Recitation Notes

Spring 16, 21-241: Matrices and Linear Transformations

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Abstract


1 Administrative Matters

• Change in Homework Problem 2: $e_i A$ does not make sense. This has been rectified to $e_i^T A$.

• In addition, $e_i$ should be in $\mathbb{R}^m$ instead of $\mathbb{R}^n$. (Do people see why?)

2 Definitions / Notation

1. Matrix transpose
2. Matrix addition
3. Matrix multiplication
4. Linear combination
5. Standard basis vectors in $\mathbb{R}^n$
6. $[ \cdot ]_{i,j}$ refers to a matrix expressed in terms of its entries
7. $u_i^T$ with respect to the matrix $U$ refers to the $i$-th row of $U$
8. $u_j$ with respect to the matrix $U$ refers to the $j$-th column of $U$

3 Problems

1. (a) What are the properties of matrix addition and scalar multiplication where the entries are in $\mathbb{R}$? (Think about the axioms on real addition and multiplication)

   **Solution.** Addition is associative, commutative, distributive. Scalar multiplication likewise.
(b) What important property does matrix multiplication NOT possess?

Solution. Commutativity.
Note: Matrix multiplication is associative!

2. What is the difference between \( u \cdot v \) and \( u^T v \), where \( u, v \) are column vectors in \( \mathbb{R}^n \)?

Solution. \( u \cdot v \) is a real number, whereas \( u^T v \) is a \( 1 \times 1 \) matrix.
Note: The entry in the matrix is the same as \( u \cdot v \).

3. Let

\[
A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

Use the row-matrix representation of the product to write each row of \( BA \) as a linear combination of the rows of \( A \).

We shall do something more general. Let

\[
B = \begin{bmatrix} b_1^T \\ \vdots \\ b_m^T \end{bmatrix}
\]

and let

\[
A = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}.
\]

Using the row-matrix representation, we have

\[
BA = \begin{bmatrix} b_1^T \\ \vdots \\ b_m^T \end{bmatrix} A = \begin{bmatrix} b_1^T A \\ \vdots \\ b_m^T A \end{bmatrix}
\]

Thus the \( i \)-th row of \( BA \) is \( b_i^T A \). Let

\[
b_i^T = \begin{bmatrix} b_{i1} & \ldots & b_{in} \end{bmatrix}
\]

Thus

\[
b_i^T A = \begin{bmatrix} b_{i1} & \ldots & b_{in} \end{bmatrix} \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} = b_{i1} a_1^T + \ldots + b_{in} a_n^T.
\]

Having this more general result, we are in good position to state the rows of \( BA \) as a linear combination of the rows of \( A \).

\[
BA_1^T = 1 \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} + 2 \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}
\]

\[
BA_2^T = 3 \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} + 4 \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}
\]
4. (a) Let $A$ be a $3 \times 2$ matrix with $A = [i + 2j]_{i,j}$

Write out $A$ explicitly.

Solution.

A = \[
\begin{bmatrix}
1 + 2 \cdot 1 & 1 + 2 \cdot 2 \\
2 + 2 \cdot 1 & 2 + 2 \cdot 2 \\
3 + 2 \cdot 1 & 3 + 2 \cdot 2
\end{bmatrix}
= \[
\begin{bmatrix}
3 & 5 \\
4 & 6 \\
5 & 7
\end{bmatrix}
\]

(b) Let $n \geq 5$. Let $B$ be an $n \times n$ matrix and the $ij$-th entry of $B$ be $b_{ij}$. Let $C$ be an $n \times n$ matrix defined based on $B$. Let $k \in \mathbb{N}$, $1 \leq k \leq n$ be a constant. What is $C^T$ given $C = [b_{5i} + b_{ij} + b_{4j}]_{i,j}$

expressed in $[\cdot]_{i,j}$ notation and in terms of $b_{ij}$? In terms of $B$ and its rows and columns?

Solution.

$C^T = [b_{5i} + b_{ij} + b_{4j}]^T = [b_{5j} + b_{ji} + b_{4i}]_{i,j}$

Note: we only swap the $i$ and $j$. Transposition is NOT about swapping the row and column indices.

\[
C^T = [b_{5j} + b_{ji} + b_{4i}]_{i,j} = [b_{5j}]_{i,j} + [b_{ji}]_{i,j} + [b_{4i}]_{i,j}
\]

\[
= \begin{bmatrix}
(b_5)^T \\
\vdots \\
(b_5)^T
\end{bmatrix} + B^T + \begin{bmatrix}
(b_1)^T \\
\cdots \\
(b_4)^T
\end{bmatrix}
\]

(c) David Poole Linear Algebra: a modern introduction (4th Ed.) Ex 3.2.21. Let $D$ be an $m \times n$ matrix. Show that $I_m A = A$.

Solution. We have

\[
I_m = [e_{ij}]_{i,j} \quad \text{where} \quad e_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\]

\[
I_m A = \sum_{k=1}^{m} e_{ik} a_{kj}
\]

\[
= \sum_{k=1}^{m} \begin{bmatrix} 1 & i = k \\ 0 & i \neq k \end{bmatrix} a_{kj}
\]

\[
= [a_{ij}]_{i,j}
\]

\[
= A
\]
5. Let $e_j$ be the $j$-th standard basis vector in $R^n$. Show that $Ae_j$ is the $j$-th column of $A$.

**Solution.** Let

$$ e_j = [e_k]_{i,j} \quad \text{where} \quad e_k = \begin{cases} 1 & k = j \\ 0 & i \neq j \end{cases} $$

Then using the column-row representation, we have

$$ Ae_j = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} e_j = \sum_{k=1}^{n} a_ke_k = a_j. $$

6. Let $A$ and $B$ be matrices such that $AB$ is a well-defined matrix multiplication.

(a) Is it always the case that we can perform the matrix multiplication $BA$?

**Solution.** No, not always. We do not know that the number of rows of $A$ is the same as the number of columns of $B$. That is required to perform the multiplication $BA$.

(b) Suppose $BA$ is well-defined matrix multiplications. What is the most precise thing we can say about the shapes of $AB$ and $BA$?

**Solution.** We have that $A$ is of size $m \times n$ and $B$ is of size $n \times m$. Thus $AB$ is of size $m \times m$ and $BA$ is of size $n \times n$. We know that both $AB$ and $BA$ are square matrices.

7. (a) Prove that the product of two $n \times n$ upper triangular matrices is upper triangular.

**Solution.** Let $A$ and $B$ be upper triangular. Then we have

$$ [a_{ij}]_{i,j} = \begin{cases} 0 & i > j \\ a_{ij} & \text{otherwise} \end{cases} \quad \text{and} \quad [b_{ij}]_{i,j} = \begin{cases} 0 & i > j \\ b_{ij} & \text{otherwise} \end{cases} $$

Now

$$ [AB]_{i,j} = \sum_{k=1}^{n} a_{ik}b_{kj} $$

$$ = \sum_{k=1}^{j} a_{ik}b_{kj} \quad \text{by looking at } b_{kj} $$

$$ = \begin{cases} 0 & i > j \\ \sum_{k=i}^{j} a_{ik}b_{kj} \end{cases} $$

Thus $AB$ is upper triangular.
(b) Let \( \{ A_m \}_{m \in \mathbb{N}} \) be a sequence of \( n \times n \) upper triangular matrices. Prove by induction that \( A_1 A_2 \cdots A_n \) is upper triangular for all \( n \in \mathbb{N} \).

**Solution.** We shall prove by induction on \( n \).

**Base cases:** When \( n = 1 \), this is trivially true.

**Induction hypothesis:** Let \( n \in \mathbb{N} \). Assume that \( A_1 A_2 \cdots A_n \) is upper triangular.

**Inductive step:** We want to show that \( A_1 A_2 \cdots A_{n+1} \) is also upper triangular. By induction hypothesis, \( A_1 A_2 \cdots A_n \) is upper triangular. By part (a), the product of two upper triangular matrices, \( (A_1 A_2 \cdots A_n) \) and \( A_{n+1} \) is upper triangular. Thus \( A_1 A_2 \cdots A_{n+1} \) is upper triangular.

**Conclusion:** By the principle of mathematical induction, we are done.

8. Consider any board of size \( 2^n \times 2^n \) \( (n \in \mathbb{N}) \) with one tile removed. Prove that this board can be tiled with L-shaped tiles of size 3.

**Solution.**

**Base case:** When \( n = 1 \), we verify by removing one tile. Certainly it has to be a corner and certainly the remainder is exactly the size of an L-shaped tile. Thus it can be tiled.

**Induction step:**

**Claim 1.** Let \( n \in \mathbb{N} \). If we can tile a \( 2^n \times 2^n \) board with one tile removed, then we can tile a \( 2^{n+1} \times 2^{n+1} \) board with one tile removed.

**Proof.** We can split the \( 2^{n+1} \times 2^{n+1} \) board into four quadrants of size \( 2^n \times 2^n \) each. Certainly one of them contains the removed tile. Without loss of generality (for certainly we can rotate our board), let that be the bottom right quadrant.

By assumption, we can tile a \( 2^n \times 2^n \) board with a tile in one corner removed. We do this for all except the bottom right quadrant. Now consider the following:
where the gray shading refers to the missing tile from each $2^n \times 2^n$ board. Note that the gray-shaded area can be tiled by exactly one L-shaped tile. Thus we have completely tiled all except the bottom right quadrant.

We know that this quadrant with the missing tile (denoted by black shading) can be tiled by assumption since it is a $2^n \times 2^n$ board with one tile removed.

We have tiled the $2^{n+1} \times 2^{n+1}$ board, thus proving the claim. \qed

**Conclusion:** By the principle of mathematical induction, we are done.

### 4 Additional Notes

1. Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$. Note the difference between $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u}^T \mathbf{v}$ even though they evaluate to the same 'value'.

2. Matrix addition is defined on matrices of the same size.

3. Matrix multiplication is defined when the number of columns of the first matrix is the same as the number of rows of the second matrix.

4. In general, matrix multiplication does not commute.

5. When using the $[ \cdot \mid \cdot ]_{i,j}$ notation, note that the transposition is simply swapping the $i$ and $j$, NOT swapping the row and column indices.

6. The general template for mathematical induction is as follows:

   - **State the statement:** Let $P(k)$ be the statement that...
   - **Base case:** Show that the base case is true: When $k = ???$, we have...
   - **Induction hypothesis:** Assume the statement is true for some $k \in \mathbb{N}$: Let $P(k)$ be true.
   - **Inductive step:** Show that $P(k+1)$ is also true
     
     Note: Remember to cite the induction hypothesis when you use it here!
   - **Conclusion:** By the principle of mathematical induction...

Mathematical induction is a **principle**. This means that there’s no fixed template to using mathematical induction. The most important thing is to understand the principle.

### 5 Exercises


I hope you see the relationship between the columns of $A$ and the rows of $B$, and of the relationship between the rows of $A$ and the columns of $B$. 

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