A New Theory of Acceptance that Solves the Lottery Paradox and Provides a Simplified Probabilistic Semantics for Adams' Logic of Conditionals

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Abstract

A class of acceptance rules is proposed to relate probabilistic degrees of belief to acceptance. The rules avoid the lottery paradox and yield a probabilistic semantics (i) that adopts Ramsey test for accepting conditionals, (ii) that defines validity as preservation of acceptance, (iii) that allows acceptance under uncertainty; and (iv) that validates exactly Adams' logic of flat conditionals. Furthermore, the rules illuminate a close relationship between two types of reasonings: the Bayesian reasoning by probabilistic conditioning can be represented as the nonmonotonic reasoning by ignoring less normal cases.

1 Introduction

If Bayesians are right, one's credal state should be a probability function p that measures one's degrees of belief. Then, it seems that one *accepts* a proposition in light of p. Acceptance of proposition A is sometimes understood as being certain of A, in the sense that one would bet one's life against nothing on the truth of A. Similarly, acceptance of proposition A is sometimes portrayed as decision to remove all doubt about A, in the sense that one changes the credal state p to make A certain (e.g., Levi 1967). But everyday acceptance of propositions is not so dire as that. Instead, I propose a more modest view of acceptance, according to which the set of propositions accepted in light of p should aptly capture some of the characteristics of the underlying credal state p.

It seems, at first glance, that some high but realistic probability suffices for accepting a proposition, a view now referred to as the *Lockean thesis*. But the Lockean thesis

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licenses acceptance of classically inconsistent sets of propositions. For suppose that one chooses some suitably high probability threshold $r < 1$ as sufficient for acceptance. There exists a fair lottery with more than $1/(1-r)$ tickets, so that for each ticket i in the lottery, the proposition that ticket i will lose has probability greater than r and is therefore accepted. The proposition that *some* ticket in the lottery wins has probability 1, so it is also accepted. Hence a classically inconsistent set of propositions is accepted.

To avoid the lottery paradox, one must give up either the Lockean thesis, or classical consistency, or acceptance of uncertain propositions. Kyburg pursues the second course through a bold logical reform. Most responses constrain the Lockean thesis in some manner: John Pollock (1995), Sharon Ryan (1996), and Igor Douven (2002) propose conditions under which the Lockean thesis is defeated; Bas van Fraassen (1995) and Horacio Arló-Costa and Rohit Parikh (2005) require that only propositions of probability 1 be accepted. Isaac Levi (1967) rejects basing acceptance on probabilities alone, insisting that acceptance be viewed as a decision informed by both probabilities and utilities.

From a broader perspective, the point of acceptance is to capture some of the structure of one's underlying credal state. The lottery paradox suggests the negative answer that the relevant structure involves something other than the probability values of the accepted propositions. In this thesis, I seek to provide a more positive account of the probabilistic structure that matters for acceptance.

First, I characterize the "geometrical shapes" of acceptance rules that are necessary to validate a widely recognized axiom system in the logic of defeasible reasoning: system P for nonmonotonic logic (Kraus, Lehmann, Magidor 1990) or, equivalently, Adams' system for the logic of flat conditionals (Adams 1975), which are extensions of classical logic. Most proposed acceptance rules do not have the right geometrical shape. Knowing the requisite geometry makes it easy to specify a class of very natural acceptance rules that not only avoids the lottery paradox but also validates system P. Due to their geometric shape, the rules I propose are called *camera shutter* rules.

Two applications lend further support to the camera shutter rules. First, the camera shutter rules yield a new probabilistic semantics for flat conditionals that improves Adams' (1975) ϵ-δ semantics and Pearl's (1989) infinitesimal semantics: (i) it defines validity as preservation of acceptance; (ii) it allows acceptance under uncertainty, without extremely high probability; and (iii) it validates *exactly* Adams' system for the logic of flat conditionals. At the same time, the semantics still employs a very direct and simple version of the Ramsey test, expressed in terms of probabilistic conditioning and acceptance of conditional-free propositions. As a second application, the camera shutter rules connect two types of reasonings: reasoning by probabilistic conditioning can be represented as "reasoning by ignoring less normal cases." The latter idea has been studied extensively in the logic of defeasible reasoning and has been shown to be equivalent to the pattern of defeasible reasoning that satisfies system P (Kraus, Lehmann, Magidor 1990).

Then, what characteristics of one's probabilistic credal state are captured by the set of accepted propositions, if one adopts a camera shutter rule for acceptance? As we will see, the camera shutter rules are based, not on probabilities alone, but on *probability ratios*. Specifically, each camera shutter rule is understood as a rule for accepting propositions in the context of a question, such that a potential answer to the question is rejected when its probability ratio to the most probable alternative is too low. Hence, in a nutshell, acceptance reflects probability ratios.

2 The Geometry of the Lottery Paradox

Let $\mathcal{E} = \{E_i : i \in I\}$ be a countable partition of an underlying set of possible worlds, whose elements E_i are called cells. Let $\mathcal{A}$ be the algebra of propositions that is constructed from partition $\mathcal{E}$ by closing it under negation, conjunction, and countable disjunction. Let $\mathcal{P}$ be the set of all countably additive probability measures on $\mathcal{A}$. Partition $\mathcal{E}$ will be understood as a *question*; the cells E_i of $\mathcal{E}$ as the *potential answers* to question $\mathcal{E}$; algebra $\mathcal{A}$ as the set of *incomplete answers* to $\mathcal{E}$; and $\mathcal{P}$ as the set of *probabilistic credal states* over the incomplete answers to $\mathcal{E}$.

For concreteness, let question $\mathcal{E}$ be ternary (i.e. $|\mathcal{E}| = 3$). Then each probability measure p on $\mathcal{A}$ can be represented uniquely by the three-vector $(p(E_1), p(E_2), p(E_3))$. The credal state space $\mathcal{P}$ corresponds, then, to the set of all 3-vectors whose components are non-negative and sum to 1, which is exactly the equilateral triangle in $\mathbb{R}^3$ whose vertices have Cartesian coordinates $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ (figure 1).

An acceptance rule specifies, for each credal state p, the propositions that are accepted in light of p. Assume that accepted propositions are closed under classical entailment. Define an *acceptance rule* to be a map

$$\alpha : \mathcal{P} \to \mathcal{A},$$

where $\alpha(p)$ is the strongest proposition accepted at credal state p. Proposition A in $\mathcal{A}$ is *accepted* by rule α at state $p \in \mathcal{P}$, written $p \Vdash_\alpha A$, if and only if $\alpha(p)$ entails A (i.e. $\alpha(p) \subseteq A$). Note that an acceptance rule is by definition relative to a partition $\{E_i : i \in I\}$. This is not merely a modeling assumption. It is necessary to avoid a trilemma (theorem 5) more fundamental than the Lottery paradox, which will be discussed in the concluding section.

The *Lockean* acceptance rule λ_r accepts the propositions having at least probability r and all of their classical logical consequences, so it can be expressed by:

$$\lambda_r(p) \quad = \quad \bigwedge\{A \in \mathcal{A} : p(A) \geq r\}. \tag{1}$$

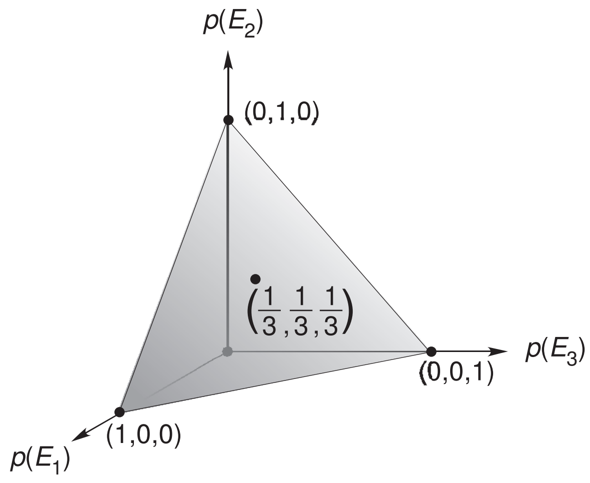

Figure 1: space of probabilistic credal states

Define the *acceptance zone* of proposition A under rule α to be the set of all credal states at which A is accepted: $\{p \in \mathcal{P} : p \Vdash_\alpha A\}$, which can be visualized as a subspace of the "triangular" state space $\mathcal{P}$. For example, the diamond-shaped zone labeled 'E_1' in figure 3.a is the acceptance zone of answer E_1 under the Lockean rule depicted.

To check that the shape is right, express the Lockean rule as follows:[1]

$$\lambda_r(p) = \bigwedge\{\neg E_i : p(\neg E_i) \geq r \text{ and } i \in I\} \tag{2}$$

$$= \bigwedge\{\neg E_i : p(E_i) \leq 1 - r \text{ and } i \in I\}. \tag{3}$$

Hence, the acceptance zone of $\neg E_1$ is the set $\{p \in \mathcal{P} : p(E_i) \leq 1 - r\}$, which is a trapezoid that results from truncating the triangular space $\mathcal{P}$ parallel to one side (figure 2). As r decreases, the trapezoid becomes thicker. The acceptance zones for all $\neg E_i$ are included in figure 3.a. Since answer E_1 is the the conjunction of the negations of the other two answers, the acceptance zone of E_1 is the intersection of the zones of $\neg E_2$ and $\neg E_3$, which is indeed diamond-shaped.

When $r \leq \frac{2}{3}$, the three trapezoidal zones of $\neg E_i$ become so thick that they overlap at the center of the triangle. In that case, $\neg E_1, \neg E_2$, and $\neg E_3$ are accepted at the credal states in the small, dark, central triangle (figure 3.b), so their conjunction—the inconsistent proposition denoted by $\perp$—is accepted as well. That is the geometry of the lottery paradox (interpret E_i as the proposition "the i-th ticket wins").

[1] Expression 1 is equivalent to expression 2 because every proposition A is equivalent to a conjunction of propositions of form $\neg E_i$ that are entailed by A and, thus, are at least as probable as A.

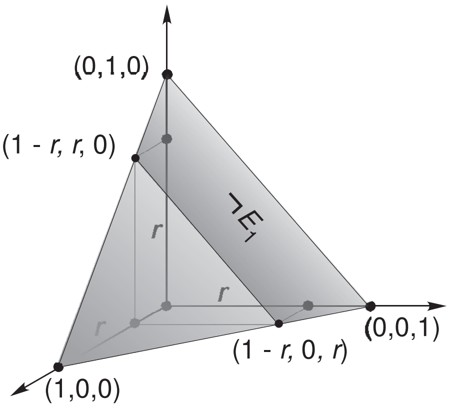

Figure 2: acceptance zone of $\neg E_1$ under Lockean rule λ_r

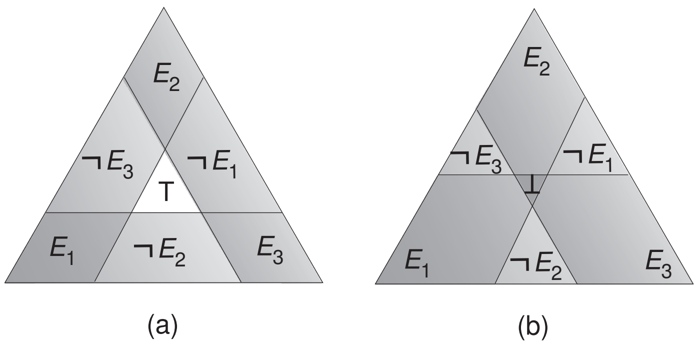

Figure 3: acceptance zones under Lockean rules

3 The Geometry of the Ramsey Test

The lottery paradox is only the most glaring logical problem with the Lockean rules. The Lockean rules, even when consistent, remain problematic in the logic of conditionals.

Frank P. Ramsey has a very influential suggestion concerning the acceptance of conditional statements, now commonly referred to as the *Ramsey test*:

> If two people are arguing 'If A, then B?' and are both in doubt as to A, they are adding A hypothetically to their stock of knowledge and arguing on that basis about B; so that in a sense 'If A, B' and 'If A, $\neg B$' are contradictories. We can say that they are fixing their degree of belief in B given A. (Ramsey 1929, footnote 1)[2]

[2] We take the liberty of substituting A, B for p, q in Ramsey's text.

Here is a very direct interpretation of that suggestion. Suppose that an idealized agent has probabilistic credal state p and adopts acceptance rule α. To determine whether the agent accepts conditional statement 'if A then B', let the agent hypothetically modify her current, probabilistic credal state p by conditioning it on A, and then see whether B would be accepted at the resulting state $p(\cdot|A)$ by the rule α she adopts.[3] In other words, define the relation $\mid\!\sim_{\alpha,p}$ on the set $\mathcal{A}$ of propositions by:

$$A \mid\!\sim_{\alpha,p} B \quad \textit{iff} \quad \begin{array}{l} \text{either } p(\cdot|A) \Vdash_\alpha B, \\ \text{or } p(A) = 0. \end{array}$$

The Ramsey test is then interpreted as saying that $A \mid\!\sim_{\alpha,p} B$ is a necessary and sufficient condition for the agent to accept conditional statement 'if A then B'.

For example, suppose that the agent adopts the consistent Lockean rule λ depicted in figure 4, with credal state p. Then the following relations hold:

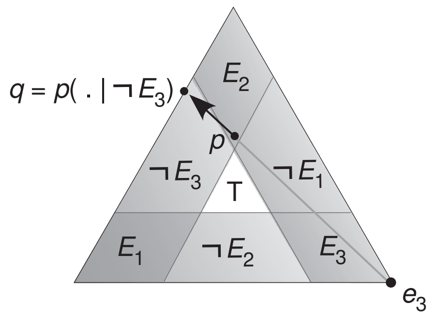

Figure 4: Cautious Monotonicity violated

$$\top \quad \mid\!\sim_{\lambda,p} \quad \neg E_3, \tag{4}$$
$$\top \quad \mid\!\sim_{\lambda,p} \quad E_2, \tag{5}$$
$$\neg E_3 \quad \not\mid\!\sim_{\lambda,p} \quad E_2. \tag{6}$$

Relations (4) and (5) follow from the fact that $p(\cdot|\top) = p$, at which E_2 and $\neg E_3$ are accepted. To establish (6), let $q = p(\cdot|\neg E_3)$. Credal state q lies on the side of the triangle opposite e_3 because $q(E_3) = 0$; and q lies on the ray from e_3 that passes through p because conditioning preserves probability ratio: $\frac{q(E_1)}{q(E_2)} = \frac{p(E_1)}{p(E_2)}$. So, as depicted in figure 4, q is the projection of p with "light source" e_3. Point q is therefore *not* in the diamond-shaped zone for accepting E_2, and hence (6).

But relations (4)-(6) violate a widely-accepted axiom in conditional/nonmonotonic logic, called *Cautious Monotonicity*:

[3] If $p(A) \neq 0$, $p(\cdot|A)$ is defined to be $\frac{p(\,\cdot\,\wedge A)}{p(A)}$; otherwise it is undefined.

$$\textbf{(Cautious Monotonicity)}\quad \frac{\begin{array}{rcl} A & \mid\!\sim & B \\ A & \mid\!\sim & C \end{array}}{\begin{array}{rcl} A \wedge B & \mid\!\sim & C \end{array}}$$

When the agent has unconditionally accepted both propositions E_2 and $\neg E_3$, Cautious Monotonicity says that supposing one proposition ($\neg E_3$) as true gives no reason for retracting the other proposition (E_2), so $\neg E_3 \mid\!\sim E_2$, which contradicts (6). In short, a Lockean rule, even if consistent, does not square with the Ramsey test and Cautious Monotonicity because of a geometrical fault: the sides of its diamond-shaped acceptance zone of E_2 meet at *too acute an angle*.

Many acceptance rules have been proposed to replace the Lockean rules, in order to respect classical logic and to guard against classical inconsistency. But nearly all of them fail to validate Cautious Monotonicity because their acceptance zones of answers E_i are shaped like those of the Lockean rules. For example, Pollock (1995), Ryan (1996), and Douven (2002) propose a type of rule that is geometrically representable by figures 3.a and 3.b, except that it produces the tautology $\top$ when the Lockean rule happens to produce the contradiction $\perp$ in figure 3.b. For another example, the acceptance rule in Levi (1967, 69) is representable by figure 3.a, except that the trapezoidal zones need not be equally thick.[4]

If the lower half of the diamond is made sufficiently blunt, Cautious Monotonicity will be satisfied. But if it is too blunt, then it can be shown to violate another widely-accepted axiom in conditional/nonmonotonic logic, called *Or*. The shape has to be "just right", as will be clear in the next section.

4 The Geometry of Nonmonotonic Reasoning

In general, a (*nonmonotonic*) *consequence relation* on the set $\mathcal{A}$ of propositions, usually denoted by $\mid\!\sim$ with suitable subscripts, is a binary relation on $\mathcal{A}$. When relation $\mid\!\sim$ captures one's pattern of reasoning, $A \mid\!\sim B$ means that one would accept B as a consequence by supposing that A is true or by "adding A hypothetically to one's stock of knowledge." The following list of axioms or rules, known as *system* P, has been recognized as central both to defeasible reasoning (Kraus, Lehmann, and Magidor 1990) and to acceptance of conditionals (Adams 1975):[5]

[4] Levi is concerned with dynamics of credal states rather than statics of accepted conditional statements, so this paragraph is no criticism of him. Although Pollock, Ryan, and Douven are concerned with statics, the logic of conditionals is not their central concern.

[5] In the nonmonotonic logic literature, conditional axioms governing the consequence relation are written like inference rules. Think of the horizontal line as material implication.

$$(\textbf{Reflexivity})\ \frac{}{A \mid\!\sim A} \qquad\qquad (\textbf{And})\ \frac{\begin{array}{lcl} A & \mid\!\sim & B \\ A & \mid\!\sim & C \end{array}}{A \mid\!\sim B \wedge C}$$

$$(\textbf{Left Equivalence})\ \frac{A \mid\!\sim B}{A' \mid\!\sim B} \text{ if } A \text{ is equivalent to } A' \text{ in classical logic.}$$

$$(\textbf{Right Weakening})\ \frac{A \mid\!\sim B}{A \mid\!\sim B'} \text{ if } B \text{ entails } B' \text{ in classical logic.}$$

$$(\textbf{Cautious Monotonicity})\ \frac{\begin{array}{rcl} A & \mid\!\sim & B \\ A & \mid\!\sim & C \end{array}}{A \wedge B \mid\!\sim C} \qquad (\textbf{Or})\ \frac{\begin{array}{rcl} A & \mid\!\sim & C \\ B & \mid\!\sim & C \end{array}}{A \vee B \mid\!\sim C}$$

Acceptance rule α *validates* system P if and only if for each state p in the domain of α, consequence relation $\mid\!\sim_{\alpha,p}$ satisfies each axiom in system P.

We already know that Lockean rules do not validate the axiom Cautious Monotonicity in system P, due to their diamond-shaped acceptance zones. To validate system P, the shape of the acceptance zone of a cell E_i must be a "blunt diamond" like that in figure 5.a, whose lower boundary lines coincide with rays from the opposite corners, respectively.

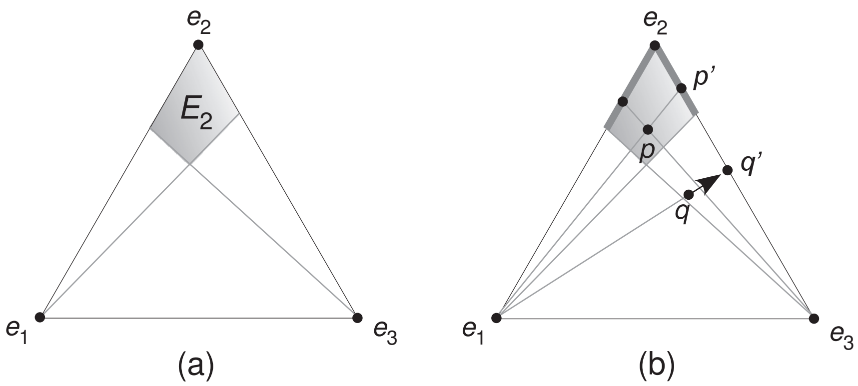

Figure 5: acceptance zone of E_2

More precisely, the acceptance zone of answer E_i under α is a *blunt diamond* if and only if it takes the following form: there exist thresholds $\{t_{ij} : j \in I \setminus \{i\}\}$ in interval $[0, \infty]$ and inequalities $\{\lhd_{ij} : j \in I \setminus \{i\}\}$ that are either $\leq$ or $<$, such that for each $p \in \mathcal{P}$:

1. $p \Vdash_\alpha E_i \iff \forall j \in I \setminus \{i\},\ \frac{p(E_j)}{p(E_i)} \lhd_{ij}\ t_{ij}$;

2. if $\lhd_{ij} = \leq$ then $t_{ij} < \infty$;

3. if $\lhd_{ij} = <$ then $t_{ij} > 0$.

Given two probabilistic states p and q, the *line segment* $\overline{p\ q}$ is defined by convex combination:

$$\overline{p\ q} \ = \ \{ap + (1-a)q : a \in [0,1]\}.$$

Say that α is *corner-convex* if and only if (i) $\alpha(e_i) = E_i$ for each $i \in I$, and (ii) for each $p \in \mathcal{P}$ such that $\alpha(p) = E_i$, we have that $\alpha(q) = E_i$ for all q in line segment $\overline{p\ e_i}$. Say that α is *everywhere consistent* if and only if $\alpha(p) = \perp$ for all p in $\mathcal{P}$.

Theorem 1 (blunt diamond). *Let $\mathcal{E}$ be finite.*[6] *Suppose that acceptance rule α is everywhere consistent, satisfies corner-convexity, and validates system* P. *Then for each answer E_i to $\mathcal{E}$, the acceptance zone of E_i under α is a blunt diamond.*

To accept cell E_i, each rival E_j ($j \neq i$) has to be rejected and, thus, the probability ratio of E_j to E_i, i.e. $\frac{p(E_j)}{p(E_i)}$, has to be sufficiently low. So acceptance depends upon probabilities as well as upon probability ratios. As in figure 5, a ray from corner e_1, for example, is the set of credal states in which the probability ratio between E_2 and E_3 is held constant. That is why the lower half of the acceptance zone of E_2 is bounded by such rays. Although the appendix contains a proof of the theorem, the argument for case $|\mathcal{E}| = 3$ is given here because we will shortly reply upon the geometric insight it offers.

Geometric Argument for Case $|\mathcal{E}| = 3$. Suppose that acceptance rule α validates system P and corner-convexity. Solve for the acceptance zone of E_2 under α, as depicted in figure 5.b. Along the side $\overline{e_2\ e_1}$ of the triangle, the states at which α accepts E_2 form a continuous, unbroken line segment with e_2 as an endpoint (by corner-convexity), which is depicted as a heavy, grey line segment lying on $\overline{e_2\ e_1}$. The same is true for side $\overline{e_2\ e_3}$.[7] Connect the endpoints of the grey line segments to the opposite corners by straight lines, which enclose the grey blunt diamond at the corner e_2.

Argue as follows that for each p in the blunt diamond, $p \Vdash_\alpha E_2$. Conditioning p on $\neg E_1$ and $\neg E_3$ results in points that lie in the two dark line segments, respectively, where E_2 is accepted by α. So $\neg E_1 \mathrel{|\!\sim}_{\alpha,p} E_2$ and $\neg E_1 \mathrel{|\!\sim}_{\alpha,p} E_2$. Then by axiom Or, $(\neg E_1 \vee \neg E_3) \mathrel{|\!\sim}_{\alpha,p} E_2$. But the left hand side equals $\top$, so $\top \mathrel{|\!\sim}_{\alpha,p} E_2$ and thus $p \Vdash_\alpha E_2$.

Argue as follows that for each q outside of the blunt diamond, $q \nVdash_\alpha E_2$. Since q is outside of the blunt diamond, the result of conditioning q either on $\neg E_1$ or on $\neg E_3$ must

[6]When $\mathcal{E}$ is countably infinite, we need to assume the infinite disjunctive generalization of axiom *Or* to prove the theorem.

[7]There is an issue whether the line segments are open or closed at the endpoints distinct from e_2, which would give rise to a possible mixture of strict and weak inequalities, as stated in the theorem. That issue is handled in the formal proof in the appendix, but ignored here.

project q to a point that lies on the corresponding side and outside of the corresponding dark line segment. Suppose without loss of generality that conditioning on $\neg E_1$ does so, as depicted in figure 5.b. Then we have that $\neg E_1 \not\mid\!\sim_{\alpha,q} E_2$. Suppose for reductio that $q \Vdash_\alpha E_2$. So $\top \mid\!\sim_{\alpha,q} E_2$ and $\top \mid\!\sim_{\alpha,q} \neg E_1$, and thus by Cautious Monotonicity, $\neg E_1 = (\top \wedge \neg E_1) \mid\!\sim_{\alpha,q} E_2$. But this contradicts $\neg E_1 \not\mid\!\sim_{\alpha,q} E_2$. $\square$

Consider a particularly simple case of the acceptance rule α in theorem 1. Let all inequalities and thresholds be the same: $\lhd_{ij} = \leq$ and $t_{ij} = 1 - r$, where r is a constant in the open unit interval $(0, 1)$. In this case the acceptance zones of E_i are symmetric, as in figure 6.a. Then we have:

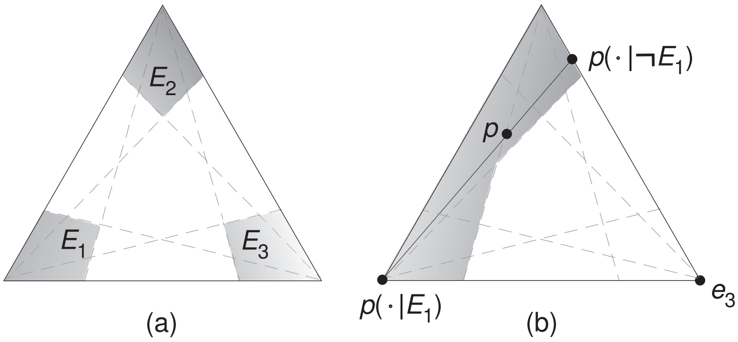

Figure 6: acceptance zone of $\neg E_3$

Claim 1. *Suppose that α is an acceptance rule whose acceptance zones of E_i, $i = 1, 2, 3$, are depicted in 6.a. Then, the acceptance of $\neg E_3$ under α includes at least the bent, grey zone depicted in figure 6.b, which lies on the side opposite corner e_3.*

Proof. Let p be an arbitrary point in the bent zone. Suppose that p is in the blunt diamond for E_1 or in that for E_2; then $p \Vdash_\alpha \neg E_3$, by closure under entailment. Suppose that p is in neither of the two blunt diamonds; by symmetry, suppose further that p lies in the upper half of the bent zone, as depicted in figure 6.(b). By conditioning, send p to the two points $p(\cdot|E_1)$ and $p(\cdot|\neg E_1)$, which are well-defined and land in the blunt diamond zones for accepting E_1 and E_2, respectively. So we have that $p(\cdot|E_1) \Vdash_\alpha \neg E_3$ and that $p(\cdot|\neg E_1) \Vdash_\alpha \neg E_3$; hence that $E_1 \mid\!\sim_{\alpha,p} \neg E_3$ and $\neg E_1 \mid\!\sim_{\alpha,p} \neg E_3$. Then, by axiom Or, $(E_1 \vee \neg E_1) \mid\!\sim_{\alpha,p} \neg E_3$. The left hand side is a tautology, so $p \Vdash_\alpha \neg E_3$, as required. $\square$

(Generalization of this claim to higher dimensions is not difficult, but the low dimensional case suffices for motivating the new acceptance rules.)

In general, $\neg E_i$ must be accepted in the bent zone lying on the side opposite corner e_i. But that is everything system P implies about what has to be accepted where.

Without violation of system P, we can add a bulge to the bent zone for accepting $\neg E_i$ as long as it does not touch the blunt diamonds. But it is theoretically simple and well-motivated to keep the acceptance zone of $\neg E_i$ minimal, without any bulge added to the bent zone. For, in that case, the acceptance zone of $\neg E_i$ coincides with the bent zone, which is given by a crisp formula: for all $p \in \mathcal{P}$,

$$p \Vdash_\alpha \neg E_i \iff \frac{p(E_i)}{\max_j p(E_j)} \leq 1 - r. \tag{7}$$

The formula expresses a natural idea: *potential answer E_i to question $\mathcal{E}$ is rejected if and only if the probability ratio of E_i to the most probable alternative is too low.* Now superpose of all the three bent zones for accepting $\neg E_i$ on the triangle, and the result is the acceptance rule depicted in figure 7.a.

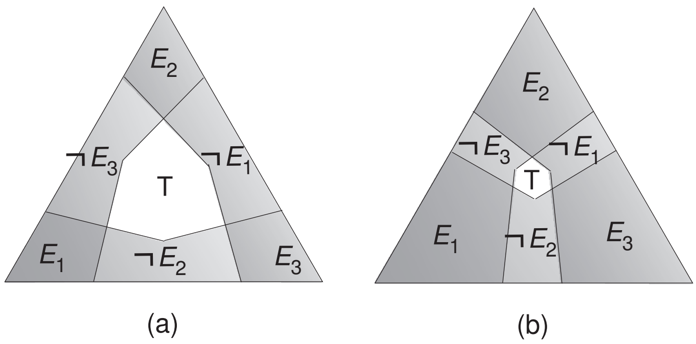

Figure 7: acceptance zones under camera shutter rules

When the standard for acceptance becomes more lenient, i.e., when r is tuned toward zero, the bent zones for rejecting the cells expand toward the center of the triangle (from figure 7.a to 7.b). But they *bend* progressively as they approach the center of the triangle so that they eventually kiss without overlapping like the leaves of a camera shutter. In other words, the zones would not crash into one another and thus the rules would not generate the lottery paradox, in contrast to what the Lockean rules do in figure 3. Hence the lottery paradox is avoided by a geometrical maneuver that is independently motivated by nonmonotonic logic.

Acceptance rules of this form can be defined in general as follows:

Definition 1 (**symmetric camera shutter rule**). *A symmetric camera shutter rule is an acceptance rule $\nu_r : \mathcal{P} \to \mathcal{A}$ defined by:*

$$\nu_r(p) \quad = \quad \bigwedge \left\{ \neg E_i \ : \ \frac{p(E_i)}{\max_j p(E_j)} \leq 1 - r \ \text{and}\ i \in I \right\},$$

where r is in $(0, 1]$.

Since the quantity $\frac{p(E_i)}{\max_j p(E_j)}$ is always in the unit interval, it seems natural to require the range of r to be the unit interval. But I disallow r to be 0, for in that case $\neg E_i$ would be accepted even if the probability of E_i is 1. The symmetric camera shutter rule is the same as the Lockean rule expressed in (3) except that, now, probability $p(E_i)$ gives way to probability ratio $\frac{p(E_i)}{\max_j p(E_j)}$. The importance of probability ratios will occur repeatedly in the following development.

The bent zones of a symmetric camera shutter rule all have the same thickness and always include their boundaries. Both of these restrictions can be relaxed without altering the crucial angles of the sides of the zones.

Definition 2 (**camera shutter rule**). *An acceptance rule ν is called a camera shutter rule*[8] *if and only if there exist thresholds r_i and inequalities $\lhd_i \in \{\leq, <\}$, for each $i \in I$, such that:*

1. $\nu(p) \;=\; \bigwedge \left\{\neg E_i : \frac{p(E_i)}{\max_j p(E_j)} \lhd_i 1 - r_i \text{ and } i \in I\right\}$*, for each p in $\mathcal{P}$,*
2. *for each $i \in I$, if $\lhd_i = \leq$ then $r_i \in (0, 1]$ (0 is omitted to make it possible to not reject E_i), and*
3. *for each $i \in I$, if $\lhd_i = <$ then $r_i \in [0, 1)$ (1 is omitted to make it possible to reject E_i).*

This geometric tree has borne logical fruit:

Theorem 2 (**validation of** P). *Each camera shutter rule ν validates system* P.

5 A New Probabilistic Semantics for Flat Conditionals

Axiom system P is characteristic of Adams' logic of flat conditionals, so it is not surprising that camera shutter rules yield a new probabilistic semantics for that logic, which in many ways improves Adams' (1975) ϵ-δ semantics and Pearl's (1989) infinitesimal semantics.

Let $\mathcal{L}$ be a set of sentences for propositional logic that is closed under conjunction, disjunction, and negation. Let $>$ be a connective standing for "if ... then ...". The

[8] Levi has formulated an acceptance rule that is almost, but not exactly, equivalent to this one. But he confessed that he has no justification for it: "I do not know how to derive it from a view of the cognitive aims of inquiry [i.e. seeking more information and avoiding error] that seems attractive." (Levi 1996: 286) On the other hand, my justification for it in this paper is based on the conditional logic it validates. I thank Teddy Seidenfeld for bringing the prior publication of the rule to my attention. Actually, Kevin Kelly and I rediscovered the rule as a consequence of our work on Ockham's razor. The problem was to extend the Ockham efficiency theorem (Kelly 2008) from methods that choose theories to methods that update probabilistic degrees of belief on theories. That required a concept of retraction of credal states, expounded in (Kelly 2010).

language for the logic of *flat conditionals*, written $\mathcal{L}^{>}$, is the set of all sentences $\phi > \psi$ with $\phi, \psi \in \mathcal{L}$. Adams' (1975) logic of flat conditionals is just system P construed as a system of rules of inference, except that the symbol for consequence relation $\mid\!\sim$ should now be replaced by connective $>$. Say that γ is *derivable* from Γ in Adams' logic of flat conditionals, written $\Gamma \vdash_{\text{Adams}} \gamma$, if and only if γ is derivable from Γ in finite steps using the rules of inference in system P.

A *probabilistic model of acceptance* for language $\mathcal{L}^{>}$ is a triple:

$$M = (\alpha, p, [\![\cdot]\!]),$$

where $\alpha : \mathcal{P} \to \mathcal{A}$ is an acceptance rule, p is a probability measure in the domain $\mathcal{P}$ of α, and $[\![\cdot]\!]$ is a classical interpretation of $\mathcal{L}$ to the codomain $\mathcal{A}$ of α. When $M = (\alpha, p, [\![\cdot]\!])$, say that the *underlying* acceptance rule of M is α. Let $\phi > \psi$ be a flat conditional in $\mathcal{L}^{>}$. *Acceptance* of flat conditional $\phi > \psi$ in model $M = (\alpha, p, [\![\cdot]\!])$, written $M \Vdash \phi > \psi$, is defined by the Ramsey test:

$$M \Vdash \phi > \psi \quad \textit{iff} \quad [\![\phi]\!] \mid\!\sim_{\alpha,p} [\![\psi]\!], \tag{8}$$

$$\textit{iff} \quad \begin{cases} \text{either} & [\![\psi]\!] \text{ is accepted by rule } \alpha \text{ at state } p(\,\cdot\,|[\![\phi]\!]) \\ \text{or} & p([\![\phi]\!]) = 0. \end{cases} \tag{9}$$

Let Γ be a set of flat conditionals in $\mathcal{L}^{>}$. Acceptance of Γ in model M is defined by: $M \Vdash \Gamma$ if and only if $M \Vdash \gamma$ for all $\gamma \in \Gamma$. Validity is defined straightforwardly, as preservation of acceptance. Let $\mathcal{C}$ be a class of acceptance rules. Say that $\mathcal{C}$ *validates* the inference from Γ to γ, written $\Gamma \Vdash_{\mathcal{C}} \gamma$, if and only if for each probabilistic model M whose underlying acceptance rule is in $\mathcal{C}$, if $M \Vdash \Gamma$, then $M \Vdash \gamma$.

By way of comparison with other semantic approaches, note that Adams' (1975) ϵ-δ semantics defines validity in a less straightforward way: inference from Γ to γ is valid in that semantics if and only if, for each probability function P defined on the propositional language $\mathcal{L}$, and for each $\epsilon > 0$, there exists $\delta > 0$ such that if every element of Γ has conditional P-probability greater than $1 - \delta$, then γ has conditional P-probability greater than $1 - \epsilon$. The proposed semantics, in contrast, employs the notion of acceptance and defines validity as preservation of acceptance. Pearl's (1989) infinitesimal semantics can be understood as defining validity to be preservation of acceptance, but then it requires that a proposition be accepted only when its probability is one minus an infinitesimal, which is only infinitesimally better than outright skepticism. The proposed semantics allows for acceptance even of propositions of fairly low probability (e.g., use a symmetric camera shutter rule ν_r with a small r). There have been non-probabilistic semantics for flat conditionals (e.g., the ranked models in Lehmann and Magidor (1992)). But either they are recognized to be translatable into the infinitesimal semantics, or they are taken as essentially non-probabilistic and thus do not capture the Ramsey test, which Ramsey states in terms of probabilistic degrees of belief. The proposed semantics explicates the Ramsey test in terms of probabilistic conditioning and acceptance of conditional-free propositions, as embodied in definition (8).

The proposed probabilistic semantics is based on the Ramsey test for accepting conditionals, defines validity as preservation of acceptance, and allows acceptance under uncertainty. Furthermore, one can still prove soundness and completeness of Adams' conditional logic, if the acceptance rules in use are the camera shutter rules:

Theorem 3 (soundness and completeness). *Let $\mathcal{N}$ be the class of camera shutter rules. Then, for each finite sentence set Γ and each sentence γ in the language $\mathcal{L}^{>}$ of flat conditionals, we have:*

$$\Gamma \vdash_{\text{Adams}} \gamma \iff \Gamma \Vdash_{\mathcal{N}} \gamma.$$

6 Reasoning by Ignoring Less Normal Cases

This section develops concepts and results that facilitate the proofs of theorems 2 and 3 in the appendix, and they are of significance in their own right. In particular, the main theorem of this section says roughly the following: for each camera shutter rule ν, although the consequence relation $\mid\!\sim_{\nu,p}$ is defined in terms of probabilistic conditioning, $\mid\!\sim_{\nu,p}$ can be taken as a special case of "reasoning by ignoring less normal cases." Reasoning of the latter kind has been studied extensively in nonmonotonic logic, and one illuminating formalization of it is in terms of the so-called *preferential models*, which are well-known for satisfying system P (Kraus, Lehmann, and Magidor 1990). So we can easily prove that each $\mid\!\sim_{\nu,p}$ satisfies system P (theorem 2) once we can show that each $\mid\!\sim_{\nu,p}$ has a preferential model—as we shall do in this section.

A *normality order* $\prec$ is a strict partial order defined on a subset of partition $\mathcal{E} = \{E_i : i \in I\}$. For any cells E_i and E_j in the domain of $\prec$, say that E_i is *more $\prec$-normal* than E_j if and only if $E_i \prec E_j$. Cells not in the domain of $\prec$ are understood as too abnormal to worth comparison. Let A be a proposition in $\mathcal{A}$. Cell E_i is a *most $\prec$-normal* case in A if and only if (i) E_i classically entails A, (ii) E_i belongs to the domain of $\prec$, and (iii) no cell that satisfies the preceding two conditions is more $\prec$-normal than E_i. The consequence relation *modeled* by normality order $\prec$, written $\mid\!\sim_{\prec}$, is defined as follows: for each propositions A, B in $\mathcal{A}$,

$$A \mid\!\sim_{\prec} B \quad \textit{iff} \quad \bigvee\{E_i : E_i \text{ is a most } \prec\text{-normal case in } A\} \subseteq B.$$

The idea on the right hand side is that for all the cells that entail proposition A, we *ignore* the ones that fail to be most normal in A, and then see whether B is entailed by the disjunction of the remaining, most normal cases in A. This formalizes the idea of “reasoning by ignoring less normal cases.” A consequence relation is said to have a *normality model* if it is modeled by some normality order.[9]

Here is a recipe for constructing a normality model of $\mathrel{|\!\sim}_{\nu,p}$ *when* ν is a camera shutter rule.

Definition 3. *Let* (ν, p) *be a pair of camera shutter rule and credal state and suppose that* ν *is expressed by*

$$\nu(p) \;=\; \bigwedge \left\{ \neg E_i : \frac{p(E_i)}{\max_j p(E_j)} \lhd_i 1 - r_i \text{ and } i \in I \right\}.$$

The normality order $\prec_{\nu,p}$ *with respect to rule* ν *and state* p *is defined as follows: the domain is the set of cells that have nonzero probability with respect to* p*; for each cells* E_j *and* E_i *in the domain,*

$$E_j \prec_{\nu,p} E_i \quad \text{iff} \quad p(E_j) \;\rhd_i\; \tfrac{1}{1-r_i}\, p(E_i),$$

where it is stipulated that $\frac{1}{0} = \infty > x$ *for each real number* x.

For example, let $\nu = \nu_{0.9}$ be a symmetric camera shutter rule with threshold $r = 0.9$. Then:

$$\begin{aligned} E_j \prec_{\nu,p} E_i \quad &\iff \quad p(E_j) \;\geq\; \tfrac{1}{1-0.9}\, p(E_i) \\ &\iff \quad p(E_j) \;\geq\; 10\, p(E_i). \end{aligned}$$

The idea is that E_j is taken as more normal than E_i if the probability of the former is greater than that of the latter by the *factor* 10. That is, the probability ratio of a more normal case to a less normal case is at least 10.

Proposition 1. $\prec_{\nu,p}$ *is a strict partial order* (*and hence a normality order by definition*).

Here is the main theorem of this section:

Theorem 4. *Let* ν *be a camera shutter rule,* p *a credal state in* $\mathcal{P}$*. Then, consequence relations* $\mathrel{|\!\sim}_{\nu,p}$ *and* $\mathrel{|\!\sim}_{(\prec_{\nu,p})}$ *are the same, as expressed by the following commutative diagram:*

$$\begin{tikzcd}
(\nu, p) \arrow[r, mapsto, "\textit{reasoning by}", "\textit{probabilistic conditioning}"'] \arrow[d, mapsto] & \mathrel{|\!\sim}_{\nu,p} \arrow[d, equal] \\
\prec_{\nu,p} \arrow[r, mapsto, "\textit{reasoning by}", "\textit{ignoring less normal cases}"'] & \mathrel{|\!\sim}_{(\prec_{\nu,p})}
\end{tikzcd}$$

[9] Normality models defined here are equivalent to the preferential models (Kraus, Lehmann, and Magidor 1990) that do not have labeling of worlds.

So, if we use “right” acceptance rules, reasoning by probabilistic conditioning (the upper edge) can be decomposed into, or reconstructed as, two consecutive processes: first, using probability ratio to explicate what it is for a case to be more normal than another (the left edge), and second, reasoning by ignoring less normal cases (the lower edge).

7 Concluding Remark: Partition Dependence

The above theorems and the definition of camera shutter rules are all based on the assumption that acceptance of a proposition is always acceptance in the context of a question or partition of possible worlds. So there is no guarantee that a proposition accepted (not accepted) by a rule with respect to a partition will remain accepted (not accepted) in a refined partition. Are there acceptance rules that are defined absolutely, so that the acceptance of a proposition is independent of which partition the proposition is in, namely independent of which question is asked? Yes, there are, but only at a severe cost. The cost includes not only the loss of the desirable theorems we have proved, but also a trilemma: give up classical logic, or be skeptical (acceptance only with certainty), or be dogmatic (necessary acceptance of a contingency). We will define the terms and prove the trilemma. The trilemma is a very hard choice, but we can easily avoid it by allowing that acceptance is question-dependent. So perhaps context-dependence is not a brute, sociological fact, but a strategy to evade the trilemma. Actually, many proposed acceptance rules are question-dependent. The rules of Levi (1967, 1969, 1996) are explicitly so. The rules of Pollock (1995), Ryan (1996), and Douven (2002) are defined with the help of quantification over a chosen, underlying algebra.

Throughout this section, assume for simplicity that there are only three possible worlds, contained in $W = \{w_i : i = 1, 2, 3\}$. (The trilemma theorem stated below can be easily generalized to each countable cardinality greater than or equal to 3, but cardinality 3 suffices for explaining the trilemma.) Let $\wp(W)$ be the power set of W. When we say of a probability measure that it is defined *over a partition*, we mean that it is defined on the algebra of sets generated by that partition. Let $\mathcal{P}(W)$ be the set of all probability measures defined over a partition—ternary, binary, or unary—of W. A (*cross-question*) *acceptance relation* $\Vdash$ is a relation on $\mathcal{P}(W) \times \wp(W)$ such that (i) for each probability measure p in $\mathcal{P}(W)$ there exists some proposition A in $\wp(W)$ such that $p \Vdash A$, and that (ii) $p \Vdash A$ only if proposition A is in the domain of probability measure p. When $p \Vdash A$, we say that A is *accepted* at p with respect to $\Vdash$. Relation $\Vdash$ is *classical* if and only if for each probability measure p in $\mathcal{P}(W)$, the propositions accepted at p with respect to $\Vdash$ are closed under classical entailment. It is *anti-classical* if not classical. For probability measures p, q in $\mathcal{P}(W)$, say that p is *refined* by q if and only if $p = q\,|_{\mathrm{dom}(p)}$, namely p is the restriction of q to the domain of p. Relation $\Vdash$ is *refinement invariant* if and only if:

$$p \text{ is refined by } q \quad \Longrightarrow \quad (p \Vdash A \iff q \Vdash A),$$

for all probability measures p, q in $\mathcal{P}(W)$ and for all propositions A in the domain of p. Let E_i be the proposition $\{w_i\}$, for $i = 1, 2, 3$. Relation $\Vdash$ is *skeptical* if and only if for each E_i, that $p \Vdash E_i$ implies that $p(E_i) = 1$. Relation $\Vdash$ is *dogmatic* if and only if there exists E_i such that for each p in $\mathcal{P}(W)$, $p \Vdash E_i$. Then one can prove the following theorem, which is due to Kevin Kelly:

Theorem 5 (trilemma). *If acceptance relation $\Vdash$ is refinement invariant, then $\Vdash$ is either anti-classical, or skeptical, or dogmatic.*

Kyburg's acceptance rule (1961) is anti-classical. Van Fraassen's rule (1995) is skeptical. No one wants to be dogmatic. Lockean rules with a fixed threshold less than 1 do not face the trilemma, so they are not refinement invariant. I recommend giving up refinement invariance, both to avoid the trilemma and to enjoy the fruits yielded by the camera shutter rules.

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A Proof of Theorem 1

Proof of Theorem 1. Suppose that acceptance rule α satisfies corner-convexity and validates system P. Fix an index i in I. The acceptance zone of E_i under α is determined as follows. (In parallel to the general argument, the special case for $|\mathcal{E}| = 3$ and $i = 2$ is illustrated in figure 5.b).

Step 1. Let j be in $I \setminus \{i\}$. Define threshold t_{ij} in the interval $[0, \infty]$ and inequality $\lhd_{ij}$ as follows:

$$\begin{aligned} S_{ij} &= \left\{ \frac{p(E_j)}{p(E_i)} \; : \; p \Vdash_\alpha E_i \text{ and } p \in \overline{e_i e_j} \right\}; \\ t_{ij} &= \sup S_{ij}; \\ \lhd_{ij} &= \begin{cases} \leq & \text{if } t_{ij} \in S_{ij}, \\ < & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $e_i \Vdash_\alpha E_i$, by corner-convexity. So set S_{ij} is nonempty, containing at least 0. Then t_{ij} is well defined and, thus, $\lhd_{ij}$ is well defined.

Step 2. To see that condition 3 is satisfied, suppose that $\lhd_{ij} = <$. Then t_{ij} is not a member of s_{ij}, and hence $t_{ij} \neq 0$. So $t_{ij} > 0$. To see that condition 2 is satisfied, suppose for reductio that $\lhd_{ij} = \leq$ but $t_{ij} = \infty$. Then $\infty \in S_{ij}$, and hence $e_j \Vdash_\alpha E_i$. But $e_j \Vdash_\alpha E_j$, by corner-convexity. So $e_j \Vdash_\alpha \bot$—contradicting the everywhere consistency of α. The following two steps jointly show that condition 1 is also satisfied.

Step 3. Using axiom Or, argue as follows that for each $p \in \mathcal{P}$,

$$p \Vdash_\alpha E_i \iff \forall j \in I \setminus \{i\}, \; \frac{p(E_j)}{p(E_i)} \; \lhd_{ij} \; t_{ij}.$$

Suppose that p satisfies the right hand side. (Geometrically, p is in the blunt diamond in figure 5.b.) For each $j \in I \setminus \{j\}$, define $p^{(j)} = p(\cdot | E_i \vee E_j)$. Since conditioning on $E_i \vee E_j$ preserves the ratio of probability between E_i and E_j, and since conditioning on $E_i \vee E_j$ always results in a credal state on the line segment $\overline{e_i e_j}$, we have that $\frac{p^{(j)}(E_j)}{p^{(j)}(E_i)} = \frac{p(E_j)}{p(E_i)} \lhd_{ij} t_{ij}$, and that $p^{(j)} \in \overline{e_i e_j}$. Then $p^{(j)} \Vdash_\alpha E_i$, by the definitions in step 1 and corner-convexity. (Geometrically, that is because $p^{(j)}$ in figure 5.b falls inside the dark line segment on $\overline{e_2 e_j}$.) Then, since $p^{(j)} = p(\cdot | E_i \vee E_j)$, we have:

$$E_i \vee E_j \mathrel{|\!\!\sim}_{\alpha, p} E_i,$$

for each $j \in I \setminus \{i\}$. Since α validates axiom Or, we have:

$$\bigvee_{j \in I \setminus \{i\}} (E_i \vee E_j) \quad \mathrel{|\!\!\sim}_{\alpha, p} \; E_i.$$

The left hand side is the tautology $\top$ in the algebra $\mathcal{A}$, so $p(\cdot | \top) \Vdash_\alpha E_i$. But $p(\cdot | \top) = p$, so $p \Vdash_\alpha E_i$, as required.

Step 4. Using axiom *Cautious Monotonicity*, argue as follows that for each $q \in \mathcal{P}$,

$$q \Vdash_\alpha E_i \implies \forall j \in I \setminus \{i\}, \; \frac{q(E_j)}{q(E_i)} \; \lhd_{ij} \; t_{ij}.$$

Suppose that q satisfies the left hand side, i.e. $q \Vdash_\alpha E_i$. (q lies outside of the blunt diamond in figure 5.b.) Suppose for reductio that p does not satisfy the right hand side, so there exists $j \in I \setminus \{i\}$ such that $\frac{q(E_j)}{q(E_i)} \not\trianglelefteq_{ij} t_{ij}$. (In figure 5.b, it is assumed that $j = 3$ without loss of generality.) Let $q^{(j)} = q(\cdot|E_i \vee E_j)$. So $q^{(j)} \in \overline{e_i e_j}$ by definition. Since conditioning preserves probability ratio, $\frac{q^{(j)}(E_j)}{q^{(j)}(E_i)} = \frac{q(E_j)}{q(E_i)} \not\trianglelefteq_{ij} t_{ij}$. So $q^{(j)} \nVdash_\alpha E_i$, By the definitions in step 1. (Geometrically, that is because in figure 5.b, $q^{(3)}$ falls outside of the dark line segment lying on $\overline{e_2 e_3}$.) That is, $q(\cdot|E_i \vee E_j) \nVdash_\alpha E_i$. So, $(E_i \vee E_j) \not\mid\!\sim_{\alpha,q} E_i$. *But*, at the same time $(E_i \vee E_j) \mid\!\sim_{\alpha,q} E_i$. For $q \Vdash_\alpha E_i$ by assumption, and thus we have both that $\top \mid\!\sim_{\alpha,q} E_i$ and that $\top \mid\!\sim_{\alpha,q} E_i \vee E_j$. Thus $\top \wedge (E_i \vee E_j) \mid\!\sim_{\alpha,q} E_i$ by Cautious follows. It follows from Left Equivalence that $(E_i \vee E_j) \mid\!\sim_{\alpha,q} E_i$. □

B Proof of Theorems 2 and 3

Note: This appendix uses theorem 4, whose concepts are explained in section 6 and whose proof is given in appendix C.

Proof of Theorem 2. Theorem 4 says that $\mid\!\sim_{\nu,p}$ is modeled by the normality order $\prec_{\nu,p}$, so it has a normality model. Normality models are a special case of preferential models, as defined in Kraus, Lehmann, and Magidor (1990). Furthermore, any consequence relation that has a preferential model satisfies system P, by the easy half of the main theorem in Kraus, Lehmann, and Magidor (1990). So $\mid\!\sim_{\nu,p}$ satisfies system P. □

Proof of Theorem 3, Soundness ($\Rightarrow$). Consider, for example, the rule of inference Cautious Monotonicity: $\{\phi > \psi, \phi > \theta\} \vdash_{\text{Adams}} (\phi \wedge \psi) > \theta$. To show that it is validated by $\mathcal{N}$, let $M = (\nu, p, [\![\cdot]\!])$ be a model in $\mathcal{N}$, where ν is a camera shutter rule. Suppose that $M \Vdash \phi > \psi$ and $M \Vdash \phi > \theta$ (and we want to show that $M \Vdash \phi \wedge \psi > \theta$). Then, by definition, $[\![\phi]\!] \mid\!\sim_{\nu,p} [\![\psi]\!]$ and $[\![\phi]\!] \mid\!\sim_{\nu,p} [\![\theta]\!]$. Since ν is a camera shutter rule, by theorem 2 we have that $[\![\phi]\!] \wedge [\![\psi]\!] \mid\!\sim_{\nu,p} [\![\theta]\!]$, and thus $[\![\phi \wedge \psi]\!] \mid\!\sim_{\nu,p} [\![\theta]\!]$ (for $[\![\cdot]\!]$ satisfies the classical semantic rules). So $M \Vdash \phi \wedge \psi > \theta$, as desired. In general, each rule of inference in Adams conditional logic takes the form of a rule in system P, and hence is validated by $\mathcal{N}$ by the same argument. This completes the soundness proof. □

Proof of Theorem 3, Completeness ($\Leftarrow$). Let $\Gamma = \{\phi_1 > \psi_1, \ldots, \phi_n > \psi_n\}$, and $\gamma = \phi > \psi$. Suppose that $\Gamma \nvdash_{\text{Adams}} \gamma$ (and we want to construct a counter-model M whose underlying acceptance rule is in $\mathcal{N}$ such that $M \Vdash \Gamma$ but $M \nVdash \gamma$). Suppose that in $\Gamma \cup \{\gamma\}$ there are exactly m sentential letters $P_1, \ldots, P_m$. Consider a partition $\{E_i : i = 1, \cdots, 2^m\}$ with exactly 2^m cells, which we close under the Boolean operations to generate algebra $\mathcal{A}$. Interpret the sentential letters P_k by distinct propositions $[\![P_k]\!]$ in $\mathcal{A}$. Extend $[\![\cdot]\!]$ by the classical semantic rules. Then, by the completeness theorem for Adams' conditional logic in Lehmann and Magidor (1992), there exists a normality order $\prec$ on a subset of the partition $\{E_i : i = 1, \cdots, 2^m\}$, such that,

1. $\prec$ is *rankable* in the sense that there exists a map κ from the domain of $\prec$ to the set of natural numbers, such that $E_i \prec E_j$ if and only if $\kappa(E_i) < \kappa(E_j)$;

2. $[\![\phi_1]\!] \mid\!\sim_{\prec} [\![\psi_1]\!]$, $\ldots$, $[\![\phi_n]\!] \mid\!\sim_{\prec} [\![\psi_n]\!]$, but $[\![\phi]\!] \not\mid\!\sim_{\prec} [\![\psi]\!]$.

So, to find a counter-model, it suffices to find a model $M = (\nu, p, [\![\cdot]\!])$ in $\mathcal{N}$ such that $\mid\!\sim_{\nu,p} = \mid\!\sim_{\prec}$. This can be done as follows.

Fix a real number $r \in (0, 1]$, and let ν_r be the symmetric camera shutter with threshold r. Let p be the unique probability measure on $\mathcal{A}$ such that for any E_i not in the domain of $\prec$, $p(E_i) = 0$, and such that for each E_i and E_j in the domain of $\prec$,

$$p(E_i) \ : \ p(E_j) \quad = \quad \left(\frac{1}{1-r}\right)^{-\kappa(E_i)} : \left(\frac{1}{1-r}\right)^{-\kappa(E_j)} .$$

The pair (ν_r, p) is constructed to ensure that $\prec\, = \,\prec_{\nu_r,p}$:

$$\begin{aligned} E_i \prec E_j &\iff \kappa(E_i) < \kappa(E_j) \\ &\iff -\kappa(E_i) + \kappa(E_j) > 0 \\ &\iff -\kappa(E_i) + \kappa(E_j) \geq 1 \\ &\iff \frac{p(E_i)}{p(E_j)} = \left(\frac{1}{1-r}\right)^{-\kappa(E_i)+\kappa(E_j)} \geq \left(\frac{1}{1-r}\right) \quad \text{[note that } \tfrac{1}{1-r} > 1\text{]} \\ &\iff p(E_i) \geq \left(\frac{1}{1-r}\right) p(E_j) \\ &\iff E_i \ \prec_{\nu_r,p} \ E_j. \end{aligned}$$

Hence, $\mid\!\sim_{\prec} \, = \, \mid\!\sim_{(\prec_{\nu_r,p})}$. But $\mid\!\sim_{(\prec_{\nu_r,p})} \, = \, \mid\!\sim_{\nu_r,p}$ by theorem 4. So $\mid\!\sim_{\prec} \, = \, \mid\!\sim_{\nu_r,p}$ and therefore $(\nu_r, p, [\![\cdot]\!])$ is a desired counter-model. $\square$

C Proof of Theorem 4

Proof of Proposition 1. Let E_i, E_j, E_k be cells that have nonzero probability with respect to p. Note the following implication:

$$p(E_j) \ \rhd_i \ \tfrac{1}{1-r_i} p(E_i) \quad \Longrightarrow \quad p(E_j) > p(E_i),$$

which follows from the joint constraint on $\rhd_i$ and r_i in the definition of camera shutter rule: when $\rhd_i = \geq$, $r_i \in (0, 1]$ and hence $\frac{1}{1-r_i} \in (1, \infty]$; when $\rhd_i = >$, $r_i \in [0, 1)$ and hence $\frac{1}{1-r_i} \in [1, \infty)$. It is stipulated that $\frac{1}{0} = \infty > x$ for each real number x. That implication property is useful for showing that $\prec_{\nu,p}$ is a strict partial order; namely, irreflexive, asymmetric, and transitive. First, $\prec_{\nu,p}$ is irreflexive, because:

$$\begin{array}{rl} & E_i \prec_{\nu,p} E_i \\ \Rightarrow & p(E_i) \ \rhd_i \ \frac{1}{1-r_i} p(E_i) \\ \Rightarrow & p(E_i) > p(E_i) \\ \Rightarrow & \text{contradiction.} \end{array}$$

Also, $\prec_{\nu,p}$ is asymmetric, because:

$$\begin{array}{rl} & E_j \prec_{\nu,p} E_i \text{ and } E_i \prec_{\nu,p} E_j \\ \Rightarrow & p(E_j) \ \rhd_i \ \frac{1}{1-r_i} p(E_i) \text{ and } p(E_i) \ \rhd_j \ \frac{1}{1-r_j} p(E_j) \\ \Rightarrow & p(E_j) > p(E_i) \text{ and } p(E_i) > p(E_j) \\ \Rightarrow & \text{contradiction.} \end{array}$$

Last, $\prec_{\nu,p}$ is transitive, because:

$$\begin{array}{rl} & E_k \prec_{\nu,p} E_j \text{ and } E_j \prec_{\nu,p} E_i \\ \Rightarrow & p(E_k) \ \rhd_j \ \frac{1}{1-r_j} p(E_j) \text{ and } p(E_j) \ \rhd_i \ \frac{1}{1-r_i} p(E_i) \\ \Rightarrow & p(E_k) > p(E_j) \ \rhd_i \ \frac{1}{1-r_i} p(E_i) \\ \Rightarrow & p(E_k) \ \rhd_i \ \frac{1}{1-r_i} p(E_i) \\ \Rightarrow & E_k \prec_{\nu,p} E_i. \end{array}$$

So we are done. □

Lemma 1. *Let A be a proposition in $\mathcal{A}$, and suppose that $p(\cdot|A)$ is defined. Then the normality order $\prec_{\nu,p(\cdot|A)}$ is the restriction of $\prec_{\nu,p}$ to the domain of the former.*

Proof. Let cells E_i and E_j be in the domain of $\prec_{\nu,p(\cdot|A)}$. So $p(E_i|A) > 0$ and $p(E_j|A) > 0$, and thus both E_i and E_j entail A (because A is a disjunction of cells). So their probability ratio is preserved by conditioning on A: $\frac{p(E_j)}{p(E_i)} = \frac{p(E_j|A)}{p(E_i|A)}$. Then we have:

$$\begin{array}{rcl} E_j \prec_{\nu,p} E_i & \iff & p(E_j) \ \rhd_i \ \frac{1}{1-r_i} p(E_i) \\ & \iff & \frac{p(E_j)}{p(E_i)} \ \rhd_i \ \frac{1}{1-r_i} \\ & \iff & \frac{p(E_j|A)}{p(E_i|A)} \ \rhd_i \ \frac{1}{1-r_i} \\ & \iff & p(E_j|A) \ \rhd_i \ \frac{1}{1-r_i} p(E_i|A) \\ & \iff & E_j \prec_{\nu,p(\cdot|A)} E_i. \end{array}$$

This establishes the lemma. □

Lemma 2. *For each state p and each proposition A such that $p(\cdot|A)$ is defined, the following two statements are equivalent:*

- *E_i is a most $\prec_{\nu,p(\cdot|A)}$-normal case in $\top$;*
- *E_i is a most $\prec_{\nu,p}$-normal case in A.*

Proof. Immediate from the last lemma. $\square$

Lemma 3. *For each credal state p, we have:*

$$\nu(p) \ = \ \bigvee \{E_i : E_i \text{ is a most } \prec_{\nu,p}\text{-normal case in } \top\} \,.$$

Proof. Calculate as follows:

$$\begin{aligned}
\nu(p) \ &= \ \bigwedge \left\{\neg E_i : \tfrac{p(E_i)}{\max_j p(E_j)} \lhd_i 1 - r_i \text{ and } i \in I\right\} && \text{by definition;}\\
&= \ \bigvee \left\{E_i : \tfrac{p(E_i)}{\max_j p(E_j)} \not\lhd_i 1 - r_i \text{ and } i \in I\right\} && \text{by the algebra of sets;}\\
&= \ \bigvee \left\{E_i : \max_j p(E_j) \not\rhd_i \tfrac{1}{1-r_i} p(E_i) \text{ and } i \in I\right\} && \text{rearranging the inequalities;}\\
&= \ \bigvee \left\{E_i : \forall j \in I, \left[p(E_j) \not\rhd_i \tfrac{1}{1-r_i} p(E_i)\right] \text{ and } i \in I\right\} &&\\
&= \ \bigvee \{E_i : E_i \text{ is a most } \prec_{\nu,p}\text{-normal case in } \top\} && \text{by definition.}
\end{aligned}$$

$\square$

Proof of Theorem 4. Let A, B be propositions in $\mathcal{A}$. We want to show that $A \mid\!\sim_{\nu,p} B$ if and only if $A \mid\!\sim_{(\prec_{\nu,p})} B$. Consider two jointly cases.

Case 1: $p(\cdot|A)$ is defined. Then, by definition, $A \mid\!\sim_{\nu,p} B$ if and only if B is entailed by $\nu(p(\cdot|A))$. By definition, $A \mid\!\sim_{(\prec_{\nu,p})} B$ if and only if B is entailed by the disjunction $\bigvee \{E_i : E_i$ is a most $\prec_{\nu,p}$-normal case in $A\}$. So it suffices to prove the the following formula:

$$\nu(p(\cdot|A)) \ = \ \bigvee \{E_i : E_i \text{ is a most } \prec_{\nu,p}\text{-normal case in } A\} \,.$$

It can be derived as follows:

$$\begin{aligned}
\nu(p(\cdot|A) \ &= \ \bigvee \left\{E_i : E_i \text{ is a most } \prec_{\nu,p(\cdot|A)}\text{-normal case in } \top\right\} && \text{by lemma 3;}\\
&= \ \bigvee \{E_i : E_i \text{ is a most } \prec_{\nu,p}\text{-normal case in } A\} && \text{by lemma 2.}
\end{aligned}$$

Case 2: $p(\cdot|A)$ is undefined, or equivalently $p(A) = 0$. Then, $A \mid\!\sim_{\nu,p} B$ by default. So it suffices to show that $A \mid\!\sim_{(\prec_{\nu,p})} B$, or by definition, that:

$$\bigvee \{E_i : E_i \text{ is a most } \prec_{\nu,p}\text{-normal case in } A\} \ \subseteq \ B.$$

So it suffices to show that the the disjunction on the left hand side is the empty set $\varnothing$. Indeed it is, because every cell E_i in the domain of $\prec_{\nu,p}$ has positive probability with respect to p and thus does not entail A, which has zero probability with respect to p. $\square$

D Proof of Theorem 5

Let $\mathcal{P} \subset \mathcal{P}(W)$ be the set of all probability measures defined over the ternary partition $\{E_i : i = 1, 2, 3\}$. Relative to an arbitrary element p of $\mathcal{P}$, define the following line segments (figure 8): let $L_p^{(i)}$ be the set of probability measures in $\mathcal{P}$ that assigns the same probability to E_i as p does, for $i = 1, 2, 3$. So, as in figure 8, $L_p^{(i)}$ is the line through p parallel to the side opposite to corner e_i. For concreteness, proposition E_2 will be the focus in the lemmas below.

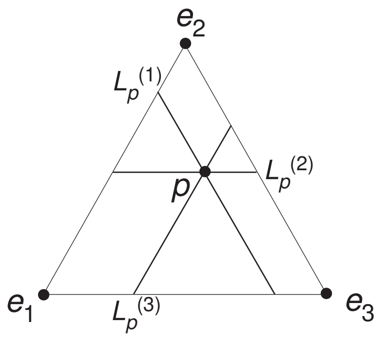

Figure 8:

Lemma 4. *Suppose that $\Vdash$ is classical and refinement invariant. For each p, q in $\mathcal{P}$, if $p \Vdash E_2$, then:*

$$\begin{aligned} q \in L_p^{(2)} \quad &\Rightarrow \quad q \Vdash E_2; \\ q \in L_p^{(i)} \quad &\Rightarrow \quad q \Vdash \neg E_i \quad \textit{for all } i \neq 2. \end{aligned}$$

Proof. Suppose $p \Vdash E_2$. To prove the first part, suppose further that q is in $L_p^{(2)}$. So q assigns the same probability to E_2 as p does. Then, the restrictions of p and q, respectively, to the algebra generated by binary partition $\{E_2, \neg E_2\}$ turn out to be the same probability measure, which we denote by p^-. So we have that p^- is refined both by p and by q. So, by refinement invariance, $p \Vdash E_2$ if and only if $p^- \Vdash E_2$ if and only if $q \Vdash E_2$. Since $p \Vdash E_2$, we have that $q \Vdash E_2$, as desired.

To prove the second, suppose that q is in $L_p^{(i)}$, $i \neq 2$. Then, consider the restrictions of p and q, respectively, to the algebra generated by binary partition $\{E_i, \neg E_i\}$, and argue as above that $p \Vdash \neg E_i$ if and only if $q \Vdash \neg E_i$. Since $p \Vdash E_2$, by classicality we have that $p \Vdash \neg E_i$, and by the biconditional we have that $q \Vdash \neg E_i$, as desired. □

A *golden triangle* for p relative to E_2 (figure 9.a) is a triangle included in $\mathcal{P}$ such that its three sides are parallel to the sides of the triangle $\mathcal{P}$, and that one of its side is part of line $L_p^{(2)}$, the "horizontal" line passing through p (figure 9). The *apex* of a golden triangle $\triangle$ for p is the vertex of $\triangle$ opposite the side of $\triangle$ incident to p.

Lemma 5. *Suppose that $\Vdash$ is classical and refinement invariant. Suppose further that p is in $\mathcal{P}$, $p(E_2) < 1$, and $p \Vdash E_2$. Let q be the apex of an arbitrary golden triangle for p. Then $q \Vdash E_2$.*

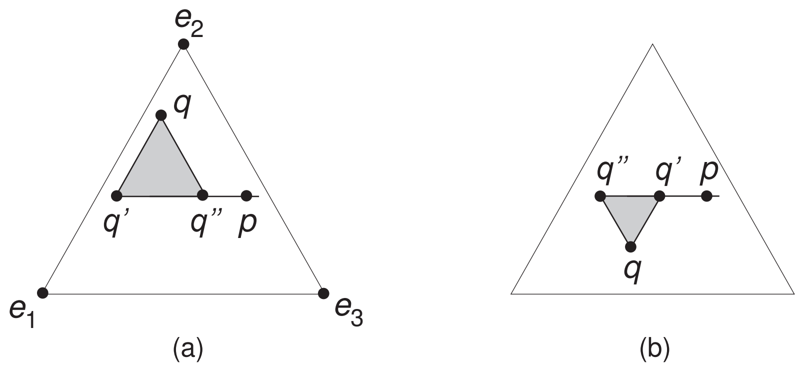

Figure 9: golden triangles, upward and downward

Proof. Let q be the apex of golden triangle $\triangle qq'q''$ for p, which may points "upward" or "downward" as depicted in figure 9.a or figure 9.b, respectively. The following argument applies to both cases. Since $p \Vdash E_2$, we have that $q' \Vdash E_2$ and $q'' \Vdash E_2$ (by applying lemma 4 to the fact that q' and q'' are on line $L_p^{(2)}$). Since E_2 classically entails both $\neg E_1$ and $\neg E_3$, we have that $q' \Vdash \neg E_3$ and $q'' \Vdash \neg E_1$ (by the classicality of $\Vdash$). It follows that $q \Vdash \neg E_3$ and $q \Vdash \neg E_1$ (by applying lemma 4 to the fact that q is on line $L_{q'}^{(3)}$ and on line $L_{q''}^{(1)}$). Then, by the classicality of $\Vdash$, $q \Vdash (\neg E_3 \wedge \neg E_1) = E_2$. $\square$

Proof of Theorem 5. Suppose that $\Vdash$ is classical, refinement invariant, and not skeptical. We want to show that $\Vdash$ is dogmatic. By non-skepticism, there exist $p \in \mathcal{P}(W)$ and E_i such that $p \Vdash E_i$ and $p(E_i) < 1$. By symmetry, suppose without loss of generality that $i = 2$. So:

$$p \Vdash E_2 \text{ and } p(E_2) < 1.$$

Also, suppose without loss of generality that p is in $\mathcal{P}$, namely defined over the ternary partition. (That is because, if p is defined over a coarser partition, then it is refined by a probability measure p' that is defined over the ternary partition, and then, by refinement invariance, $p' \Vdash E_2$ and $p'(E_2) < 1$). To show that $\Vdash$ is dogmatic, it suffices

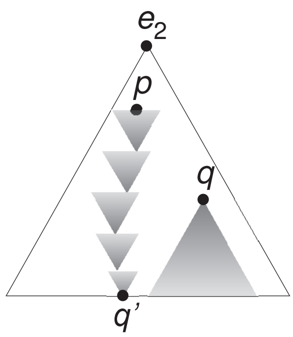

Figure 10: chain of golden triangles

to show that $q \Vdash E_2$ for all $q \in \mathcal{P}(W)$. Since every such q is refined by a probability measure in $\mathcal{P}$, by refinement invariance it suffices to show that $q \Vdash E_2$ for all $q \in \mathcal{P}$.

Let q be in $\mathcal{P}$, and we want to show that $q \Vdash E_2$. Construct a finite chain of (downward) golden triangle, $\triangle_0, \ldots, \triangle_i, \ldots, \triangle_n$, such that $\triangle_0$ is for p and $\triangle_{i+1}$ is for the apex of $\triangle_i$ for each i such that $0 \leq i < n$ (figure 10), until a golden triangle $\triangle_n$ is constructed with its apex q' lying on line $\overline{e_1 e_3}$. (That can always be done in finite steps, because $p \neq e_2$.) Then, construct an (upward) golden triangle $\triangle_{n+1}$ for q' whose apex is q. Note that $p \Vdash E_2$. So, by repeated applications of lemma 5 to the golden triangles $\triangle_0, \ldots, \triangle_n, \triangle_{n+1}$ in the order of construction, we have that $q \Vdash E_2$, as desired. $\square$