80-310/610 Logic and Computation

Exercise Set 6

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Exercise 1 (2.5.2.i) *Let $x \notin FV(\psi)$.*

$$\begin{aligned} \forall x\phi \to \psi &\approx \neg\forall x\phi \vee \psi \\ &\approx \exists x\neg\phi \vee \psi \\ &\approx \exists x(\neg\phi \vee \psi) \\ &\approx \exists x(\phi \to \psi). \end{aligned}$$

Exercise 2 (2.5.3.i) *Suppose that $x \in FV(\psi)$. Recall that by the definition of $\models$, free variables get interpreted as universally quantified variables. With that in mind, other solutions are possible. Observe that:*

$$\begin{aligned} \exists x(P(x) \to Q(x)) &\approx \exists x(\neg P(x) \vee Q(x)) \\ &\approx \exists x\neg P(x) \vee \exists x Q(x). \end{aligned}$$

Also observe that:

$$\begin{aligned} \forall x P(x) \to Q(x) &\approx \forall x(\forall x P(x) \to Q(x)) \\ &\approx \forall x P(x) \to \forall x Q(x) \end{aligned}$$

Now choose structure $\mathcal{A}$ with domain $\{a, b\}$ such that P holds of both a, b but Q holds of a but not of b. Then the first statement is true because $\exists x Q(x)$ is true but the second statement is false because $\forall x P(x)$ is true and $\forall x Q(x)$ is false.

Exercise 3 (2.5.4)

$$\not\models \forall x\exists y\phi \leftrightarrow \exists y\forall x\phi.$$

Again, many answers are possible. Bertrand Russell's example: maybe everybody loves someone, but it doesn't follow that somebody is loved by everyone. The hint was: every number is equal to some number, but there is no number every number is equal to.

Exercise 4 (2.5.5)

$$\begin{aligned} \models \phi(x) &\Rightarrow \models \forall x\phi(x) \\ &\Rightarrow \models \exists x\phi(x). \end{aligned}$$

The first implication is because free variables are interpreted as universally quantified by the definition of $\models$. The second implication follows from the first because relational structures have non-empty domains, so if everything satisfies $\phi(x)$ then something does.

Exercise 5 (2.5.6) *This is easy. Some number is zero but not every number is zero.*

Exercise 6 (2.5.7) *Some number is equal to zero and some number is the successor of zero but no number is both.*

Exercise 7 (2.5.9.i) *Suppose that* $\mathcal{A} \models \forall x(\phi \to \psi)$. *Then for each* $a \in |\mathcal{A}|$, $\mathcal{A} \models (\phi \to \psi)[a/x]$. *So* $\mathcal{A} \models (\phi[a/x] \to \psi[a/x])$. *Now suppose that* $\mathcal{A} \models \forall x\phi$. *Then for all* $b \in |\mathcal{A}|$, $\mathcal{A} \models \phi[b/x]$. *Let* $b \in |\mathcal{A}|$ *be arbitrary. Since* $\mathcal{A} \models \phi[b/x]$, *it follows that* $\mathcal{A} \models \psi[b/x]$. *Thus, for all* $b \in |\mathcal{A}|$, $\mathcal{A} \models \psi[b/x]$. *So* $\mathcal{A} \models \forall x\phi \to \forall x\psi$. *Hence,* $\mathcal{A} \models \forall x(\phi \to \psi) \to \forall x\phi \to \forall x\psi$.

Exercise 8 (2.5.14.a) *Prenex normal form exercise.*

$$\begin{aligned}
\neg((\neg\forall x\phi(x) \vee \forall x\psi(x)) \wedge (\exists x\sigma(x) \to \forall x\tau(x))) &\approx \neg((\neg\forall x\phi(x) \vee \forall x\psi(x)) \wedge (\neg\exists x\sigma(x) \vee \forall x\tau(x))) \\
&\approx \neg(\neg\forall x\phi(x) \vee \forall x\psi(x)) \vee \neg(\neg\exists x\sigma(x) \vee \forall x\tau(x)) \\
&\approx (\neg\neg\forall x\phi(x) \wedge \neg\forall x\psi(x)) \vee (\neg\neg\exists x\sigma(x) \wedge \neg\forall x\tau(x)) \\
&\approx (\forall x\phi(x) \wedge \exists x\neg\psi(x)) \vee (\exists x\sigma(x) \wedge \exists x\neg\tau(x)) \\
&\approx (\forall x\phi(x) \wedge \exists y\neg\psi(y)) \vee (\exists z\sigma(z) \wedge \exists w\neg\tau(w)) \\
&\approx \forall x\exists y\exists z\exists w((\forall x\phi(x) \wedge y\neg\psi(y)) \vee (\sigma(z) \wedge \neg\tau(w))).
\end{aligned}$$

Exercise 9 (bonus exercise 2.5.12) *Suppose that* $\mathcal{A} \models \exists y\forall x(S(y,x) \leftrightarrow \neg S(x,x))$. *Then for some* $a \in |\mathcal{A}|$ *it is the case that* $\mathcal{A} \models \forall x(S(a,x) \leftrightarrow \neg S(x,x))$. *So in particular,* $\mathcal{A} \models S(a,a) \leftrightarrow \neg S(a,a)$, *which contradicts the definition of* $\models$ *in the double arrow case. So by reductio ad absurdum,* $\mathcal{A} \not\models \exists y\forall x(S(y,x) \leftrightarrow \neg S(x,x))$. *So* $\mathcal{A} \models \neg(\exists y\forall x(S(y,x) \leftrightarrow \neg S(x,x))$. *Since* $\mathcal{A}$ *is arbitrary,* $\models \neg(\exists y\forall x(S(y,x) \leftrightarrow \neg S(x,x))$.