

## Chapter 11

# Productive Sets

The theory of computability was originally invented to unpack the full significance of Gödel's incompleteness theorems. As we have seen, the main issue was to arrive at a comprehensive mathematical explication of “effectiveness” that is not subject to diagonalization.

But one might also wish to study the Gödel phenomenon, itself, in the newly erected, general framework. The “Gödel phenomenon” may be expressed by saying that every attempt to effectively enumerate the truths of arithmetic is bound to fail. Fix a Gödel numbering of formulas in the language of arithmetic. Let  $A$  denote the set of all Gödel numbers of truths of arithmetic. Then:

$$\forall n (A \neq W_n)$$

So for each proposed recursive enumeration  $W_n$  of  $A$ , either some falsehood is included (unsoundness) or some truth is missed (incompleteness). Equivalently, for each enumeration  $W_n$ , if  $W_n$  doesn't include a falsehood, then  $W_n$  misses a truth.

$$\forall n [(W_n \subseteq A) \rightarrow \exists x (A(x) \wedge \neg W_n(x))]$$

But the situation is more interesting than that. Gödel's construction allows one to effectively produce the missing truth from a sound enumeration. In other words, the existence claim is uniform:

$$\exists \text{ total recursive } f \forall n [(W_n \subseteq A) \rightarrow (A(f(x)) \wedge \neg W_n(f(x)))]$$

This situation is computationally stronger than the non-uniform version, which states only that  $A$  is not recursively enumerable. What more, beyond non r.e.-ness does it imply? This encourages us to formulate the Gödel phenomenon as a general, recursion theoretic property of sets:

$$\begin{aligned} S \text{ is } \textit{productive} &\iff \\ \exists \text{ total recursive } f \forall n &[(W_n \subseteq S) \rightarrow (S(f(x)) \wedge \neg W_n(f(x)))] \end{aligned}$$

Then  $f$  is called the *productive function* for  $S$ .

Everybody who hears about Gödel has thought of the following idea. Each r.e. approximation to a productive set misses something. What if we keep adding the new things to what we started with?

**Proposition 11.1** *Each productive set has an infinite r.e. subset.*

Proof: there is a total recursive  $f$  such that for each  $x, y$ ,

$$W_{f(x,y)} = W_x \cup \{y\}$$

**Exercise 11.1** *Prove it.*

Let  $g$  be the productive function of  $S$ . Let  $W_a = \emptyset$ . Now do what everybody wants to:

$$\begin{aligned} h(0) &= a \\ h(n+1) &= f(h(n), g(h(n))) \\ d(n) &= g(h(n)) \end{aligned}$$

We know that  $\text{rng}(d)$  is infinite because  $W_{h(n)} \subseteq S$  (by induction on  $n$ ), so  $\neg W_{h(n)}(g(n))$ . And since  $d$  is total recursive,  $\text{rng}(d)$  is r.e.  $\dashv$

Now we may ask: is productivity (a.k.a. the Gödel phenomenon) just the same as non-r.e.-ness or is it something more? Our first clue is that  $\overline{K}$ , our favorite example of a non-r.e. set, is productive.

**Proposition 11.2**  *$\overline{K}$  is productive.*

Proof: the identity function  $p_1^1$  serves as a productive function.

For suppose that  $W_n \subseteq \overline{K}$ .

We need to show that  $\overline{K}(n) \wedge \neg W_n(n)$ .

Suppose, for reductio, that  $W_n(n)$ .

Then  $K(n)$ , by the definition of  $K$ .

But this contradicts  $W_n \subseteq \overline{K}$ .

So  $\neg W_n(n)$ , i.e.,  $\overline{W}_n(n)$ .

Now suppose, for reductio, that  $K(n)$ .

Then  $W_n(n)$ , by the definition of  $K$ .

But this again contradicts  $W_n \subseteq \overline{K}$ .

So  $\neg K(n)$ , i.e.,  $\overline{K}(n)$ .  $\dashv$

Roger Penrose turned this argument into a second career by claiming it proves that humans are not machines. It can't be producing the missing elements of  $\overline{K}$  that makes us superior, since that is a matter of computing the identity function! It can't be that we see that the produced object belongs to  $\overline{K}$ , for arbitrary  $n$ , since that depends on seeing first that  $W_n \subseteq \overline{K}$ , and it isn't likely we can "see" that. If we could, then cracking the Pentagon's web pages would be a cake walk in comparison. I think the linchpin of the argument is his cryptic discussion that complex systems for which we couldn't "see"  $W_n \subseteq \overline{K}$  wouldn't count as systems of mathematics (which must be simple and

intersubjective), so the range of  $n$  for which we must decide the question is radically restricted.

The next result is that productivity is closed upward in the many-one reducibility ordering

**Proposition 11.3** *( $S$  is productive  $\wedge S \leq_M Q \Rightarrow Q$  is productive).*

Proof: let  $g$  be the productive function and let  $f$  witness the reduction relation. We need to construct from these a total recursive productive function  $h$  for  $Q$ . This function must have the property:

$$\forall n (W_n \subseteq Q \rightarrow (Q(h(x)) \wedge \neg W_n(h(x))))$$

Here's an idea for how to proceed. First we construct a total recursive  $d$  that computes the index of the inverse image of an r.e. set under the reduction function  $f$ :

$$\forall n (W_{d(n)} = f^{-1}(W_n))$$

This can be done as follows. Define partial recursive

$$\psi(x, y) \approx (\mu z) U(x, (z)_0, (z)_1, \langle f(y) \rangle)$$

Applying  $s$ - $m$ - $n$ , we obtain total recursive  $d$  such that

$$\phi_{d(x)}(y) \approx \psi(x, y)$$

Thus:

$$\begin{aligned} W_{d(x)}(y) &\iff \psi(x, y) \downarrow \\ &\iff (\mu z) U(x, (z)_0, (z)_1, \langle f(y) \rangle) \downarrow \\ &\iff (f^{-1}(W_n))(y) \end{aligned}$$

We now propose that  $h(x) = f(g(d(x)))$  is a productive function for  $Q$ .

For suppose that  $W_n \subseteq Q$ .

Then  $W_{d(n)} = f^{-1}(W_n) \subseteq S$ , since  $f$  witnesses  $S \leq_M Q$ .

So since  $g$  is productive for  $S$ ,  $S(g(d(n))) \wedge \neg W_{d(n)}(g(d(n)))$ .

Since  $f$  witnesses  $S \leq_M Q$ , we have  $Q(f(g(d(n))))$ .

Since  $W_{d(n)} = f^{-1}(W_n)$  and  $\neg W_{d(n)}(g(d(n)))$ , we have  $\neg W_n(f(g(d(n))))$ .

So  $Q(f(g(d(n)))) \wedge \neg W_n(f(g(d(n))))$ .  $\dashv$

**Corollary 11.4**

- All co-r.e.-hard problems are productive.
- All index sets that are not experimentally verifiable are productive.

Proof: the first follows from the fact that is co-r.e.-complete. The second follows from the Rice-Shapiro theorem.  $\dashv$

So we know that everything from  $\overline{K}$  upward is productive. What about sets below  $\overline{K}$ ? Clearly, the recursive ones cannot be productive because productivity implies non-r.e. But even if there were non-recursive co-r.e. sets below  $\overline{K}$ , they couldn't be productive.

Before we can show this, we require a strengthened version of the recursion theorem.

**Proposition 11.5** (*The strengthened recursion theorem*)

$$\forall \text{ total recursive } f \exists \text{ total recursive } g (\phi_{f(g(x),x)} = \phi_{g(x)})$$

Proof: let total recursive  $f$  be given. Using the universal and  $s$ - $m$ - $n$  theorems, construct total recursive  $s$  such that

$$\phi_{s(x,y)}(z) \approx \phi_{f(\phi_x(x),y)}(z)$$

By  $s$ - $m$ - $n$  again we obtain total recursive  $m$  such that:

$$\phi_{m(y)}(x) \approx s(x, y)$$

So

$$\phi_{\phi_{m(y)}(x)}(z) \approx \phi_{f(\phi_x(x),y)}(z)$$

Now substitute  $m(y)$  for  $x$ .

$$\phi_{\phi_{m(y)}(m(y))}(z) \approx \phi_{f(\phi_{m(y)}(m(y)),y)}(z)$$

Now define

$$g(y) = \phi_{m(y)}(m(y)) = s(m(y), m(y))$$

This is total recursive since  $s, m$  are.

So we have

$$\phi_{g(y)}(z) \approx \phi_{f(g(y),y)}(z)$$

$\dashv$  Now we may show the following characterization of productivity. According to this result, productivity is equivalent to non-r.e.-ness just in case there are no incomplete r.e. degrees. The proof gets the Deviousness Award for the day.

**Proposition 11.6** (*Myhill's theorem*)  $A$  is productive  $\iff \overline{A}$  is r.e.-hard.

Proof: by corollary 11.4, co-r.e.-hardness implies productivity. But by proposition 8.1,  $\overline{A}$  is r.e.-hard just in case  $A$  is co-r.e.-hard.

Conversely, suppose  $A$  is productive with productive function  $g$ .

Let arbitrary r.e. set  $W_n$  be given. Using the universal construction and  $s$ - $m$ - $n$ , obtain total recursive  $s$  such that:

$$\phi_{s(x,y)}(z) \approx 0((\mu w)(g(x) = z \wedge U(n, (z)_0, (z)_1, \langle y \rangle)))$$

Thus

$$\begin{aligned} W_{s(x,y)} &= \{g(x)\} \text{ if } W_n(y) \\ &\text{and} \\ W_{s(x,y)} &= \emptyset \text{ otherwise.} \end{aligned}$$

Applying the strengthened recursion theorem we obtain a total recursive  $h$  such that

$$W_{h(y)} = W_{s(h(y),y)}$$

Now I claim that the composition  $C(g, h)(x) = g(h(x))$  witnesses  $W_n \leq_M \bar{A}$ .

For suppose  $W_n(y)$ .

Then  $W_{h(y)} = W_{s(h(y),y)} = \{g(h(y))\}$ .

Suppose for reductio that  $A(g(h(y)))$ .

Then  $W_{h(y)} = \{g(h(y))\} \in A$ .

So since  $g$  is the productive function of  $A$ ,  $A(g(h(y)))$ . Contradiction.

So  $\bar{A}(g(h(y)))$ .

Now suppose that  $\bar{W}_n(y)$ . Then  $W_{h(y)} = W_{s(h(y),y)} = \emptyset \in \bar{A}$ .

So since  $g$  is the productive function of  $\bar{A}$ ,  $A(g(h(y)))$ .  $\dashv$

## 11.1 Simple Sets and Incomplete Degrees

Our original question was whether the Gödel phenomenon is the same as being non-r.e. We now know that all productive sets are co-r.e.-hard. This sounds like something different from being non-r.e., since there might be non-r.e.-complete, non-recursive, co-r.e. degrees, and these would be non-r.e. sets that fail to be productive. But are there co-r.e., non-recursive, co-r.e.-incomplete degrees? We haven't seen one yet since all of our non-r.e. sets have resulted from reducing  $\bar{K}$ . So maybe productivity is the same as non-r.e.-ness after all!

But maybe we can turn the issue around. Maybe we can use what we have learned to find out whether there are incomplete co-r.e. (and hence incomplete r.e.) degrees. Since such degrees would not reduce  $\bar{K}$ , that would be something new! Turning matters around, we now know that every complete r.e. set has a productive complement. Productive complements all contain infinite r.e. sets. So how about trying to construct an r.e. set  $S$  whose complement  $\bar{S}$  contains no infinite r.e. set? That's not quite enough, because the complement  $\bar{S}$  might be finite, and all co-finite sets are recursive (use a finite lookup table for the complement). But it is enough if we also require that the complement  $\bar{S}$  is infinite.

$S$  is **simple**  $\leftrightarrow$

1.  $S$  is r.e.
2.  $\bar{S}$  is infinite
3.  $\bar{S}$  has no infinite r.e. subset.

.

Now I state the obvious:

**Proposition 11.7**

1. Simple sets are r.e.
2. Simple sets are not r.e.-complete.
3. Simple sets are not recursive.

Proof:

**Exercise 11.2** *Make sure that these are obvious.* †

So if they exist, simple sets witness the difference between the Gödel phenomenon and mere non-r.e.-ness. More fundamentally, they represent a new way to prove non-verifiability, since they can't reduce  $\bar{K}$ . This idea is due to Emil Post.

**Proposition 11.8 Post's Theorem** *There exists a simple set.*

Proof: using the universal construction, define

$$\phi_a(x) \approx \phi_x((\mu z)(\phi_x(z) > 2x))$$

We know that ranges of partial recursive functions are r.e., so

$$E_a \text{ is r.e.} \tag{11.1}$$

Next, let  $S$  be an infinite r.e. set.

Then for some  $b$ ,  $S = E_b$ .

Then since  $E_b$  is infinite,  $(\mu z)(\phi_b(z) > 2b) \downarrow$ .

So  $\phi_x((\mu z)(\phi_x(z) > 2x)) \downarrow$ .

Hence,  $\phi_x((\mu z)(\phi_x(z) > 2x)) \in E_a \cap E_b$ .

We have just shown that  $E_a$  catches part of each infinite, r.e. set  $E_b$ , so

$$\bar{E}_a \text{ contains no infinite r.e. set.} \tag{11.2}$$

Finally, for each such  $x$ ,  $\phi_a(x) > 2x$ .

Hence,

$$\bar{E}_a \text{ is infinite.} \tag{11.3}$$

†

Now we know that every *complete* r.e. set has a productive complement, which implies that the complement is infinite.

**Exercise 11.3** *Which of the examples prior to proposition 8.4 are productive? Prove your answer, including positive and negative claims.*

**Exercise 11.4** *(Rogers exercise 7-25) Show that*

$$(A \text{ is productive} \wedge B \text{ is r.e.} \wedge B \subseteq A) \Rightarrow A - B \text{ is productive.}$$