1. (a) i. True or False? Provide a counterexample if false, and correct the statement.

ii. Given matrices $A$ and $B$, the matrix multiplication $AB$ is always a valid operation.

iii. Given matrices $A$ and $B$, and that $AB$ is a valid matrix multiplication, $BA$ is also a valid operation.

iv. Given matrices $A$ and $B$, and that $AB$ and $BA$ are valid matrix multiplications, then $AB = BA$.

v. Given square matrices $A$ and $B$, if $AB = BA$, then $ABA^{-1} = B$.

(b) Consider $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -2 & 5 & 7 \\ 4 & 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$.

i. What is the Row-Column Representation of matrix multiplication? Express $[AB]_{2,1}$ as the product of a row vector and column vector, then evaluate it.

ii. What is the Matrix-Column Representation of matrix multiplication? Express the second column of $AB$ as the product of a matrix and a column vector, then evaluate it.

iii. What is the Row-Matrix Representation of matrix multiplication? Express the third row of $AB$ as the product of a row and a matrix, then evaluate it.

iv. Evaluate $AB$.

v. What is the Column-Row Representation of matrix multiplication? Verify your answer in part (iv) by evaluating $AB$ using the Column-Row Representation.

2. In this problem, we are interested in properties of matrix transposition and symmetric matrices. A matrix is said to be symmetric if and only if $A = A^t$.

(a) Let $C = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ be a $2 \times 2$ matrix. If $D = [c_{ji} + j]_{ij}$, write out the matrix $D$.

(b) Using the $[ ]_{ij}$ notation, prove that $(A^t)^t = A$.

(c) Using the $[ ]_{ij}$ notation, prove that $(AB)^t = B^tA^t$. (Hint: Let $A = [a_{ij}]_{ij}$. Then what is $A^t$?)

(d) Using parts (b) and (c), show that $AA^t$ and $A^tA$ are symmetric matrices.

(e) Prove or disprove: $AA^t = A^tA$.

(f) State the most general condition(s) under which $A + A^t$ is symmetric. Then prove that $A + A^t$ is symmetric with these/this condition(s).

3. (a) Let $A$ be an invertible matrix.

i. What can we say about the size of $A$?

ii. What is the inverse of $A^{-1}$?

iii. What is the inverse of $A^n$ where $n \in \mathbb{Z}$?

(b) State the Fundamental Theorem of Invertible Matrices.

(c) Define the term elementary matrices.

(d) Let $B$ be a matrix with 5 rows. Write down the matrices corresponding to the following row operations on $B$:

i. $r_2 \rightarrow r_2 + 2r_3$. 


ii. \( r_1 \leftrightarrow r_4 \).

iii. \( r_3 \rightarrow cr_3 \) for non-zero scalar \( c \).

(e) What is the domain of the det function?

(f) What are the axioms of the det function?

(g) State 7 other properties of the det function?

(h) What are the determinants of the matrices obtained in part (d)?

(i) Show using part (h) how performing the different row operations on a matrix change the determinant.

(j) Let \( B = \begin{pmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{pmatrix} \)

i. Compute \( \det B \).

ii. For what values of \( k \) is \( B \) invertible?

(k) Compute \( \det \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 5 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 6 & 5 & 4 \end{pmatrix} \).

4. Consider \( n \times n \) square matrices \( A(k) \) of the form

\[
a_{i,j} = \begin{cases} 1 & \text{if } j - i = k \\ 0 & \text{otherwise} \end{cases}
\]

(a) For \( n = 4 \), write out \( A_1, A_2, A_3 \).

(b) Using induction, prove that \( A(1)^k = A(k) \) for all \( k \in \mathbb{N}_0 \).

(c) Does this hold for negative integer \( k \)?

(d) Given \( x, y \in \mathbb{N}_0 \), prove that \( A(x) \) and \( A(y) \) commute.

(e) Given \( x, y \in \mathbb{N}_0 \), hypothesize what conditions are necessary for \( A(x)A(y) = 0 \).

(f) Prove your hypothesis in part (e).

5. A matrix \( [a_{i,j}]_{ij} \) is called a diagonal matrix if it is of the form

\[
a_{i,j} = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\]

Suppose \( A \) is a diagonal square matrix. Prove by induction that \( A^k = [a_{i,j}^k]_{ij} \) for \( k \in \mathbb{N} \).

6. (a) Define the following

i. span,

ii. eigenvalue,

iii. eigenvector, and

iv. eigenspace.

(b) Let \( A \) be a matrix with eigenvalue \( \lambda \) and corresponding eigenvector \( \mathbf{v} \neq \mathbf{0} \).

i. What is the relationship between \( A^n \), \( \lambda \), and \( \mathbf{v} \) where \( n \in \mathbb{N} \)?

ii. Suppose \( A \) is invertible. \( \lambda \) cannot take on a certain value. What is it?

iii. Suppose \( A \) is invertible. What is the relationship between \( A^n \), \( \lambda \), and \( \mathbf{v} \) where \( n \in \mathbb{Z} \)?
(c) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

i. Verify that $-1$ and $2$ are eigenvalues of $A$.

ii. Let $E_{-1}$ be the eigenspace of $-1$ and $E_2$ be the eigenspace of $2$. Express $E_{-1}$ as the span of $\{v_1, v_2\}$ and $E_2$ as the span of $\{v_3\}$ where $v_1$, $v_2$, $v_3$ are vectors.

iii. Let $S = (v_1 \ v_2 \ v_3)$ be a matrix. Prove that $S$ is invertible.

iv. Find $S^{-1}$.

v. Compute $S^{-1}AS$.

vi. Find a closed form for $A^n$ where $n \in \mathbb{N}$.

7. (a) Prove that if an $n \times n$ square matrix $A$ has $n$ eigenvalues (not necessarily distinct), then $\det A$ is equal to the product of its eigenvalues.

(b) If $A$ does not have $n$ eigenvalues, then $\det A$ may or may not be equal to the product of its eigenvalues. Give an example where it still is, and an example where it isn’t.

8. (Bonus Problem) In this problem, we explore the determinant property of diagonal square hypermatrices.

(a) Let $m = m_1 + m_2$, $n = n_1 + n_2$, $p = p_1 + p_2$. Let $A_{i,j}$ be an $m_i \times n_j$ matrix and $B_{i,j}$ be an $n_i \times p_j$ matrix. Define $C_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j}$. Prove that

$$
\begin{pmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2}
\end{pmatrix}
\begin{pmatrix}
B_{1,1} & B_{1,2} \\
B_{2,1} & B_{2,2}
\end{pmatrix} = 
\begin{pmatrix}
C_{1,1} & C_{1,2} \\
C_{2,1} & C_{2,2}
\end{pmatrix}
$$

(Note that this property extends to hypermatrices of other sizes.)

(b) Let $n = n_1 + n_2$. Let $A_1$ be an $n_1 \times n_1$ matrix and $A_2$ be an $n_2 \times n_2$ matrix. Prove that

$$
\det \begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix} = \det A_1 \cdot \det A_2
$$

(c) Now let $n = n_1 + n_2 + \ldots + n_m$ and $A_i$ be an $n_i \times n_i$ matrix for $i = 1, 2, \ldots, m$. Prove that

$$
\det \begin{pmatrix}
A_1 & A_2 & \ldots & A_m
\end{pmatrix} = \det A_1 \cdot \det A_2 \cdot \ldots \cdot \det A_m
$$

END OF REVIEW PAPER