80-310/610 Logic and Computation
Exercise Set 4
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Exercise 1 *Van Dalen 1.5.2. The following are equivalent:*

1. $\{\phi_1, \ldots, \phi_n\} \nvdash \bot$;
2. $\nvdash \neg(\phi_1 \wedge \ldots \wedge \phi_n)$;
3. $\nvdash \phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_{n-1} \rightarrow \neg\phi_n$.

Answer: Do it by contraposition so you can get your hands on the derivations witnessing the derivability claims and move from claim to claim by modifying the derivation given. When you prove the contrapositive of a circuit of claims, it doesn't matter how you go around the circuit since every statement implies every other.

For $\neg 1 \Rightarrow \neg 2$, suppose $\{\phi_1, \ldots, \phi_n\} \vdash \bot$. Then there exists derivation

$$\phi_1, \ldots, \phi_n \mathcal{D} \bot.$$

So using the horizontal notation from the preceding exercise solutions, there exists derivation

$$[\phi_1, \ldots, \phi_n]\mathcal{D}\bot|((\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \bot),$$

whose last rule application is $\rightarrow I$. So $\vdash (\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \bot$, which is abbreviated as $(\phi_1 \wedge \ldots \wedge \phi_n)$.

Suppose $\vdash \neg(\phi_1 \wedge \ldots \wedge \phi_n)$. So $\vdash ((\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \bot$. So there exists derivation

$$[(\phi_1 \wedge \ldots \wedge \phi_n)]\mathcal{D}\bot|((\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \bot)$$

So, removing the last step, there exists derivation:

$$(\phi_1 \wedge \ldots \wedge \phi_n)\mathcal{D}\bot.$$

So by $\wedge$-I, there exists derivation

$$(\phi_1 \wedge \ldots \wedge \phi_{n-1}), \phi_n | \phi_1 \wedge \ldots \wedge \phi_n \mathcal{D}\bot.$$

So by $\rightarrow$I, there exists derivation:

$$(\phi_1 \wedge \ldots \wedge \phi_{n-1}), [\phi_n] | \phi_1 \wedge \ldots \wedge \phi_n \mathcal{D}\bot|(\phi_n \rightarrow \bot).$$

By $\rightarrow$I again, there exists derivation:

$$[(\phi_1 \wedge \ldots \wedge \phi_{n-1})], [\phi_n] | \phi_1 \wedge \ldots \wedge \phi_n \mathcal{D}\bot|(\phi_n \rightarrow \bot)|((\phi_1 \wedge \ldots \wedge \phi_{n-1}) \rightarrow (\phi_n \rightarrow \bot)).$$

The conclusion $((\phi_1 \wedge \ldots \wedge \phi_{n-1}) \rightarrow (\phi_n \rightarrow \bot))$ is abbreviated as $(\phi_1 \wedge \ldots \wedge \phi_{n-1}) \rightarrow \neg\phi_n$.
$\dashv$

Exercise 2 *Van Dalen 1.5.6. Show that if* Γ *is consistent and complete then* Γ *is maximally consistent.*

Answer: Suppose that $\Gamma \nvdash \perp$ and for all $\phi \in \text{PROP}$, $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$. By Lemma 1.5.8, Γ is closed under $\vdash$, so $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. Suppose that $\Gamma' \nvdash \perp$ and $\Gamma \subseteq \Gamma'$. It suffices to show that $\Gamma = \Gamma'$. Suppose for reductio ad absurdum that $\Gamma \neq \Gamma'$. Then since $\Gamma \subseteq \Gamma'$, there exists ϕ such that $\phi \in \Gamma' \setminus \Gamma$. By lemma 1.5.8 again, $\Gamma \nvdash \phi$. Since Γ is complete, $\Gamma \vdash \neg\phi$, so by lemma 1.5.8. $\neg\phi \in \Gamma$. Since $\Gamma \subseteq \Gamma'$, $\neg\phi \in \Gamma'$. So $\neg\phi, \phi \in \Gamma'$. Apply lemma 1.5.3.ii to conclude that Γ' is inconsistent. Contradiction. So $\Gamma = \Gamma'$. $\dashv$

Exercise 3 *Van Dalen 1.5.7. Show that* ATOM *is complete.*

Answer: By induction on PROP, using only connectives $\perp, \wedge, \rightarrow$. In the base case, for each atom p_i, ATOM $\vdash p_i$. Consider $\perp$. Since $\vdash \perp \rightarrow \perp$, it follows that $\Gamma \vdash (\perp \rightarrow \perp) = \neg\perp$. By IH, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$ and either $\Gamma \vdash \psi$ or $\Gamma \vdash \neg\psi$.

Consider $\phi \rightarrow \psi$. If $\Gamma \vdash \neg\phi$ then there exists derivation

$$\Gamma\mathcal{D}(\phi \rightarrow \perp),$$

so there exists derivation

$$[\phi], \Gamma\mathcal{D}(\phi \rightarrow \perp)|\perp|\psi|(\phi \rightarrow \psi),$$

by the $\rightarrow E$, $\rightarrow I$, and $\perp$ rules, so $\Gamma \vdash \phi \rightarrow \psi$. If $\Gamma \vdash \Psi$, then there exists derivation

$$\Gamma\mathcal{D}\psi,$$

so there exists derivation $\Gamma, [\phi]\mathcal{D}\psi|(\phi \rightarrow \psi)$. The only remaining case is that in which $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$. Then there exist derivations

$$\text{(i)}\Gamma\mathcal{D}\phi$$

and

$$\text{(ii)}\Gamma\mathcal{D}'(\psi \rightarrow \perp).$$

From (i), construct via $\rightarrow E$:

$$[\phi \rightarrow \psi], \Gamma\mathcal{D}'\phi|\psi.$$

Put this together with (ii) to obtain a derivation of $\perp$ by $\rightarrow E$ and then apply $\rightarrow I$ to derive $(\phi \rightarrow \psi) \rightarrow \perp$.

Finally, by a similar argument, one can construct a derivation of $\neg(\phi \wedge \psi)$ if either $\Gamma \vdash \neg\phi$ or $\Gamma \vdash \neg\psi$) (only one case has to be worked out because the two are symmetrical) and one can construct a derivation of $\phi \wedge \psi$ if $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$ using $\wedge$I. $\dashv$