1. (a) Fill in the blanks.

\[
\begin{array}{cccccc}
A & B & \neg A & A \land B & A \lor B & A \Rightarrow B & A \Leftarrow B \\
T & T & & & & & \\
T & F & & & & & \\
F & T & & & & & \\
F & F & & & & & \\
\end{array}
\]

(b) State the definition of *even* and *odd* used in class.

(c) In each of the following statements, circle the terms (e.g. \(x, y\)) that the bolded quantifiers refer to.

i. \(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x, y) \Rightarrow Q(x)\)

ii. \(\forall x \in \mathbb{Z}, \left(\exists y \in \mathbb{Z}, P(x, y)\right) \lor \left(\exists y \in \mathbb{Z}, Q(x, y)\right)\)

(d) Negate the following statements:

i. \(\forall x, P(x)\)

ii. \(\exists x, P(x)\)

iii. \(P \land R\)

iv. \(P \lor R\)

v. \(P \Rightarrow R\)

vi. \(P \Leftarrow R\)

(e) Consider the statement:

\(\forall x \in \mathbb{Z}, \left(\exists y \in \mathbb{Z}, x = 2y + 1\right) \Rightarrow \left(\exists y \in \mathbb{Z}, 5x^2 = 4y + 1\right)\)

i. If we denote the statement as \(\forall x \in \mathbb{Z}, P(x)\), write down what \(P(x)\) is.

ii. If we denote \(P(x)\) as \(Q(x) \Rightarrow R(x)\), write down what \(Q(x)\) and \(R(x)\) are.

iii. What is the negation of \(R(x)\)?

iv. Using the above parts (or not), negate the original statement. Be sure to show all your working.

(f) Prove or disprove the statement in 1(e).

2. In this problem, we are interested in properties of matrix transposition and symmetric matrices. A matrix is said to be symmetric if and only if \(A = A^t\).

(a) Let \(C = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\) be a \(2 \times 2\) matrix. If \(D = \begin{bmatrix} c_{ji} + j \end{bmatrix}_{ij}\), write out the matrix \(D\).

(b) Using the \([ \ ]_{ij}\) notation, prove that \((A^t)^t = A\).

(c) Using the \([ \ ]_{ij}\) notation, prove that \((AB)^t = B^tA^t\). *(Hint: Let \(A = [a_{ij}]_{ij}\). Then what is \(A^t\)?)

(d) Using 2(a), show that \(AA^t\) and \(A^tA\) are symmetric matrices.

(e) Prove or disprove: \(AA^t = A^tA\).

(f) State the most general condition(s) under which \(A + A^t\) is symmetric. Then prove that \(A + A^t\) is symmetric with these/this condition(s).

3. (a) Solve the following linear systems by Gaussian Elimination.

\[
\begin{align*}
a + 2b + 3c &= 1 \\
b + 4c &= 0 \\
5a + 6b &= 0
\end{align*}
\]

\[
\begin{align*}
d + 2e + 3f &= 0 \\
e + 4f &= 1 \\
5d + 6e &= 0
\end{align*}
\]

\[
\begin{align*}
g + 2h + 3j &= 0 \\
h + 4j &= 0 \\
5g + 6h &= 1
\end{align*}
\]

*(Hint: Solve them all together.)*
(b) Compute \[
\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix}.
\]

(c) Compute \[
\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}.
\]

(d) Find values of \(x\) and \(y\) for which \[
\begin{pmatrix} 2 & 2 & 4 \\ 3 & x & -1 \\ -1 & 0 & 5 \end{pmatrix} \]

i. a unique solution
ii. infinitely many solutions
iii. no solution.

4. (a) State all the axioms concerning vector spaces.

(b) Let \(M\) be the set of all \(2 \times 2\) square matrices with real entries. Show that \(M\) is a vector space under matrix addition and scalar multiplication over the reals.

(c) State the definition of a linear transformation as in the lecture notes/textbook.

(d) Prove that the following is an equivalent definition for linear transformation:
Let \(T : U \rightarrow V\) be a function where \(U\) and \(V\) are vector spaces. \(T\) is a linear transformation if for \(x, y \in U\) and scalar \(c\), \(T(x + cy) = T(x) + cT(y)\).

(e) Express \[
\begin{pmatrix} 13 \\ 9 \\ 22 \end{pmatrix}
\]
as a linear combination of \[
\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}.
\]

(f) Let \(T : \mathbb{R}^3 \rightarrow M\) be a linear transformation such that \[
T\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},
T\left(\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},
T\left(\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}\right) = \begin{pmatrix} 4 & -3 \\ 5 & 9 \end{pmatrix}.
\]
Find \(T\left(\begin{pmatrix} 13 \\ 9 \\ 22 \end{pmatrix}\right)\).

(g) You may have observed that \(T : \mathbb{R}^3 \rightarrow M\) is actually defined by \[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \\ z \end{pmatrix}.
\]
Prove that this is indeed a linear transformation.

5. (a) Fill in the blanks.

<table>
<thead>
<tr>
<th>Object</th>
<th>Normal form</th>
<th>General form</th>
<th>Vector form</th>
<th>Parametric form</th>
</tr>
</thead>
<tbody>
<tr>
<td>line in (\mathbb{R}^2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>line in (\mathbb{R}^3)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>plane in (\mathbb{R}^3)</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

(b) Write down a system of 2 linear equations that gives the plane:
\[
\mathcal{P}_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}, \quad s, t \in \mathbb{R}
\]
(c) Find the equation of the line $\ell$ that passes through $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and is perpendicular to the normal of the plane and the direction $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$. Express your final answer in the parametric form.

6. True or False? If false, rectify the statement.

(a) Differentiation is a linear transformation.
(b) To prove that $\mathbb{R}^2$ restricted to $x \geq y$ is not a vector space, we want to show that the closure property does not hold. It suffices to show that there is a $\mathbf{v} = (x_v, y_v)$ with $x_v \geq y_v$, and its inverse $-\mathbf{v} = (-x_v, -y_v)$ is not in the space since $-x_v \leq -y_v$.
(c) Given $A$ and $B$ are matrices, and $AB$ is a well-defined matrix multiplication, we can conclude that $BA$ is also a well-defined matrix multiplication.
(d) The dot product operation on two vectors in $\mathbb{R}^n$ can be expressed as the matrix multiplication of a $1 \times n$ matrix and an $n \times 1$ matrix.
(e) If $A$ and $B$ are square $n \times n$ matrices, then $A + B = B + A$ and $AB = BA$.

Bonus. We have never actually proven that $-\mathbf{u} = (-1)\mathbf{u}$. Prove it.

(Hint: If that statement seems very obvious, here is the question rephrased: ‘why is the inverse of a vector $\mathbf{v}$ equals to $-1$ times $\mathbf{v}$?’)

END OF REVIEW PAPER