80-310/610 Logic and Computation

Exercise Set 2

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Study van Dalen sections 1.2, 1.3. Keep in mind that you will be called upon to know any of the tautologies in section 1.3 from now on, even if they aren't involved in these exercises.

Exercise 1 *Do van Dalen exercise 1.2.1 (b).*

Exercise 2 *Consider the conditional "if the assignment command* $X := 6$ *is executed then* $X = 5$*" and also the conditional "if the assignment command* $X := 5$ *is executed then* $X = 5$*". Show that the intuitive truth conditions for these conditionals are incompatible with any truth functional definition of* $\rightarrow$*. So truth-functional semantics (defining the truth value of a composite sentence in terms of the truth values of its components) is not sufficient to model the semantics of the conditional, in spite of van Dalen's apologies. You are invited, but not required, to speculate about the truth values of conditionals like the ones in the example.*

Exercise 3 *Do van Dalen exercise 1.2.2 For this you require the definition of* $\models$ *but not induction on PROP.*

Exercise 4 *The Sheffer stroke is defined as* $\phi|\psi \equiv \neg(\phi \wedge \psi)$*. Show that for each proposition there is a logically equivalent proposition whose only connectives are Sheffer strokes. Don't assume that* $\wedge, \neg$ *are expressively complete in this sense, so you will have to carry out the solution to van Dalen's exercise 7. So you may as well answer the observation about disjunctive normal form while you are at it.*

Exercise 5 *Do van Dalen exercise 1.3.1.*

Exercise 6 *Do van Dalen exercise 1.3.2.*

The algebraic approach to logic lends itself to explanatory pictures, which I encourage you to study at length on your own. Let the propositional atoms be just $\{p, q\}$. Now all the logical equivalence classes $[\phi]$ over these atoms can be viewed as nodes of the graph depicted in figure 1, where $\perp$ is conventionally at the bottom and each upward line corresponds to $\models$. This is called the Lindenbaum-Tarski algebra. For two propositional variables, the Lindenbaum-Tarski algebra is what is called the 4-element Boolean algebra or the 4-dimensional cube in graph theory. The four "elements" are not propositional variables (there are only two) but the four **atoms** or nodes immediately above $\perp$. Think of the atoms of a Boolean algebra as the "possible ways things can be"; in this case they correspond to assignments of the two propositional variables p, q. The least upper bound of a set of nodes is the lowest node connected to all of them from above and the greatest lower bound is the highest node connected to all of them from below. God can see the full, infinite picture for infinitely many propositional variables in the same way. We need computers, proofs, and logic teachers because our brains are not infinite. That generates a lot of redundant names for each vertex.

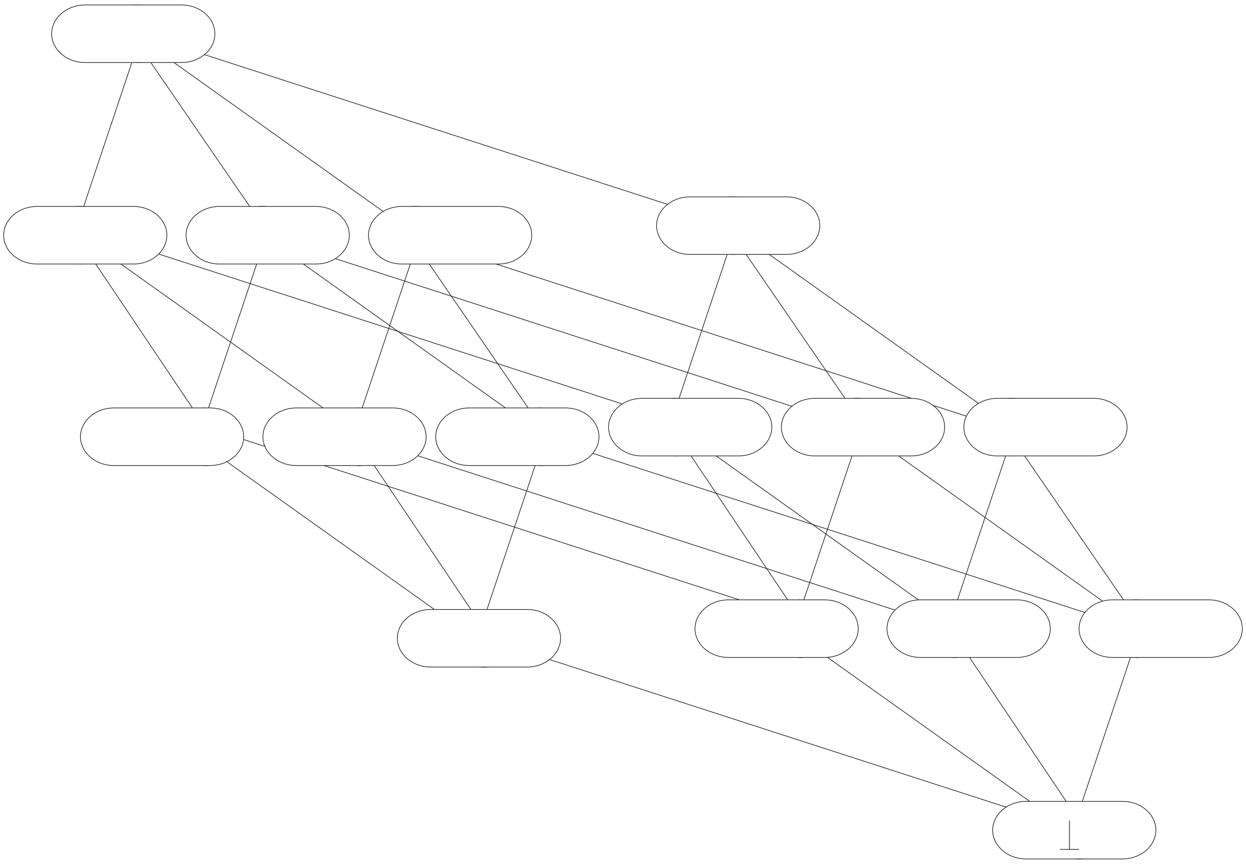

Figure 1: Boolean algebra

Exercise 7 *Label each node in the diagram with a shortest representative of the corresponding logical equivalence class $[\phi]$ over propositions involving only atoms p, q so that lines upward correspond to the entailment relation $\phi \models \psi$.*

1. *What truth function do least upper bounds correspond to in the diagram?*
2. *What truth function do greatest lower bounds correspond to in the diagram?*
3. *What symmetries (reflections about an axis) does duality correspond to in the diagram?*
4. *How can you read the disjunctive normal form for ϕ off of the diagram?*
5. *What does rank (height) in the diagram correspond to?*
6. *The diagram for zero atomic propositions consists of two nodes, $\perp$ and $\top$ with no propositions in between. A single atomic proposition p leads to a square and two atomic propositions yield the 4-dimensional cube. That leaves out the 3-cube. Give a logical interpretation of the 3-cube.*

7. *How many nodes are there in the algebra if there are n atomic sentences?*

Exercise 8 *Do van Dalen's exercise 1.3.14.ii. Remember that there are infinitely many propositional variables. Note that this implies that the Lindenbaum-Tarski algebra for infinitely many propositional variables is very different than for any finite number of propositional variables. How? How many atoms does the Lindenbaum-Tarski algebra have when there are infinitely many propositional variables?*