Continuous equilibria in incomplete markets

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Agenda

1. Equilibrium
2. Asset pricing puzzles
3. Taylor approximation
The individual investor’s problem

- Investor $i$, $i = 1, 2, ..., I$, is modeled by ($I < \infty$)
  - Beliefs: $\mathbb{P}^{(i)}$: We consider $\mathbb{P}^{(i)} = \mathbb{P}$ for all $i$
  - Preferences: $U^{(i)}(c)$ - utility function over running consumption $c_t$
  - Income: $Y_t^{(i)}$ - income process
- The financial market is modeled by
  - Money market account $S^{(0)}_t$: interest rate
  - Stock $S_t$: drift and volatility
  - Dividends $D_t$: We consider no dividends
- Given these quantities, how should the investor optimally consume $(\hat{c}^{(i)}_t)$ and invest $(\hat{\rho}^{(i)}_t, \hat{\theta}^{(i)}_t)$ in $(S^{(0)}_t, S_t)$ order to maximize expected utility?
- Main obstacle: Market completeness - or rather the lack of it
Utility maximization with income

- The problem without income, $Y := 0$, has been studied in a very general setting and we now know basically everything.
- If the income process $Y$ is spanned by the market $(S^{(0)}, S)$, the problem can be treated as a no income problem: simply increase the investor’s initial endowment appropriately.
- For unspanned income, the individual investor’s problem is hard.
- There are only few concrete examples.
  - For exponential utility, see Henderson (2005).
Equilibrium (Radner type)

- How do the equilibrium price processes $S_t^{(0)}$ and $S_t$ look like?
  - The aggregate demand for $S^{(0)}$ - money market - has to be zero
  - The aggregate demand for the stock $S$ has to equal the supply
  - All investors have to follow utility maximizing strategies

- Existence and uniqueness
  - For the complete case the martingale method explicitly gives us the individual demand functions
  - The incomplete case is much less developed, see Cuocu and He (1994), Cheridito et al (2009), and Žitković (2010)

- Examples of tractable models
  - In the complete case the representative agent can be used
  - Infinite horizon discrete time models with a continuum of agents with homogeneous preferences: see Constantinides and Duffie (1996) and Wang (2003) - Bewley models
Clearing conditions

- We need to find the dynamics of \((S^{(0)},S)\), i.e., we need the interest rate, the stock’s drift, and the stock’s vol such that
  - Clearing in the stock market (no dividends)
    \[
    \sum_{i=1}^{I} \hat{\theta}_t^{(i)} = 0
    \]
  - Clearing in the money market (zero supply)
    \[
    \sum_{i=1}^{I} \hat{\rho}_t^{(i)} = 0
    \]
  - Clearing in the goods market
    \[
    \sum_{i=1}^{I} \hat{c}_t^{(i)} = \sum_{i=1}^{I} Y_t^{(i)}
    \]
Exogenous parameters

- CARA preferences over running consumption
  \[ U^{(i)}(c) := -\exp(a^{(i)} c), \quad c \in \mathbb{R} \]

- Investor specific risk preference parameters \( a^{(i)} > 0 \)

- Heston-type of stochastic vol model (Feller process)
  \[ dv_t := \kappa_v (\theta_v - v_t) dt + \sigma_v \sqrt{v_t} dW_t \]

- Income rate dynamics
  \[ dY^{(i)}_t := \mu^{(i)}_Y dt + \sqrt{v_t} \left( \sigma^{(i)}_Y dW_t + \beta^{(i)}_Y dZ^{(i)}_t \right) \]

Since \( W \perp Z^{(1)} \perp ... \perp Z^{(l)} \), the market \((S^{(0)}, S)\) cannot span \((Y^{(i)})_{i=1}^l\), i.e., the market \((S^{(0)}, S)\) is incomplete
Individual optimization problems

- The investor chooses a strategy $\theta$ to be invested in $S$ (where the investment $\rho$ in $S(0)$ is given by the self-financing condition) and generates the wealth dynamics

$$dX^{(c,\theta)}_t := \rho_t dS_t^{(0)} + \theta_t dS_t + (Y_t - c_t)\Gamma(dt), \quad X^{(c,\theta)}_0 := 0$$

- Seek $(\hat{\theta}, \hat{c})$ maximizing expected utility over running consumption

$$\sup_{(\theta, c) \in A^{(i)}} \mathbb{E} \left[ \int_0^T U^{(i)}(c_u)\Gamma(du) \right],$$

where $\Gamma(dt)$ is a finite measure on $[0, T]$

- Continuous trading is always allowed. $\Gamma$ allows us to consider both discrete and continuous income/consumption
Equilibrium

- The money market account: $S^{(0)}$ has equilibrium dynamics
  \[ dS_t^{(0)} = S_t^{(0)} r_t dt, \]
  for a stochastic interest rate $r_t$ adapted to $\sigma(v_u)_{u \in [0, t]}$

- The risky security $S$ has equilibrium dynamics
  \[ dS_t = \left( S_t r_t + \mu_S(t) v_t \right) dt + \sqrt{v_t} dW_t, \]
  for a deterministic drift function $\mu_S$

- All European claims paying out $f(S_T) - f$ sufficiently smooth and uniformly bounded - are replicable

1. Conjecture the above form and derive the optimal consumption processes $\hat{c}_t^{(i)}$ for $i = 1, ..., l$

2. Aggregation produces dynamics consistent with the conjecture
Naive Representative Agent

- The representative agent’s utility function is defined by

\[ U_{\text{rep}}(c) := -e^{-\frac{1}{\tau} c}, \quad c \in \mathbb{R}, \quad \tau := \sum_{i=1}^{l} \frac{1}{a(i)} \]

- Because of CARA-preferences there are no Negishi-weights appearing in \( U_{\text{rep}} \) (Gorman aggregation)

- Aggregate endowment is defined by \( E_t := \sum_{i=1}^{l} Y_{t}^{(i)} \)

- The welfare theorems state (under additional assumptions)

1. In a complete setting all equilibria are Pareto efficient, meaning any re-distribution \( \mathcal{E} \) which increases one investor’s expected utility will lower some other investor’s expected utility

2. In a complete setting any equilibrium is implemented by the representative agent (see the next slide)
The argument from Breeden (1986)

- In a complete market the state-price density has the dynamics
  \[ d\xi_t^{\text{rep}} = -\xi_t^{\text{rep}} \left( r_t^{\text{rep}} dt + \lambda_t^{\text{rep}} dW_t + \ldots dZ_t^{(1)} + \ldots dZ_t^{(l)} \right) \]

- The martingale method produces the proportionality relation
  \[ U'_t(\mathcal{E}_T) \propto \xi_T^{\text{rep}} \]

- This produces the equilibrium interest rate \( r_t^{\text{rep}} \) and the equilibrium market price of risk
  \[ \lambda_t^{\text{rep}} = \mu_{S_t}^{\text{rep}}(t) \sqrt{v_t} \]

- How does this naive approach compare to the actual incomplete equilibrium quantities \( r_t \) and \( \mu_S(t) \sqrt{v_t} \)? This is the basis for our definition of asset pricing puzzles
Continuous case

• Let the income/consumption rate be continuous, i.e., $\Gamma(dt) := dt$
• Risk-free puzzle: Our model explicitly quantifies that $r_t < r_t^{rep}$
• Equity premium puzzle: Our model produces $\mu_S(t) = \mu_S^{rep}(t)$
• Under fairly general conditions we prove that all Brownian based income models (and exponential utilities) produce the same market price of risk process as the naive representative agent
Discrete case

- As before consider the income model

\[ dY_t^{(i)} := \mu_Y^{(i)}(t)dt + \sqrt{v_t}\left(\sigma_Y^{(i)}dW_t + \beta_Y^{(i)}dZ_t^{(i)}\right) \]

with \( v \) being the common Feller process

\[ dv_t := \kappa_v(\theta_v - v_t)dt + \sigma_v\sqrt{v_t}dW_t \]

- Consider only consumption at \( t = 0 \) and \( t = T \) (for simplicity)
- Let \( \sigma_v < 0 \) (Gatheral)
- Risk-free puzzle: Our model explicitly quantifies that \( r_0 < r_0^{rep} \)
- Equity premium puzzle: Our model produces \( \mu_S(t) > \mu_S^{rep}(t) \)
Continuous case - redefined

- Let the income/consumption rate be continuous, i.e., $\Gamma(dt) := dt$ and let the income processes $Y^{(i)}$ be as before.
- We have measured risk-premium via
  
  $$dW^\text{min}_t := dW_t + \mu_S(t)\sqrt{v_t}dt,$$

  which is needed to obtain martingality of $S$ under the minimal martingale measure $Q^\text{min}$.
- Zero coupon bonds are spanned by $(S^{(0)}, S)$ and have price
  
  $$B(t, T) := \mathbb{E}^\text{min} [\exp(-\int_t^T r_u du)|\mathcal{F}_t], \quad t \in [0, T].$$

- The forward measure $Q^T$ is defined such that $S^{(0)}_t/B(t, T)$ and $S_t/B(t, T)$ are driftless under $Q^T$. 
Continuous case - redefined continued

- $Q^T_{\text{rep}}$ denotes the analogue for the naive representative agent defined via

$$B^{\text{rep}}(t,T) := E^{Q^\text{rep}_{\text{min}}} \left[ \exp \left( - \int_t^T r_u du \right) | \mathcal{F}_t \right], \quad t \in [0,T].$$

- Using the explicit expressions for $Q^\text{min}$ and $Q^\text{min}_{\text{rep}}$ we can find processes $b$ and $b^{\text{rep}}$ such that

$$dW^{Q^T}_{t} := dW_t + b_t dt, \quad dW^{Q^T}_{\text{rep}} := dW_t + b^{\text{rep}}_t dt$$

- If - as before - $\sigma_v < 0$ our model produces $b_t > b^{\text{rep}}_t$. 
The idea

- Use simple tractable models as approximation for general models
- The general setting (only terminal consumption)

\[
\sup_{\theta \in \mathcal{A}^{(i)}} \mathbb{E} \left[ U^{(i)} \left( X^\theta_T + f_i(B_T, W^{(i)}_T) \right) \right]
\]

- Žitković (2010) as well as Žitković and Yingwu (201x) study the existence of a function \( \lambda = \lambda(t, B, W) \) such that

\[
dS_t = \lambda(t, B_t, W^{(1)}_t, \ldots, W^{(I)}_t) dt + dB_t
\]

- No uniqueness of dynamics
- The main complication is that this requires us to study a nonlinear coupled system of \( I \) second order PDEs
The idea continued

- For $f_i = \tilde{f}_i$ where

  \[ \tilde{f}_i(B, W) = \alpha_0^{(i)} + \alpha_1^{(i)} B + \alpha_2^{(i)} B^2 + \alpha_3^{(i)} W + \alpha_4^{(i)} W^2 + \alpha_5^{(i)} B W \]

  this PDE system can be solved in closed form
- $\tilde{f}_i$ is a 2nd order Taylor approximation of $f_i$
- We need to perform stability analysis of the PDE system with respect to variations of the terminal conditions