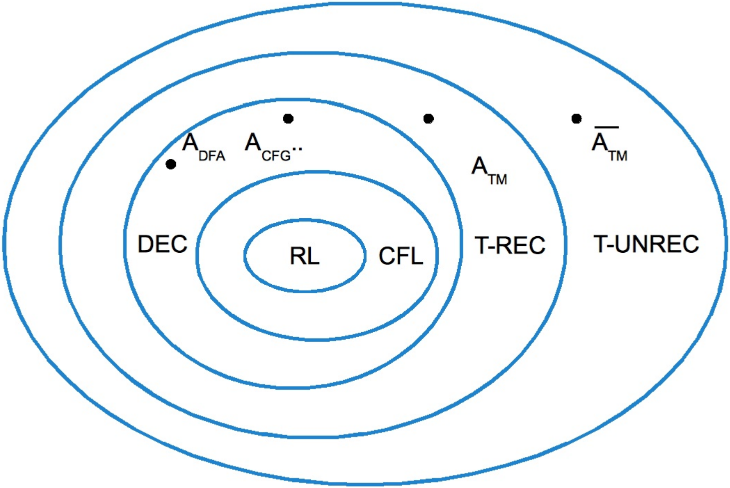
Formal Languages, Automata and Computation

Reducibility

The Landscape of the Chomsky Hierarchy



REDUCIBILITY

- A reduction is a way of converting one problem to another problem, so that the solution to the second problem can be used to solve the first problem.
    - Finding the area of a rectangle, reduces to measuring its width and height
    - Solving a set of linear equations, reduces to inverting a matrix.
- Reducibility involves two problems *A* and *B*.
    - If *A* reduces to *B*, you can use a solution to *B* to solve *A*
- When *A* is reducible to *B* solving *A* can not be "harder" than solving *B*.
- If *A* is reducible to *B* and *B* is decidable, then *A* is also decidable.
- If *A* is undecidable and reducible to *B*, then *B* is undecidable.

PROVING UNDECIDABILITY VIA REDUCTIONS

THEOREM 5.1

$HALT_{TM} = \{\langle M, w \rangle \mid M$ is a TM and M halts on input $w\}$ is undecidable.

PROOF

- Use the idea that " If A is undecidable and reducible to B, then B is undecidable."
- Suppose R decides $HALT_{TM}$. We construct S to decide A_{TM}.
- $S =$ "On input $\langle M, w \rangle$
  1. Run R on input $\langle M, w \rangle$.
  2. If R rejects *reject*.
  3. If R accepts, simulate M on w until it halts.
  4. If M has accepted, *accept*; If M has rejected, *reject*."
- Since A_{TM} is reduced to $HALT_{TM}$, $HALT_{TM}$ is undecidable.

THEOREM 5.2

$E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Phi\}$ is undecidable.

- Suppose R decides E_{TM}. We try to construct S to decide A_{TM} using R.
    - Note that S takes $\langle M, w \rangle$ as input.
- One idea is to run R on $\langle M \rangle$ to check if M accepts some string or not – but that that does not tell us if M accepts w.
- Instead we modify M to M_1. M_1 rejects all strings other than w but on w, it does what M does.
- Now we can check if $L(M_1) = \Phi$.

Proving Undecidability via Reductions

Theorem 5.2

$E_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M) = \Phi\}$ is undecidable.

Proof

- For any w define M_1 as
  M_1 = "On input x:
  1. If $x \neq w$, *reject*.
  2. If $x = w$, run M on input w and *accept* if M does."
- Note that M_1 either accepts w only or nothing!

PROOF CONTINUED

- Assume R decides E_{TM}
- S defines below uses R to decide on A_{TM}
  $S =$ "On input $\langle M, w\rangle$
  1. Use $\langle M, w\rangle$ to construct M_1 above.
  2. Run R on input $\langle M_1\rangle$
  3. If R accepts, *reject*, if R rejects, *accept*.
- So, if R decides M_1 is empty,
  - then M does NOT accept w,
  - else M accepts w.
- If R decides E_{TM} then S decides A_{TM} – Contradiction.

TESTING FOR REGULARITY (OR OTHER PROPERTIES)

- Can we find out if a language accepted by a Turing machine M is accepted by a simpler computational model?
  - Is the language of a TM actually a regular language? ($REGULAR_{TM}$)
  - Is the language of a TM actually a CFL? (CFL_{TM})
  - Does that language of a TM have an "interesting" property?
    - Rice's Theorem.

$REGULAR_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ is a regular language $\}$ is undecidable.

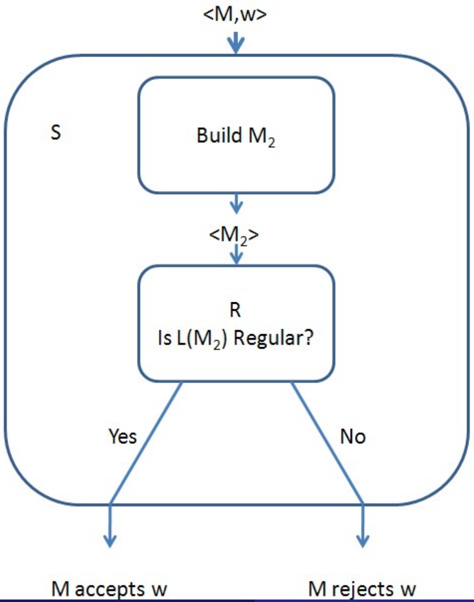
Proof Idea

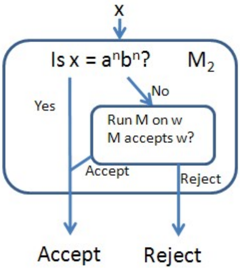
- We assume $REGULAR_{TM}$ is decidable by a TM R and use this assumption to construct a TM S that decides A_{TM}.
- The basic idea is for S to take as input $\langle M \rangle$ and modify M into M_2 so that the resulting TM recognizes a regular language if and only if M accepts w.
- M_2
  - accepts $\{0^n 1^n \mid n \geq 0\}$ if M does not accept w,
  - but recognizes Σ^* if M accepts w.

PROOF IDEA –CONTINUED

- M_2 accepts $\{0^n1^n \mid n \geq 0\}$ if M does not accept w, but recognizes Σ^* if M accepts w.
- What does M_2 look like?
- $M_2 =$ "On input x
  1. If x has the form 0^n1^n, *accept*.
  2. If x does not have this form, run M on input w and *accept* if M accepts w."
- All strings x (that is Σ^*) are accepted if M accepts w.

Testing for Regularity





So $L(M_2)$ is = Σ^* if M accepts w

$L(M_2)$ is = $\{a^n b^n\}$ otherwise

Testing for Regularity

Proof

- $S =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:
  1. Construct the following TM M_2.
  2. $M_2 =$ "On input x
     1. If x has the form $0^n 1^n$, *accept*.
     2. If x does not have this form, run M on input w and *accept* if M accepts w."
  3. Run R on $\langle M_2 \rangle$
  4. If R accepts, *accept*, if R rejects, *reject*.
- So, R will say M_2 is a regular language, if M accepts w.
- S says "M accepts w" if R decides M_2 is regular – Contradiction!

Theorem 5.4

$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$ is undecidable.

Proof Idea

- We reduce E_{TM} (the emptiness problem) to this problem.
- If one of the languages is empty, determining equality is the same as determining if the second language is empty!
- In fact, the E_{TM} is a special case of the EQ_{TM} problem!!

Testing for Language Equality

Theorem 5.4

$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$ is undecidable.

Proof

- Assume R decides EQ_{TM}
- S = "On input $\langle M\rangle$ where M is a TM:
  1. Run R on input $\langle M, M_1\rangle$ where M_1 is a TM that rejects all inputs.
  2. If R accepts, *accept*; if R rejects *reject*"
- Thus, if R decides EQ_{TM}, then S decides E_{TM}
- But E_{TM} is undecidable, so EQ_{TM}, must be undecidable.

Reductions via Computation Histories

- An accepting computation history for a TM is a sequence of configurations

$$C_1, C_2, \ldots, C_l$$

  such that
  - C_1 is the start configuration for input w
  - C_l is an accepting configuration, and
  - each C_i follows legally from the preceding configuration.
- A rejecting computation history is defined similarly.
- Computation histories are finite sequences – if M does not halt on w, there is no computation history.
- Deterministic v.s nondeterministic computation histories.

Linear Bounded Automaton

- Suppose we cripple a TM so that the head never moves outside the boundaries of the input string.
- Such a TM is called a linear bounded automaton (LBA)
- Despite their memory limitation, LBAs are quite powerful.

Lemma

Let M be a LBA with q states, g symbols in the tape alphabet. There are exactly qng^n distinct configurations for a tape of length n.

Proof.

- The machine can be in one of q states.
- The head can be on one of the n cells.
- At most g^n distinct strings can occur on the tape.

□

Decidability of LBA Problems

Theorem 5.9

$A_{LBA} = \{\langle M, w\rangle \mid M \text{ is an LBA that accepts string } w\}$ is decidable.

Proof Idea

- We simulate LBA M on w with a TM L (which is NOT an LBA!)
- If during simulation M accepts or rejects, we accept or reject accordingly.
- What happens if the LBA M loops?
  - Can we detect if it loops?
- M has a finite number of configurations.
  - If it repeats any configuration during simulation, it is in a loop.
  - If M is in a loop, we will know this after a finite number of steps.
  - So if the LBA M has not halted by then, it is looping.

Decidability of LBA Problems

Theorem 5.9

$A_{LBA} = \{\langle M, w\rangle \mid M$ is an LBA that accepts string $w\}$ is decidable.

Proof

- The following TM decides A_{LBA}.
- $L =$ "On input $\langle M, w\rangle$
  1. Simulate M on for qng^n steps or until it halts.
  2. If M has halted, *accept* if it has accepted, and *reject* if it has rejected. If it has NOT halted, *reject*."
- LBAs and TMs differ in one important way. A_{LBA} is decidable.

COMPUTATION OVER "COMPUTATION HISTORIES"

- Now for a really wild and crazy idea!
- Consider an accepting computation history of a TM M, $C_1, C_2, \ldots, C_l$
- Note that each C_i is a string.
- Consider the string

$$\# \underbrace{\qquad\qquad}_{C_1} \# \underbrace{\qquad\qquad}_{C_2} \# \underbrace{\qquad\qquad}_{C_3} \# \cdots \# \underbrace{\qquad\qquad}_{C_l} \#$$

- The set of all valid accepting histories is also a language!!
- This string has length m and an LBA B can check if this is a valid computation history for a TM M accepting w.
  - Check if $C_1 = q_0 w_1 w_2 \cdots w_n$
  - Check if $C_l = \cdots q_{accept} \cdots$
  - Check if each C_{i+1} follows from C_i legally.
- Note that B is not constructed for the purpose of running it on any input!
- If $L(B) \neq \Phi$ then M accepts w

Decidability of LBA Problems

Theorem 5.10

$E_{LBA} = \{\langle M \rangle \mid M \text{ is an LBA and } L(M) = \Phi\}$ is undecidable.

Proof.

- Suppose TM R decides E_{LBA}, we can construct a TM S which decides A_{TM}
- $S =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string
  1. Construct LBA B from M and w as described earlier.
  2. Run R on $\langle B \rangle$.
  3. If R rejects, *accept*; if R accepts, *reject*."
- So if R says $L(B) = \Phi$, the M does NOT accept w.
- If R says $L(B) \neq \Phi$, the M accepts w.
- But, A_{TM} is undecidable – contradiction.

□