A Topological Theory of Empirical Simplicity

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1 Introduction

These notes present a new, axiomatic theory of Ockham's razor and empirical simplicity, with applications to the philosophy of science, statistical model selection, and machine learning. The theory, itself, is simple, as is its motivation. Suppose that the truth is an unknown polynomial law of form:

$$y = \sum_{i \leq N} \alpha_i x^i.$$

Let S denote the set of all $i \leq N$ such that α_i is non-zero. Then:

$$y = p_S(x) = \sum_{i \leq N} \alpha_i x^i.$$

Most would agree that p_S is more complex than $p_{S'}$ if $S' \subset S$. Is that a mere, aesthetic judgment or does it have a deeper, epistemological basis? We recommend the latter alternative. It is well-known that inductive inference is *non-monotonic*, in the sense that more premises may result in the *retraction* of conclusions based on less information. Consider the linear polynomial $y = \alpha_1 x + \alpha_0$ and the quadratic polynomial $y = \alpha_2 x^2 + \alpha_1 x + \alpha_0$. Since $i \in S$ implies that α_i is non-zero, each $i \in S$ is detected *eventually*, as experience increases, but might be undetectable in the short run. Scientists refer to the detection of $i \in S$ as an ith-order *effect*. The coefficients α_i might be arbitrarily small, making the ith-order effect arbitrarily hard to detect, but it is detectable eventually. Assuming that the scientist can converge to the true polynomial form from increasing data, nature has a strategy to force the scientist to ascend through an arbitrary, ascending sequence $S_1 \subset \ldots \subset S_N$ of finite sets of effects. For since science can converge to the truth, nature can present data from the zero constant function $S_1 = \emptyset$ until science decides that $S = \emptyset$. Then nature can choose $\alpha_0 > 0$ sufficiently small that the data presented before might have been true, so science will pick up the 0-order effect eventually and retract to $S_2 = \{0\}$. Then nature can set $\alpha_1 > 0$ to be so small that the data presented before might have been true, etc., up to arbitrary N. In philosophical parlance, each retraction corresponds to a *problem of induction* and the problem of inferring S poses a *nested* problem of induction. The proposal is that empirical simplicity is a *map* of the nested problems of induction inherent in a given theoretical inference problem. It is widely thought that simplicity is a context-dependent whim, exhibiting itself sometimes as minimization of entities, sometimes as smoothing of curves, sometimes as minimizing causes, sometimes as maximizing symmetries, etc. But that pessimistic conclusion is premature. Simultaneity depends upon reference frame, but that does not make it a subjective will-o-the wisp—Einstein isolated the exact laws by which simultaneity depends upon one's inertial reference frame. In a similar spirit, it is proposed that empirical simplicity is a *univocal* concept that *manifests* itself differently depending systematically upon the semantic and informational structure of the theoretical problem under consideration.

2 Information, Questions and Problems

Let W be a set of parameters describing the relevantly possible states of the world, so far as inquiry is concerned. We refer to them as *possible worlds*, without taking any metaphysical stand on whether there *exist* possible worlds distinct from the actual world. In probability theory, one speaks rather misleadingly of the *sample space*.

Information states are propositional, so they are modeled as sets of possible worlds. It is not assumed that information states are closed under finite conjunction (intersection) or disjunction (union), since information may arrive in a particular way— e.g., as a rectangle or a metric ball in the joint variable space—and a finite intersection of balls is not a ball unless one ball is included in all the others. Instead, we require that for each true, finite conjunction of information states, there is a true information state that entails the conjunction. An information basis $\mathcal{I}$ is a collection of subsets of W such that

1. For each $w \in W$, there is information state $E \in \mathcal{I}$ such that $x \in E$.
2. If $w \in E_1 \cap E_2$ for $E_1, E_2 \in \mathcal{I}$ then there is information $E_3 \in \mathcal{I}$ such that $w \in E_3$ and $E_3 \subset E_1 \cap E_2$.

Let $\mathcal{O}$ be the closure of $\mathcal{I}$ under arbitrary union. Let $\mathfrak{T}$ denote the topological space $(W, \mathcal{O})$. Call $\mathfrak{T}$ the *information topology* generated by $\mathcal{I}$. The elements of $\mathcal{O}$ are called *open sets* or, more appropriately, *verifiable propositions*, since elements of $\mathcal{O}$ are the propositions that, if true, are logically entailed, eventually, by true information. Let $\mathcal{I}_w$ denote the set of all $E \in \mathcal{I}$ such that $w \in E$. In other words, $\mathcal{I}_w$ is the set of all possible information states true in w.

A *question* $\mathcal{Q}$ is a partition of W into *answers*. Let H_w be the unique answer to the question that contains (i.e., is true in) w. An *empirical problem* $\mathfrak{P}$ is a triple $(W, \mathcal{I}, \mathcal{Q})$, such that $\mathcal{I}$ is an information base and $\mathcal{Q}$ is a question.

3 Restrictions

Restrictions capture the acquisition of new information. For $E \in I$:

Definition Restrictions

$W|_E = E$;

$\mathcal{I}|_E = \{E' \in \mathcal{I} : E' \subseteq E\}$;

$\mathcal{Q}|_E = \{H \cap E : H \in \mathcal{Q}\}$;

Lemma 1. *$\mathcal{I}|_E$ is an information basis for $W|_E$.*

Proof. Let $w \in W|_E$. Then $w \in E \in \mathcal{I}|_E$ as required. Suppose $w \in E_1 \cap E_2$ for $E_1, E_2 \in \mathcal{I}|_E$. Since E_1, E_2 are also in $\mathcal{I}$, there is $E_3 \in \mathcal{I}$ such that $w \in E_3$ and $E_3 \subset E_1 \cap E_2 \subseteq E$. Therefore $E_3 \in \mathcal{I}|_E$. $\square$

Lemma 2. *$\mathcal{I}|_E$ is a basis for the subspace topology $\mathfrak{T}_E$.*

Proof. Let O be an open set in $\mathfrak{T}$ and $w \in O \cap E$. $O = \bigcup_{\alpha \in J} E_\alpha$ for some index set J. Therefore, we have, for some α, that $w \in E_\alpha \cap E$. So there exists $E' \in \mathcal{I}$ such that $E' \subset E_\alpha \cap E$. Furthermore, since $E' \subseteq E$, $E' \in \mathcal{I}|_E$. Therefore $\mathcal{I}|_E$ is a basis for the subspace topology $\mathfrak{T}_E$. $\square$

Now we are in a position to define restrictions for problems:

Definition

$\mathfrak{P}|_E = (\mathfrak{T}_E, Q|_E)$.

4 Arrows

The basic idea is that empirical simplicity is an apt representation or road map of the theory choice problem under consideration. Coarse-grain W into *simplicity degrees* and draw arrows between such degrees to map the problem of induction in the following sense. Draw a *skeptical* arrow from C to D if and only if science faces the problem of induction in the given problem $\mathfrak{P}$ from C to D. Draw a *benign* arrow from cell C to cell D if and only if the problem of induction obtains from C to D but the true answer to $\mathcal{Q}$ does not. Arrows capture changes of *epistemic state* during the process of inquiry. Traversing a skeptical arrow corresponds to observing an empirical effect that may force a retraction of the scientist's answer to $\mathcal{Q}$. Traversing a benign arrow corresponds to a change of epistemic state that does not require a retraction relative to $\mathcal{Q}$. A *generic* arrow is an arrow of either type. Arrows are problem-relative in two ways: the problem topology determines where arrows exist; the question determines which arrows are skeptical and which benign. Let $\mathfrak{T}_{\mathfrak{P}}$, be the topology for the problem $\mathfrak{P}$.

Skeptical, Benign and Generic Arrows

$S_{\mathfrak{P}}(w, X)$ iff $w \in \mathrm{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(X \setminus H_w)$;

$B_{\mathfrak{P}}(w, X)$ iff $\neg S_{\mathfrak{P}}(w, X)$ and $w \in \mathrm{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(X) \setminus X$;

$A_{\mathfrak{P}}(w, X)$ iff $S_{\mathfrak{P}}(w, X)$ or $B_{\mathfrak{P}}(w, x)$.

We will omit the problem subscript where there is no ambiguity.

Remark 1. *We have that $A_{\mathfrak{P}}(w, X) \to w \in bdry_{\mathfrak{T}_{\mathfrak{P}}}(X)$ but not the converse. However, $w \in bdry_{\mathfrak{T}_{\mathfrak{P}}}(X) \setminus X \to A_{\mathfrak{P}}(w, X)$.*

Lemma 3. *If $E \in \mathcal{I}_w$ then $A_{\mathfrak{P}}(w, D|_E)$ if and only if $A_{\mathfrak{P}|_E}(w, D|_E)$. Moreover, the type of the arrow is preserved.*

Proof.

(*Skeptical Case*) Suppose $S_{\mathfrak{P}}(w, D|_E)$. Then for all open nbhds O of w in $\mathfrak{T}$, $O \cap E \cap D \setminus H_w$ is nonempty. But since $O \cap E$ is an arbitrary open set of $\mathfrak{T}_E$, we have that $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}|_E}}(D \setminus H_w)$ and therefore $S_{\mathfrak{P}|_E}(w, D|_E)$. The converse is symmetrical.

(*Benign Case*) Suppose $B_{\mathfrak{P}}(w, D|_E)$. Then $w \notin D|_E$ and for every nbhd O of w open in $\mathfrak{T}$, $O \cap D \cap E$ is non-empty. As above, since $O \cap E$ is an arbitrary open set in $\mathfrak{T}_E$, we have that $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}|_E}}(D|_E) \setminus D|_E$ as well. By the skeptical case, if $\neg S_{\mathfrak{P}}(w, D|_E)$ then $\neg S_{\mathfrak{P}|_E}(w, D|_E)$. So we have that $B_{\mathfrak{P}|_E}(w, D|_E)$. The converse is symmetrical. □

The previous lemma allows us to use the original problem for the restricted problem freely.

Lemma 4. *If $E \in \mathcal{I}_w$ then $A_{\mathfrak{P}}(w, D)$ if and only if $A_{\mathfrak{P}}(w, D|_E)$. Moreover, the type of the arrow is preserved.*

Proof. $\Rightarrow$ Let $A(w, D)$ and $E \in \mathcal{I}_w$.

(*Skeptical Case*) Suppose $S(w, D)$ and therefore $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D \setminus H_w)$. Now suppose for a contradiction that $w \notin \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E \setminus H_w)$. So there is a nbhd O of w that catches E and $D \setminus H_w$ but no part of their intersection. But since E is also a nbhd of w, $O \cap E$ is a nbhd of w that does not intersect $D \setminus H_w$, contradicting $S(w, D)$. So it must be that $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E \setminus H_w)$ and therefore $S(w, D|_E)$. Since E was arbitrary, we have shown that skeptical arrows are preserved by information.

(*Benign Case*) Now suppose $B(w, D)$ and therefore $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D)$ but $w \notin D$ and $w \notin \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D \setminus H_w)$. Clearly, $w \notin D|_E$. If $w \notin \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E)$ then there is a neighborhood O of w such that $O \cap E \cap D = \emptyset$. But $O \cap E$ is also a nbhd of w, contradicting $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D)$. So $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E)$. Furthermore, if $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E \setminus H_w)$ then clearly $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D \setminus H_w)$, contradicting $B(w, D)$. So $w \notin \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E \setminus H_w)$ and we have that $B(w, D|_E)$. Since E was arbitrary, we have shown that benign arrows are preserved by information.

$\Leftarrow$ (*Skeptical Case*) Suppose $S(w, D|_E)$. Then $\neg B(w, D|_E)$, $w \notin D|_E$ and $w \in \text{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D|_E)$. Since w is in the boundary of the subset $D|_E$, it is in the boundary of D as well. Since w is not in the subset $D|_E$ it is not in D

either. By contraposition of $\Rightarrow$, we have $\neg B(w, D)$, therefore $S(w, D)$. (*Benign Case*) Suppose $B(w, D|_E)$. Then $w \in \text{bdry}_{\mathfrak{I}_{\mathfrak{P}}}(D|_E \setminus H_w)$. Since w is in the boundary of the subset $D|_E \setminus H_w$, it is in the boundary of the superset $D \setminus H_w$ as well. Therefore $B(w, D)$.

□

5 Factorizations

It is time to draw the epistemic map of problem $\mathfrak{P} = (W, \mathcal{I}, \mathcal{Q})$. Let $\mathcal{F}$ partition W. Call the elements of this partition *simplicity degrees.* Arrows are lifted to simplicity degrees $D, D' \in \mathcal{F}$ as follows:

Arrows for Reasons

$S(D, D')$ iff $S(w, D')$ for some $w \in D$;

$B(D, D')$ iff $B(w, D')$ for some $w \in D$;

$A(D, D')$ iff $A(w, D')$ for some $w \in D$.

We will make use of the set of arrow-minimal simplicity degrees. These are the simplicity degrees for which no world has a problem of induction.

The Set of Arrow-Minimal Reasons

$\text{Min}(\mathcal{F}) = \{D : (\forall w)\ \neg A(w, D)\}$

Restrictions of Factorizations

$\mathcal{F}|_E = \{D \cap E : D \in \mathcal{F}\}$.

Factorization

Let the binary relation $\textsf{Fac}(\mathcal{F}, \mathfrak{P})$ hold if and only if for every $w \in W$:

Homogeneity

$$w, w' \in D \to (S(w, D') \to S(w', D')) \ \ \wedge \ \ (B(w, D') \to B(w', D')).$$

Minimality

$$(\exists E \in \mathcal{I}_w)\ (\{D_w \cap E\} \in \text{Min}(\mathcal{F}|_E)).$$

Theorem 1. $\textsf{Fac}(\mathcal{F}, \mathfrak{P}) \to \textsf{Fac}(\mathcal{F}|_E, \mathfrak{P}|_E)$.

Proof.

(*Homogeneity*) Let $w, w' \in D$ and $E \in \mathcal{I}_w \cap \mathcal{I}_{w'}$. Suppose $S(w, D')$. By **Homogeneity**, $S(w', D')$. By Lemma 4, $S(w, D'|_E)$ and $S(w', D'|_E)$. The case of the benign arrow is identical. So information preserves Homogeneity.

(*Minimality*) By **Minimality**, for all $w \in W$ there is information $O \in \mathcal{I}_w$ such that for all $w' \in W|_O$, $\neg A(w', D|_O)$. By Lemma 4, for all $w' \in W|_{O \cap E}$, $\neg A(w', D|_{O \cap E})$ and Minimality is satisfied.

□

Theorem 2. *Arrows are a strict partial order over cells.*

Proof.

(*Asymmetry*) Suppose $A(D, D')$ and $A(D', D)$. Let $w \in D$ and $E \in \mathcal{I}_w$. By **Homogeneity**, we have that $A(w, D')$. By the arrow we know that E catches some point $w' \in D'$. By **Homogeneity**, we have that $A(w', D)$. Since $E \in \mathcal{I}_{w'}$ as well, we have by Lemma 4 that $A(w', D|_E)$. So for all E, we have that $A(D'|_E, D|_E)$ and therefore $D \cap E \notin \mathrm{Min}(\mathcal{F}|_E)$, contradicting **Minimality**.

(*Transitivity*) Suppose $A(D, D')$ and $A(D', D'')$. By Asymmetry, $D \neq D''$. Let $w \in D$. By **Homogeneity**, we have that $A(w, D')$. So every $E \in \mathcal{I}_w$ catches some $w' \in D'$. By **Homogeneity**, $A(w', D'')$. Since E is a nbhd of w' it catches some point in D''. Since E arbitrary, we have that $w \in \mathrm{bdry}_{\mathfrak{T}_{\mathfrak{P}}}(D'') \setminus D''$. By the Remark, $A(w, D'')$ and $A(D, D'')$. $\square$

Theorem 3. *For all $w \in W$ there exists $E \in \mathcal{I}_w$ such that $D_w \cap E \subset H_w$.*

Proof. Suppose for a contradiction that for all $E \in \mathcal{I}_w$, $D_w \cap E \cap H_{z \neq w}$ is nonempty. But then $w \in D \backslash H_w$ and therefore $S(w, D)$. Then by **Homogeneity** $A(D, D)$, a contradiction by asymmetry of the partial order. $\square$

Theorem 4. *Finite anti-chains are decidable.*

Proof. Suppose $D_1, D_2, D_3, ...$ is a finite collection of distinct cells, unordered by arrows. Let $w \in D_i$. By **Homogeneity**, for all $j \neq i$ we have $\neg A(w, D_j)$. So for all $j \neq i$ there exist nbhds O_j of w such that $O_j \cap D_j = \emptyset$. Since the collection is finite, $\bigcap_{j \neq i} O_j$ is a nbhd of w that intersects no $D_{j \neq i}$. Furthermore, there must be $E \in \mathcal{I}_w$ such that $w \in E$ and $E \subseteq \bigcap_{j \neq i} O_j$. So D_i is separable from the collection of cells $D_{j \neq i}$. $\square$