80-310/610 Logic and Computation

Exercise Set 2

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Study van Dalen sections 1.2, 1.3.

Exercise 1 *Do van Dalen exercise 1.2.1 (b).*

Exercise 2 *Consider the conditional "if the assignment command $X := 6$ is executed then $X = 5$" and also the conditional "if the assignment command $X := 5$ is executed then $X = 5$". Show that the intuitive truth conditions for these conditionals are incompatible with any truth functional definition of $\rightarrow$. So truth-functional semantics (defining the truth value of a composite sentence in terms of the truth values of its components) is not sufficient to model the semantics of the conditional, in spite of van Dalen's apologies. You are invited, but not required, to speculate about the truth values of conditionals like the ones in the example.*

Solution. *The first conditional is false under every truth assignment. The second is true under every truth assignment except for the one in which the antecedent is true and the conclusion false. No fixed truth function can satisfy both constraints. The way to fix it is to interpret the conditional more generally in terms of operations on valuations: $v(\phi \rightarrow \psi) = 1$ if and only if $f(v, \phi)(\psi) = 1$, where f satisfies $f(v, \phi)(\phi) = 1$. This is the Stalnaker/Thomason semantics for conditionals.*

Exercise 3 *Do van Dalen exercise 1.2.2 For this you require the definition of $\models$ but not induction on PROP.*

Solution. *If $v(\phi) = 1$ then $v(\phi) = 1$, so $\phi \models \phi$. Suppose $\phi \models \psi$ and $\psi \models \sigma$. Then for all v, $v(\phi) = 1$ implies $v(\psi) = 1$ and $v(\psi) = 1$ implies $v(\sigma) = 1$. Then $v(\phi) = 1$ implies $v(\psi) = 1$. Hence, $\phi \models \sigma$. Finally, suppose that for each v, $v(\phi \rightarrow \psi) = 1$. Then for each v, $v(\phi) = 1$ implies $v(\psi) = 1$. So $\phi \models \psi$.*

Exercise 4 *The Sheffer stroke is defined as $\phi|\psi\neg(\phi \wedge \psi)$. Show that for each proposition there is a logically equivalent proposition whose only connectives are Sheffer strokes. Don't assume that $\wedge, \neg$ are expressively complete in this sense, so you will have to carry out the solution to van Dalen's exercise 7. So you may as well answer the observation about disjunctive normal form while you are at it.*

Solution. $\neg\phi \equiv \phi|\phi$, $\phi \wedge \psi \equiv (\phi|\phi)|(\psi|\psi)$. $\phi \vee \psi \equiv \neg(\neg\phi \wedge \neg\psi)$. *Now for an arbitrary proposition δ, write out the truth table for δ. For each row in which δ is true, conjoin the atoms true in the row and the negations of atoms false in the row and disjoin the resulting conjunctions. This (conjunctive normal form) proposition is true if and only if δ is and involves only connectives expressible in terms of Sheffer stroke.*

Exercise 5 *Do van Dalen exercise 1.3.1.*

Exercise 6 *Do van Dalen exercise 1.3.2.*

The algebraic approach to logic lends itself to explanatory pictures, which I encourage you to study at length on your own. Let the propositional atoms be just $\{p, q\}$. Now all the logical equivalence classes $[\phi]$ over these atoms can be viewed as nodes of the graph depicted in figure 1, where $\perp$ is conventionally at the bottom

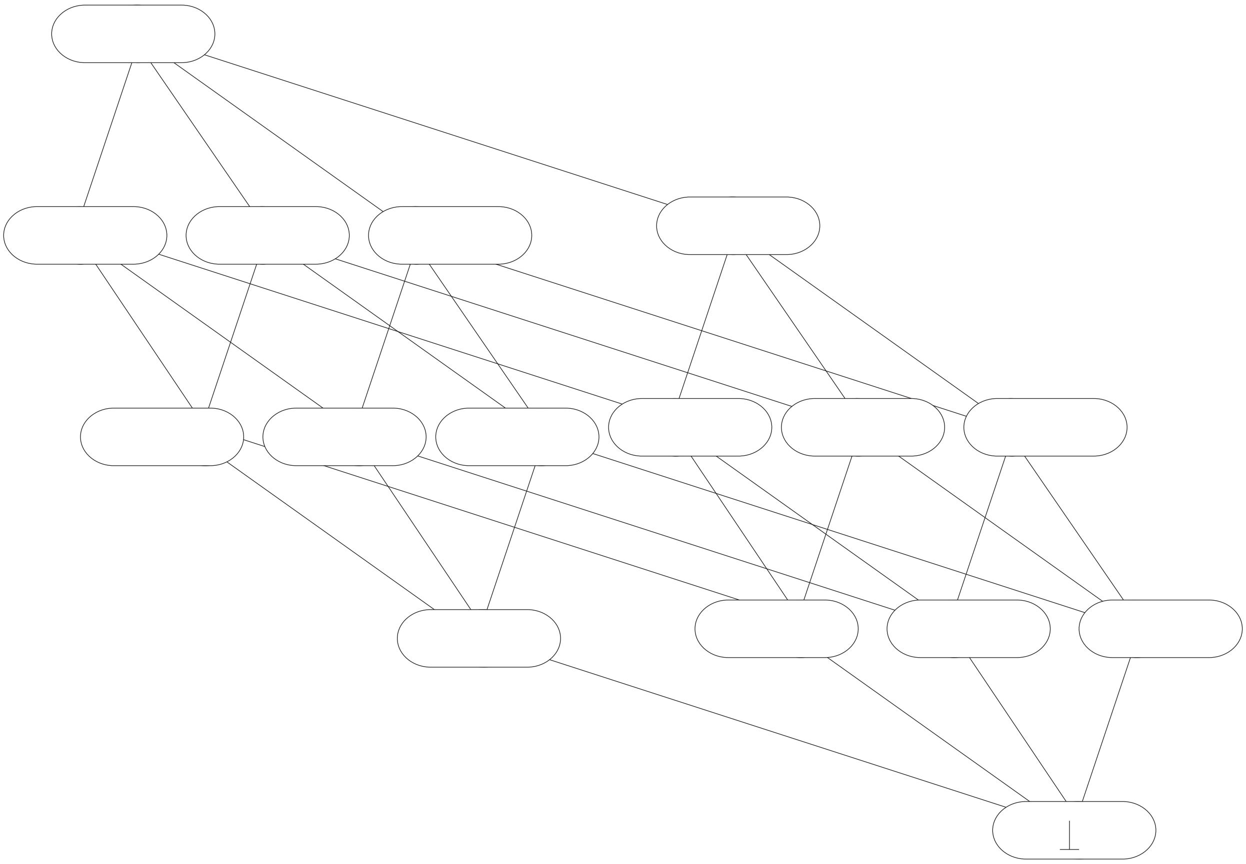

Figure 1: Boolean algebra

and each upward line corresponds to $\models$. This is called the Lindenbaum-Tarski algebra (cf. http://en.wikipedia.org/wiki/LindenbaumFor two propositional variables, the Lindenbaum-Tarski algebra is what is called the 4-element Boolean algebra or the 4-dimensional cube in graph theory. The four "elements" are not propositional variables (there are only two) but the four **atoms** or nodes immediately above $\perp$. Think of the atoms of a Boolean algebra as the "possible ways things can be"; in this case they correspond to assignments of the two propositional variables p, q. The least upper bound of a set of nodes is the lowest node connected to all of them from above and the greatest lower bound is the highest node connected to all of them from below. God can see the full, infinite picture for infinitely many propositional variables in the same way. We need computers, proofs, and logic teachers because our brains are not infinite. That generates a lot of redundant names for each vertex.

Exercise 7 *Label each node in the diagram with a shortest representative of the corresponding logical equivalence class* $[\phi]$ *over propositions involving only atoms* p, q *so that lines upward correspond to the entailment relation* $\phi \models \psi$.

1. *What truth function do least upper bounds correspond to in the diagram?*
2. *What truth function do greatest lower bounds correspond to in the diagram?*
3. *What symmetries (reflections about an axis) does duality correspond to in the diagram?*
4. *How can you read the disjunctive normal form for* ϕ *off of the diagram?*
5. *What does rank (height) in the diagram correspond to?*
6. *The diagram for zero atomic propositions consists of two nodes,* $\perp$ *and* $\top$ *with no propositions in between. A single atomic proposition* p *leads to a square and two atomic propositions yield the 4-dimensional cube. That leaves out the 3-cube. Give a logical interpretation of the 3-cube.*
7. *How many nodes are there in the algebra if there are* n *atomic sentences?*

Solution.

Conjunction.

Disjunction.

Negation flips both horizontally and vertically, so disjunction exchanges with conjunction.

The disjunction of all the algebra atoms below the proposition for which DNF is taken.

Rank is the number of lines in a truth table for p, q *in which the propositions in the equivalence class are true.*

The lattice of all propositions entailed by any proposition of the third rank (e.g., $(p \vee q)$*).*

There are 2^n *atoms (each atomic proposition negated or non-negated) and then each proposition corresponds to the disjunction of some of some set of atoms, there being* 2^{2^n} *such sets.*

Exercise 8 *Do van Dalen's exercise 1.3.14.ii. Remember that there are infinitely many propositional variables. Note that this implies that the Lindenbaum-Tarski algebra for infinitely many propositional variables is very different than for any finite number of propositional variables. How? How many atoms does the Lindenbaum-Tarski algebra have when there are infinitely many propositional variables?*

Answer. *Set* $\sigma = \phi \vee (\psi \wedge q)$, *where* q *does not occur in* ϕ *or* ψ.

The entailment ordering is dense, so there is no rank.

There are no atoms, since there is always another interpolating proposition between ϕ *and* $\perp$.

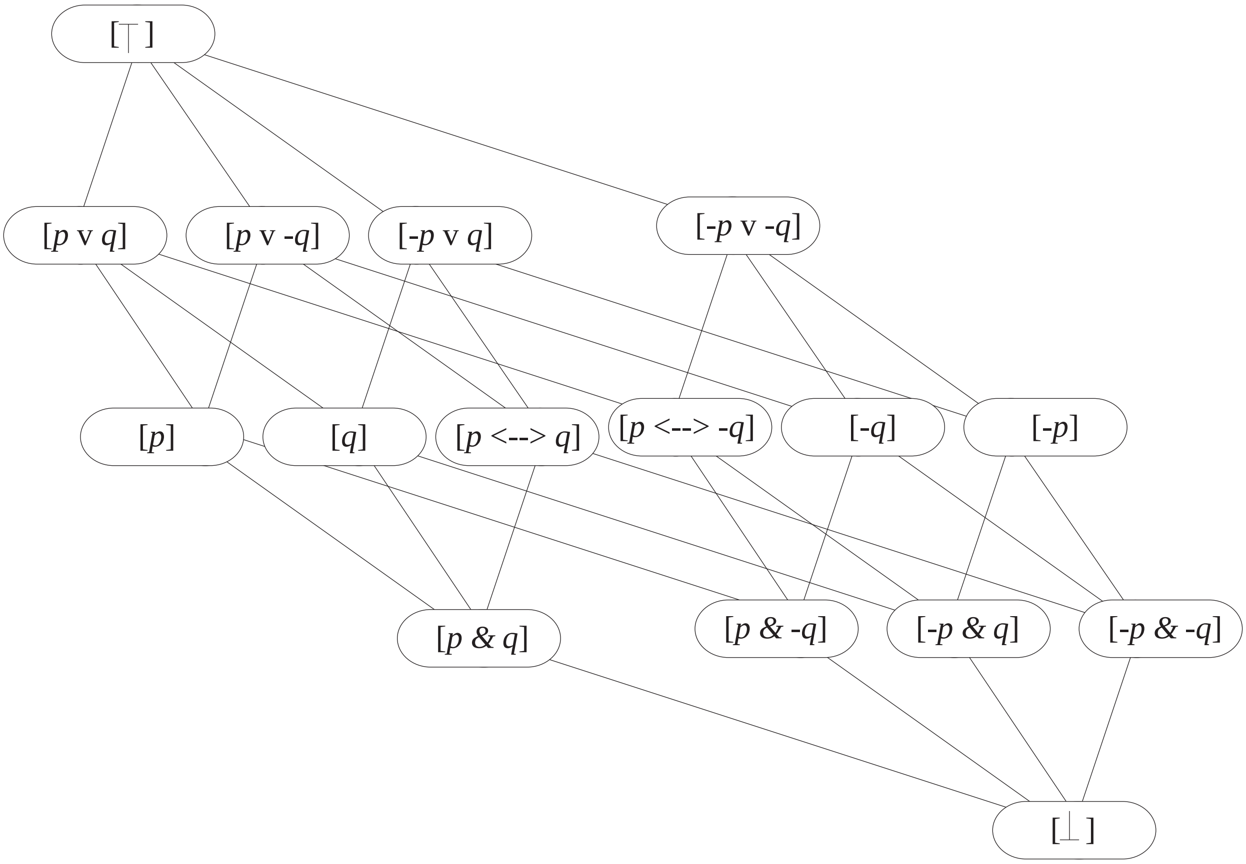

Figure 2: Solution