Recitation Notes
Spring 16, 21-241: Matrices and Linear Transformations
March 22, 2016

Abstract

1 Administrative Matters
2 Definitions / Notation
   1. Dimension
   2. Rank
3 Problems
   1. Let
      \[
      A = \begin{bmatrix}
      1 & 1 & 0 & 1 \\
      0 & 1 & -1 & 1 \\
      0 & 1 & -1 & -1 \\
      \end{bmatrix}
      \]
      (a) David Poole \textit{Linear Algebra: a modern introduction} (4th Ed.) Ex 3.5.17. Give a basis for \( \mathcal{R}(A) \) and \( \mathcal{C}(A) \).
      (b) David Poole \textit{Linear Algebra: a modern introduction} (4th Ed.) Ex 3.5.23. Find bases for \( \mathcal{R}(A) \) and \( \mathcal{C}(A) \) using \( A^T \) this time.

      \[
      \begin{bmatrix}
      1 & 1 & 0 & 1 \\
      0 & 1 & -1 & 1 \\
      0 & 1 & -1 & -1 \\
      \end{bmatrix}
      \rightarrow
      \begin{bmatrix}
      1 & 1 & 0 & 1 \\
      0 & 1 & -1 & 1 \\
      0 & 0 & 0 & -2 \\
      \end{bmatrix}
      \]
      Thus
      \[
      \mathcal{R}(A) \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\}
      \]
where we simply read off the nonzero rows of the reduced matrix. Also, 

$$C(A) \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

where we find the columns in $A$ corresponding to the pivot columns in the reduced matrix. 

(b) We shall row reduce $A^T$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_4 \rightarrow R_4 - R_1, R_2 \rightarrow R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1, R_4 \rightarrow R_4 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\text{some row swaps}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we read off the rows to get a basis of

$$\mathcal{R}(A^T) = C(A) \text{ to be } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\},$$

and the columns in $A^T$ corresponding to the pivot columns in the reduced matrix form a basis for

$$C(A^T) = \mathcal{R}(A) \text{ to be } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$ 

*Note: Although the basis derived in parts (a) and (b) are different, they span the same space.*

2. David Poole *Linear Algebra: a modern introduction* (4th Ed.) Ex 3.5.33. Prove that if $R$ is a matrix in echelon form, then a basis for $\mathcal{R}(A)$ consists of the nonzero rows of $R$.

**Solution.** Let $r_1, \ldots, r_k$ be the nonzero rows of $R$ starting from the 1st row to the $k$-th row.

**Step 1:** We want to show that

$$\mathcal{R}(R) = \text{span}\{r_1, \ldots, r_k\}$$
For any vector \( v \in \mathcal{R}(R) \), we may write
\[
v = c_1 r_1 + c_2 r_2 + \cdots + c_k r_k + c_{k+1} 0 + \cdots + c_m 0.
\]
Then
\[
v = c_1 r_1 + c_2 r_2 + \cdots + c_k r_k,
\]
so
\[v \in \text{span}\{r_1, \ldots, r_k\}.
\]
Thus
\[
\mathcal{R}(R) \subseteq \text{span}\{r_1, \ldots, r_k\}.
\]
But trivially,
\[
\text{span}\{r_1, \ldots, r_k\} \subseteq \text{span}\{r_1, \ldots, r_k, 0, \ldots, 0\} = \mathcal{R}(R),
\]
so
\[
\text{span}\{r_1, \ldots, r_k\} = \mathcal{R}(R).
\]

**Step 2:** We want to show that \( r_1, \ldots, r_k \) are linearly independent. Suppose otherwise, then we can write
\[
c_1 r_{i_1} + \cdots + c_\ell r_{i_\ell} = 0
\]
where \( c_1, \ldots, c_\ell \neq 0 \) and \( \{r_{ij}\}_{1 \leq j \leq \ell} \subseteq \{r_j\}_{1 \leq j \leq k} \). Let the leading entry of row \( r_{i_1} \) be in the \( j \)-th column. Then
\[
c_1 r_{i_1,j} + \cdots + c_\ell r_{i_\ell,j} = 0
\]
where we look only at the \( j \)-th column. However, since all entries below the pivot entry are 0, we just have
\[
c_1 r_{i_1,j} + 0 + \cdots + 0 = 0,
\]
which is a contradiction since \( c_1 \neq 0 \) and \( r_{i_1,j} \neq 0 \). Thus it must be that \( r_1, \ldots, r_k \) are linearly independent.

From Steps 1 and 2, we have proven that the nonzero rows of \( R \) form a basis for the row space.

3. David Poole *Linear Algebra: a modern introduction* (4th Ed.) Ex 3.5.59a. Prove that \( \text{rank}(AB) \leq \text{rank}(B) \).

**Solution.** Let \( v_1, \ldots, v_n \) be the rows of \( B \). We shall show that the row space of \( AB \) is a subspace of the rowspace of \( B \).

\[
AB = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad B = \begin{bmatrix} a_1^T B \\ \vdots \\ a_m^T B \end{bmatrix}
\]
Now observe that in the $i$-th row,

$$a_i^T B = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} = a_{i1}b_1^T + \cdots + a_{in}b_n^T$$

Since every vector in $\mathcal{R}(AB)$ is a linear combination of the rows of $AB$, which in turn are linear combinations of the rows of $B$, we conclude that $\mathcal{R}(AB)$ is a linear combination of the rows of $B$. (Rigorous proof in previous recitation.) Thus for every $v \in \mathcal{R}(AB)$,

$$v \in \text{span}(b_1, \ldots, b_n) = \mathcal{R}(B).$$

Thus

$$\mathcal{R}(AB) \leq \mathcal{R}(B).$$

Let $B$ be a basis for $\mathcal{R}(AB)$ and let $\text{rank}(AB) = k$. Then $B$ is a linearly independent set in $\mathcal{R}(B)$. Thus any basis for $\mathcal{R}(B)$ must have $\geq k$ vectors. Thus $\text{rank}(B) \geq k$.

$$\therefore \text{rank}(AB) \leq \text{rank}(B).$$

4. David Poole *Linear Algebra: a modern introduction* (4th Ed.) Ex 3.5.64. Prove that, for $m \times n$ matrices $A$ and $B$,

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

**Solution.** Let $k := \text{rank}(A)$ and $\ell := \text{rank}(B)$. Let $\{v_1, \ldots, v_k\}$ be a basis for $\mathcal{R}(A)$ and $\{u_1, \ldots, u_\ell\}$ be a basis for $\mathcal{R}(B)$. We shall show that

$$\mathcal{R}(A + B) \leq \text{span}\{v_1, \ldots, v_k, u_1, \ldots, u_\ell\}.$$ 

Consider specifically row $i$. We may write

$$a_i = \sum_{\alpha=1}^{k} c_\alpha v_\alpha \text{ and } b_i = \sum_{\beta=1}^{\ell} d_\beta u_\beta.$$ 

Thus we may write the $i$-th row of $A + B$ as

$$a_i + b_i = \sum_{\alpha=1}^{k} c_\alpha v_\alpha + \sum_{\beta=1}^{\ell} d_\beta u_\beta.$$ 

Now since every row of $A + B$ can be written as a linear combination of $v_1, \ldots, v_k, u_1, \ldots, u_\ell$, thus every linear combination of the rows of $A + B$ can be written as a linear combination of $v_1, \ldots, v_k, u_1, \ldots, u_\ell$ (Rigorous proof in previous recitation.) Thus

$$\mathcal{R}(A + B) \leq \text{span}\{v_1, \ldots, v_k, u_1, \ldots, u_\ell\} \Rightarrow \text{rank}(A + B) \leq k + \ell.$$ 

$$\therefore \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$
4 Additional Notes

1. To prove that $\dim V \leq \dim U$ where $U$ and $V$ are vector spaces, you need to show that the number of vectors in a basis of $V$ is less than or equal the number of vectors in a basis of $U$.

2. The column space of matrix $A$ consists of all vectors of the form $Ax$, whereas the row space of $A$ consists of all vectors of the form $y^TA$.

5 Exercises

1. (a) Let $v_1, \ldots, v_n$ be linearly independent vectors. Find a counterexample to the following statement:

   Then we can express $v_1$ as a linear combination of the other vectors $v_2, \ldots, v_n$, i.e.

   $$v_1 = c_2v_2 + \cdots + c_nv_n.$$  

   (b) Why is this exercise important?