent ML (DML) [Xi and Pfenning 1998] is a family of dependently-typed languages. Types in DML depend not on terms, but on elements of some index language $\mathcal{L}$, a parameter to the system. This constraint language must include booleans and a binary function $\doteq_s$ which must return a boolean for every sort of the language. The constraint relation $\phi; \vec{P} \models P$, which states when proposition P about $\mathcal{L}$ is derivable from assumptions, is likewise a parameter to the system. This relation must satisfy a number of *regularity* rules, somewhat analogous to the **isEq** properties in Figure 8. Xi points out that this constraint relation may be undecidable, but discourages undecidable instances of it.

However, because DML is phase-sensitive, the index language $\mathcal{L}$ is *not* the computation language, and is not computationally relevant. Therefore, there is no analogue of Property IsEqBeta for the constraint relation as the index language is never evaluated. To program in DML, singleton types must be used to make a connection between the index language and computations, leading to redundancy. Programmers must understand two different definitions of equivalence to program in DML, one for $\mathcal{L}$, used for type checking, and another for the computation language, to understand what their program does. In contrast, $\lambda^{\cong}$ is a *full-spectrum* language, indexing types by actual computation. As far as we know, $\lambda^{\cong}$ is the only such language to parameterize term equivalence.

Pattern matching with dependent types Languages that support dependently-typed pattern matching, such as Epigram, Coq and Agda, typically specify the rules for pattern matching using some variant of unification to represent the static information gained during case analysis.

Of these languages, Agda's specification of pattern matching is the most sophisticated [Norell 2007]. Agda uses unification to match the index of the scrutinee's type and the index of the pattern's type. The unification algorithm will simply give up when the unification is hard; for instance, unifying a function application with a term. As a result, Agda's type checking algorithm is not substitutive; unification between a variable y and an arbitrary term always succeeds, however after substituting $f\,x$ for y, the unification algorithm might fail. In our system, instead of solving a unification problem, we add the assumption that the indices are equivalent in the context. Consequently, substitution is available in $\lambda^{\cong}$.

There are some languages that use equivalence assumptions to specify dependently-typed case analysis. A notable example is Altenkirch and Oury's core dependently-typed language ΠΣ [Altenkirch and Oury 2008]. However, as their goal is decidable type checking, they design an specific equivalence algorithm. This algorithm β-reduces terms to normal forms, then rewrites using equations from the context. To make sure that their algorithm terminates they place restrictions on the equations that could be used and use boxed terms to control the unrolling of general recursive functions. Such ideas could be used in an instantiation of **isEq**.

Likewise, some specifications of *generalized algebraic datatypes* (GADTs, aka guarded recursive datatypes) use equivalence assumptions [Xi et al. 2003, Pottier and Régis-Gianas 2006]. GADTs add index equivalences (but not scrutinee/pattern equivalences) to the context when type checking pattern matching. In these settings, the index language is restricted so that there is an effective algorithm for using these assumptions during type checking. As a result of this restriction, this specification is no more expressive than one that uses unification.

9. Conclusion

In this paper, we have explored the trade-off between decidable type checking and the complexity of the design of $\lambda^{\cong}$, an expressive, dependently-typed language. Because we have not insisted in the former, we are able to give a simple specification to $\lambda^{\cong}$, despite its advanced features, that permits straightforward, modular proof of type safety. We view this *simplicity* as a contribution of our approach.

The second significant contribution of our work is the *uniformity* of semantics. Although many different instantiations of **isEq** are valid, we have worked hard to ensure that **isEqCA** is one of them. Therefore, the same semantics can be used to reason about the program both statically and dynamically.

The final contribution of our design is its *generality*. We can view $\lambda^{\cong}$ with **isEqCA** as an ideal goal for the design of a dependently-typed language, much as System F is an ideal model of a polymorphic functional language. Of course, we can never implement a *complete* type checker for $\lambda^{\cong}$ with **isEqCA**; the problem is undecidable. We can however, specify and implement complete type checkers for decidable sublanguages, as any equivalence dominated by **isEqCA** defines a type safe language.

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