1 Review of Maxwell’s equations

Maxwell’s Equations

\[
\begin{align*}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0} \rho \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\frac{1}{\mu_0} \nabla \times \vec{B} &= \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]

Lorentz Force

\[
\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})
\]

2 Coulomb’s Law

Force between two charge particles,

\[
\vec{F}_{12} = \frac{q_1 q_2}{4\pi \varepsilon_0 r_{12}^2} \hat{r}_{12}
\]

Electric field: (action at distance) for a charge \( q \) at the origin,

\[
\vec{E}(\vec{r}) = \frac{q}{4\pi \varepsilon_0 r^2} \hat{r}
\]

and the force on a charge \( q' \) is

\[
\vec{F} = q' \vec{E}
\]

Remarks:

1. \( \vec{E} \) is in the radial direction, \( \nabla \times \vec{E} = 0 \).
2. There are two different types of charges—opposite charges attract and same charges repulse.
3. The force is long range—no intrinsic length scale.

Principle of linear supposition: Electric field due to a collection of charges is just the sum of electric fields due to each charge,

\[
\vec{E}(\vec{r}) = \sum_i \vec{E}_i(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \sum_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}
\]

For continuous charge distribution,

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d^3r'
\]

Identity

\[
\frac{\vec{r}}{r^3} = -\nabla \left( \frac{1}{r} \right)
\]
The electric field can then be written as,
\[ \vec{E} (\vec{r}) = -\nabla \phi \]

where
\[ \phi (\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho (\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' \quad \text{Coulomb potential} \]

**Example**: Spherical Charge distribution \( \rho (\vec{r}) = \rho (r) \), for \( r \leq a \).

\[ \phi (r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho (\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' = \frac{1}{4\pi \varepsilon_0} \int r'^2 dr' \rho (r') \int \frac{\sin \theta' d\theta' d\phi}{|\vec{r} - \vec{r}'|} \]

Take \( \vec{r} \) to be the z-axis for \( \vec{r}' \), then
\[ |\vec{r} - \vec{r}'| = (r^2 + r'^2 - 2rr' \cos \theta')^{\frac{1}{2}} \]

and
\[ I (r, r') = \int \frac{\sin \theta' d\theta' d\phi}{|\vec{r} - \vec{r}'|} = 2\pi \int \frac{dz}{(r^2 + r'^2 - 2rr'z)^{\frac{1}{2}}} \quad \text{where } z = \cos \theta' \]

\[ = 2\pi \frac{2}{(-2rr')} \left[ |r - r'| - (r + r') \right] = 4\pi \left\{ \begin{array}{ll} \frac{1}{r'} & r > r' \\ \frac{1}{r} & r < r' \end{array} \right. \]

1. \( r > a \)
\[ \phi (r) = \frac{Q}{4\pi \varepsilon_0 r}, \quad \text{where } Q = 4\pi \int \rho (r) r^2 dr \quad \text{total charge} \]

and
\[ \vec{E} (\vec{r}) = \frac{1}{4\pi \varepsilon_0 r^2} \hat{r} \]

2. \( r < a \)
\[ \phi (r) = \frac{1}{\varepsilon_0} \left[ \frac{1}{r} \int_0^r \rho (r') r'^2 dr' + \int_r^a \rho (r') r'^2 dr' \right] \]

and
\[ \vec{E} (\vec{r}) = \frac{1}{4\pi \varepsilon_0 r^2} \frac{Q (r)}{r^2} \hat{r} \]

where
\[ Q (r) = 4\pi \int_0^r \rho (r') r'^2 dr' \]

is the charge inside the spherical surface of radius \( r \).

**3 Dirac Delta Function \( \delta (x) \)**

**Formal definition**
\[ \delta (x) = 0, \text{ for } x \neq 0 \]

and
\[ \int_{-\infty}^{\infty} \delta (x) f (x) dx = f (0) \]

**\( \delta (x) \) as a limit of a series of functions**

1. \[ \delta_n (x) = \left\{ \begin{array}{ll} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \frac{1}{2n} \leq x \end{array} \right. \]
2. \[ \delta_n(x) = \frac{n}{\sqrt{\pi}} \exp \left( -n^2 x^2 \right) \]

3. \[ \delta_n(x) = \frac{n}{\pi (1 + n^2 x^2)} \]

\[ \delta_n(x) = \frac{\sin n\pi x}{\pi x} = \frac{1}{2\pi} \int_{-\infty}^{n} \exp (ixt) dt \]

**Remark:** \[ \lim_{n \to \infty} \delta_n(x) \] does not exist. But we do have

\[ \int_{-\infty}^{\infty} \delta_n(x) \, dx = 1, \quad \text{and} \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) \, dx = f(a) \]

\( \delta(x) \) is called generalized function or distribution.
\\
\[ \delta'(x) \] is defined by

\[ \int_{-\infty}^{\infty} \delta'(x) f(x) \, dx = -f'(a) \]

**Useful properties of \( \delta \)-functions**

1. \[ \int_{-\infty}^{\infty} \delta(x - a) f(x) \, dx = f(a), \quad \int_{-\infty}^{\infty} \delta'(x - a) f(x) \, dx = -f'(a) \]

2. \[ x \delta(x) = 0 \]

3. \[ \int_{-\infty}^{\infty} \delta(ax) f(x) \, dx = \frac{1}{a} f(a) \]

4. \[ \delta(x) = \delta(-x), \quad \delta'(-x) = -\delta'(x) \]

5. \[ \delta(g(x)) = \frac{1}{|g'(x)|} \delta(x - x_0), \quad \text{where} \quad g(x_0) = 0 \]

6. \[ \delta(x) = \frac{d}{dx} \theta(x) \]

7. \[ \delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp (-ikx) \]

\[ x \delta'(x) = -\delta(x) \]

**\( \delta \)-function and completeness**

\( \{ f_n \} \) complete set of orthonormal functions. For arbitrary function \( f(x) \),

\[ f(x) = \sum_{n} c_n f_n(x), \quad \text{with} \quad c_n = \int_{a}^{b} f_n^*(y) f(y) \, dy \]

We can write

\[ f(x) = \sum_{n} c_n f_n(x) = \int_{a}^{b} dy f(y) \sum_{n} f_n^*(y) f_n(x) \]

Compare with

\[ f(x) = \int_{a}^{b} dy f(y) \delta(x - y) \]
we get
\[ \sum_n f_n^* (y) f_n (x) = \delta (x - y) \] (completeness relation)

**Charge density for a pointed particle**

Using δ-function we can write the charge density for a pointed particle as
\[ \rho (\vec{r}) = q \delta^3 (\vec{r} - \vec{s}(t)) \]

where \( \vec{s}(t) \) is the trajectory of the particle. Then
\[ Q = \int \rho (\vec{r}) d^3r = q. \]

Note that the current density can be written as,
\[ \vec{J} (\vec{r}) = q \vec{v} \delta^3 (\vec{r} - \vec{s}(t)) \]

where \( \vec{v} = \frac{d\vec{s}(t)}{dt} \) is the velocity of the particle.

4 **Gauss Law**

Electric field from a point charge \( q \) at origin,
\[ \vec{E} (\vec{r}) = \frac{q}{4\pi \varepsilon_0} \frac{\hat{r}}{r^2} \]

Flux of \( \vec{E} \) through an infinitesimal surface with unit normal \( \vec{n} \) and area \( da \) is,
\[ \vec{E} \cdot \vec{n} da = \frac{q}{4\pi \varepsilon_0} \frac{\hat{r} \cdot \vec{n}}{r^2} da = \frac{q}{4\pi \varepsilon_0} \frac{\cos \theta}{r^2} da = \frac{q}{4\pi \varepsilon_0} d\Omega \]

where \( \theta \) is the angle between \( \vec{E} \) and \( \vec{n} \) and \( d\Omega \) is the solid angle subtended by the infinitesimal surface with respect to the charge \( q \). The flux through a closed surface \( S \) is then,
\[ \oint_S \vec{E} \cdot \vec{n} da = \frac{q}{4\pi \varepsilon_0} \int d\Omega = \frac{q}{\varepsilon_0} \]

It is clear that this can be generalized to a continuous charge distribution \( \rho (\vec{r}) \),
\[ \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_V \rho (\vec{r}) d^3r \]

where \( V \) is the volume enclosed by the closed surface \( S \). This is usually called the **Gauss law**.

**Gauss theorem** : For arbitrary vector field \( \vec{V} \) the following relation holds,
\[ \oint_S \vec{V} \cdot d\vec{S} = \int_V \nabla \cdot \vec{V} d^3r \]

Then
\[ \int_V \nabla \cdot \vec{E} d^3r = \frac{1}{\varepsilon_0} \int_V \rho (\vec{r}) d^3r \]

Since we can make \( V \) arbitrary small, we get a local relation,
\[ \nabla \cdot \vec{E} = \frac{\rho (\vec{r})}{\varepsilon_0} \]

This is the differential form of the Gauss law.

**Example** : Spherical charge distribution centered at origin \( , \rho (\vec{r}) = \rho (r) \), for \( r \leq a \)
From symmetry, it is easy to see that \( \vec{E} \) is in the radial direction, \( \vec{E}(r) = E(r) \hat{r} \). Take a spherical surface with radius around the origin and apply the Gauss law, we get for \( r > a \),

\[
E(r)A_\pi r^2 = \frac{Q}{\varepsilon_0}
\]

or

\[
\vec{E}(r) = \frac{Q}{4\pi r^2 \varepsilon_0} \hat{r}
\]

where \( Q \) is the total charge.

Note that Gauss's law is more general than Coulomb's law and includes Coulomb's law as a special case. To see this we use the vector identity,

\[
\vec{\nabla} \times \left( \vec{\nabla} \times \vec{A} \right) = \vec{\nabla} \left( \vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A}
\]

to write formally,

\[
\vec{A} = \frac{1}{r^2} \left[ \vec{\nabla} \left( \vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla} \times \left( \vec{\nabla} \times \vec{A} \right) \right]
\]

This implies that one needs both \( \vec{\nabla} \cdot \vec{A} \) and \( \vec{\nabla} \times \vec{A} \) to reconstruct the vector field \( \vec{A} \). For the case of Coulomb field, \( \vec{\nabla} \times \vec{E} = 0 \) because it is in the radial direction.

**Scalar potential**

Any vector fields which satisfies

\[
\vec{\nabla} \times \vec{E} = 0
\]

can be written as,

\[
\vec{E} = -\vec{\nabla} \phi
\]

This is the case for the static electric field (from Faraday’s law). In this case, Gauss’s law becomes,

\[
\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad \text{Poisson equation}
\]

The general solution is of the form,

\[
\phi(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(r')}{|\vec{r} - \vec{r}'|} d^3r' + u(r)
\]

where \( u(r) \) is an arbitrary solution to the Laplace equation,

\[
\nabla^2 u(r) = 0.
\]

This can be verified by using the relation,

\[
\nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta^3 \left( \vec{r} - \vec{r}' \right).
\]

Thus, up to a solution to the Laplace equation, the Coulomb potential is the most general solution to Gauss’s law for the case of the static electric field.

**Example**: Potential from a uniformly charged disk at distance \( z \) from the center

\[
\phi = \frac{1}{4\pi \varepsilon_0} \int \frac{\sigma r' dr' d\theta'}{|\vec{x} - \vec{x}'|}
\]

where \( \sigma \) is the surface charge density. From

\[
|\vec{x} - \vec{x}'| = \left( r'^2 + z^2 \right)^{\frac{3}{2}}
\]
we get
\[
\phi = \frac{\sigma}{4\pi \varepsilon_0} \int_0^{2\pi} d\theta \int_0^a \frac{r' dr'}{(r'^2 + z^2)^{1/2}} = \frac{\sigma}{2\varepsilon_0} \left[ \sqrt{z^2 + a^2} - z \right]
\]

The \(z\)-component of electric field is
\[
E_z = -\frac{\partial \phi}{\partial z} = \frac{\sigma}{2\varepsilon_0} \left[ \frac{-z}{\sqrt{z^2 + a^2}} + 1 \right]
\]

## 5 Potential from Surface charge distribution

Let \(\sigma(\vec{x})\) be the surface charge distribution (charge per unit area). The potential is then
\[
\phi(\vec{x}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\sigma(\vec{x}') ds'}{|\vec{x} - \vec{x}'|}
\]

**Example**: Uniform charge distribution on a plane \((z = 0)\), \(\sigma(\vec{x}) = \sigma_0\)
\[
|\vec{x} - \vec{x}'| = \left[ (x - x')^2 + (y - y')^2 + z^2 \right]^{1/2}
\]
Then
\[
\phi(\vec{x}) = \frac{\sigma_0}{4\pi \varepsilon_0} \int \frac{dx'dy'}{\left[ (x - x')^2 + (y - y')^2 + z^2 \right]^{1/2}}
\]
Introduce the polar coordinates,
\[
x - x' = \rho \cos \theta, \quad y - y' = \rho \sin \theta
\]
then
\[
\phi(z) = \frac{\sigma_0}{2\varepsilon_0} \int_0^\infty \frac{\rho d\rho}{(\rho^2 + z^2)^{1/2}}
\]
This is a divergent integral. However the electric field is finite,
\[
E_z = -\frac{\partial \phi}{\partial z} = \frac{\sigma_0}{\varepsilon_0} \int_0^\infty \frac{z \rho d\rho}{(\rho^2 + z^2)^{3/2}} = \frac{\sigma_0}{\varepsilon_0} \left[ \frac{z}{|z|} \right] = \begin{cases} \frac{\sigma_0}{\varepsilon_0}, & z > 0 \\ \frac{\sigma_0}{\varepsilon_0} \frac{z}{|z|}, & z < 0 \end{cases}
\]

**Boundary conditions**

In this example, we see that the surface charge density gives rise to discontinuity in the electric field. More generally we have from Gauss’s law
\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}
\]
Take a pill box with height infinitesimally small across the boundary of charge distribution. Using Gauss’s theorem, we get
\[
\int \left( \vec{\nabla} \cdot \vec{E} \right) d^3x = \int_S \vec{E} \cdot d\vec{S} \simeq \left(E^{(1)}_{\perp} - E^{(2)}_{\perp}\right) A = \frac{\sigma}{\varepsilon_0} A
\]
or
\[
\left| E^{(1)}_{\perp} - E^{(2)}_{\perp} \right| = \frac{\sigma}{\varepsilon_0}
\]
Since this follows directly from the Gauss’s law, this holds whether electric field is static or not. But for the static electric field we have in addition,
\[
\vec{\nabla} \times \vec{E} = 0
\]
Take a rectangular surface across the surface and apply Stokes’s theorem,
\[
\int_A \left( \vec{\nabla} \times \vec{E} \right) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{l} \simeq \vec{E}^{(1)}_{\parallel} - \vec{E}^{(2)}_{\parallel} = 0
\]
Thus the parallel components of the electric field are continuous across the surface.

**Dipole and dipole layers**

Electrostatic potential from a pair of charges \( q \) and \(-q\), separately by a distance \( a \), is of the form

\[
\phi(x) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{|x-x'|} - \frac{1}{|x-x'+a|} \right]
\]

When \(|x - x'| \gg |a|\),

\[
\frac{1}{|x-x'+a|} \approx \frac{1}{|x-x'|} + \left( \vec{a} \cdot \vec{\nabla} \right) \left( \frac{1}{|x-x'|} \right) + O(a^2)
\]

Then

\[
\phi(x) = -\frac{1}{4\pi\varepsilon_0} \left( \frac{\vec{p} \cdot \vec{\nabla}}{|x-x'|} \right) \left( \frac{1}{|x-x'|} \right)
\]

where \( \vec{p} = q\vec{a} \) is the dipole moment of the charge pair. We can generalize this to a layer of dipoles to get

\[
\phi(x) = -\frac{1}{4\pi\varepsilon_0} \int \left( \vec{D}(\vec{x'}) \cdot \vec{n} \cdot \vec{\nabla} \right) \left( \frac{1}{|x-x'|} \right) \, da
\]

where \( \vec{D}(\vec{x}) \) is the dipole per unit area and \( \vec{n} \) is the unit normal to the surface.

**Example**: uniform layer of dipole on \( xy-\)plane

Again

\[
|\vec{x} - \vec{x}'| = \left[ (x-x')^2 + (y-y')^2 + z^2 \right]^{1/2}
\]

and

\[
\phi(x) = \frac{D_0}{4\pi\varepsilon_0} \int \frac{z dx' dy'}{\left[ (x-x')^2 + (y-y')^2 + z^2 \right]^{3/2}}
\]

\[
= \frac{D_0 z}{2\varepsilon_0} \int_0^\infty \frac{z d\rho}{\rho^2 + z^2}^{3/2} = \frac{D_0 z}{2\varepsilon_0 |z|} \left\{ \begin{array}{ll} D_0 & z > 0 \\ -\frac{D_0}{2\varepsilon_0} & z < 0 \end{array} \right.
\]

where \( D_0 \) is the dipole density. Thus layer of dipoles will give rise to a discontinuity in the static potential. More generally we have,

\[
\phi_2(x) - \phi_1(x) = \frac{D_0}{\varepsilon_0} \left( \vec{n} \cdot \vec{\nabla} \right) \left( \frac{1}{|x-x'|} \right) \, da = \frac{\vec{n} \cdot (\vec{x} - \vec{x}')}{|x-x'|^3} \, da = -d\Omega
\]

Note that

\[
\left( \vec{n} \cdot \vec{\nabla} \right) \left( \frac{1}{|x-x'|} \right) \, da = \frac{\vec{n} \cdot (\vec{x} - \vec{x}')}{|x-x'|^3} \, da = -d\Omega
\]

where \( d\Omega \) is the solid angle subtended by the surface \( da \) with respect to \( x \). Then we have

\[
\phi(x) = -\frac{1}{4\pi\varepsilon_0} \int D(\vec{x'}) \, d\Omega
\]
6 Electrostatic Energy

Electric force on a charge particle is

$$\vec{F} = q\vec{E}$$

For the case where $\vec{E}$ is derived from a potential,

$$\vec{E} = -\nabla \phi$$

the equation of motion is then

$$m \frac{d^2 \vec{x}}{dt^2} = -q \nabla \phi$$

Take scalar product with $\frac{d\vec{x}}{dt}$,

$$\frac{d}{dt} \left[ \frac{m}{2} \left( \frac{d\vec{x}}{dt} \right)^2 \right] = -q \frac{d\vec{x}}{dt} \cdot \nabla \phi$$

or

$$\frac{d}{dt} \left[ \frac{m}{2} \left( \frac{d\vec{x}}{dt} \right)^2 + q \phi \right] = 0$$

We recognize this is the energy conservation and $q\phi$ is just the potential energy associated with the static electric field. Another way to get the same result is to look at the work done in moving charge $q$ in the external electric field from $A$ to $B$,

$$W = -\int_A^B \vec{F} \cdot d\vec{l} = q \int_A^B \vec{E} \cdot d\vec{l} = q \int_A^B d\phi = q (\phi_B - \phi_A)$$

So $q\phi$ is seen to be the potential energy. Note that this result is independent of the path it takes from $A$ to $B$. Thus for a closed path $A = B$, and

$$0 = \oint_C \vec{E} \cdot d\vec{l} = \int_S \left( \nabla \times \vec{E} \right) \cdot d\vec{S}$$

Since $S$ is arbitrary, we get

$$\nabla \times \vec{E} = 0$$

Suppose we bring charges $q_1, \ldots q_n$, in succession from infinity to a localized region. The work done on charge $q_i$ is

$$W_i = q_i \phi_i \left( \vec{x}_i \right)$$

where $\phi_i \left( \vec{x}_i \right)$ is the potential due to charges $q_1, \ldots q_{i-1}$,

$$\phi_i \left( \vec{x} \right) = \frac{1}{4\pi\varepsilon_0} \sum_{j<i} \frac{q_j}{|\vec{x} - \vec{x}_j|}$$

The total potential energy is then,

$$W = \sum_i W_i = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_0 |\vec{x}_i - \vec{x}_j|}$$

This generalize to the case of continuous charge distribution,

$$W = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \int d^3x d^3y \frac{\rho \left( \vec{x} \right) \rho \left( \vec{y} \right)}{|\vec{x} - \vec{y}|}$$

Or

$$W = \frac{1}{2} \int d^3x \rho \left( \vec{x} \right) \phi \left( \vec{x} \right)$$

From Gauss’s law, $\rho \left( \vec{x} \right) = -\varepsilon_0 \nabla^2 \phi$, we can write

$$W = -\frac{\varepsilon_0}{2} \int d^3x \phi \nabla^2 \phi = \frac{\varepsilon_0}{2} \int d^3x E^2$$

We can identify the energy density as

$$u \left( \vec{x} \right) = \frac{\varepsilon_0}{2} \frac{E^2}{E}$$